SENSITIVITY STUDY OF ESTIMATION METHODS OF THE TWO-DIMENSIONAL AUTOREGRESSIVE MODEL

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Abstract: In this paper we present an estimator of the parameters of an AR-2D model that is an extension of an estimator presented for autoregressive models in time series. It uses an auxiliary model (BIP-AR) that limits the propagation of noise in an AR process. In addition, we present an analysis of the behavior of these new estimator (BMM-2D) and others estimators for the case of AR-2D processes contaminated by Gaussian noise. We also show an application to the image processing obtaining favorable results for our estimator. Computational implementation is carried out by R statistical software.

Keywords: *Robust Estimation, Autoregressive Process, Image Processing*

1 INTRODUCTION

During the last decade, the representation and recovery of images have been one of the topics of greater interest in the treatment of images in general. The spatial autoregressive model (AR-2D) has been extensively used because of its ability to represent a variety of real scenarios without the need of using a large number of parameters. Different robust estimators for the parameters of this model have been studied. Kashyap and Eom [4] proposed the class of M-estimators, Allende et al. [1] introduced the GM-2D estimators and Ojeda [6] the RA-2D estimators. The high expressiveness of the AR-2D model justifies the development of new robust estimators of its parameters. In this sense, this work proposes a new class of robust estimators for the AR-2D model: the BMM-2D estimators; these are a two-dimensional version of the BMM estimators, defined for time series ARMA models in 2009 by Muler et al. [5].
In this paper we intend to show the performance of the BMM-2D estimator in comparison with other estimators (LS, M, GM and RA) and exhibit its capacity for image restoration and segmentation. The proposal is of interest in the field of Statistical Image Processing, understanding an image as the realization of a two-dimensional autoregressive random process.

2 METHODOLOGY

2.1 AR-2D MODELS

Consider a spatial process with an associated random variable $Y_{i,j}$ defined in each place (i, j) of the rectangular grid in two dimensions. One of models studied in spacial data analysis has been autoregressive two-dimensional model (AR-2D) (Whittle, [9]) which is defined as

$$\Phi(B_1, B_2)Y_{i,j} = \varepsilon_{i,j}$$

where $\Phi(B_1, B_2) = \sum_k \sum_l \phi_{k,l} B_1^k B_2^l$ with $B_1 Y_{i,j} = Y_{i-1,j}$ and $B_2 Y_{i,j} = Y_{i,j-1}$, and $\varepsilon_{i,j}$'s are i.i.d. random variables with symmetric distribution, $E(\varepsilon_{i,j}) = 0$ and $Var(\varepsilon_{i,j}) = \sigma^2$. A particular case is AR unilateral model of first order given by:

$$Y_{i,j} = \phi_1 Y_{i-1,j} + \phi_2 Y_{i,j-1} + \phi_3 Y_{i-1,j-1} + \varepsilon_{i,j} \tag{1}$$

with $\Phi(B_1, B_2) = 1 - (\phi_1 B_1 + \phi_2 B_2 + \phi_3 B_1 B_2)$.
In this paper we will work with the stationary version of (1).

2.2 BIP-AR 2D MODELS

A new class of bounded nonlinear AR-2D models is presented in this work, the bounded innovation propagation AR-2D model (BIP-AR 2D). The model is a generalization in two dimensions of the model

presented for time series by Muler et al. [5].
Given a unilateral two-dimensional autoregressive process of first order like in (1) stationary and invertible, it supports an moving average representation as follows:

$$Y_{i,j} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty} \lambda_{klr}\varepsilon_{i-k-r,j-l-r} = \Lambda(B_1, B_2)\varepsilon_{i,j}$$

where $\lambda_{klr} = \frac{(k+l+r)!}{k!l!r!}\phi_1^k\phi_2^l\phi_3^r$ with $k, l, r \in \mathbb{N}_0$ and $\Lambda(B_1, B_2) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty} \lambda_{klr} B_1^{k+r} B_2^{l+r}$ with $B_1^{k+r} Y_{i,j} = Y_{i-k-r,j}$ and $B_2^{l+r} Y_{i,j} = Y_{i,j-l-r}$.
Since the goal is to estimate the best possible parameters of a autoregressive central model when a contaminated process is observed, it appealed to BIP-AR 2D auxiliary model defined by:

$$Y_{i,j} = \sum_{(k,l,r)\in D} \lambda_{klr}\sigma\eta\left(\frac{\varepsilon_{i-k-r,j-l-r}}{\sigma}\right) + \varepsilon_{i,j}$$

with $D = \{k \geq 0, l \geq 0, r \geq 0\} \setminus \{k = 0 = l = r\}$, $\varepsilon_{i,j}$'s are i.i.d. random variables with symmetric distribution, $\lambda_{i,j}$ are coefficients of $\Phi^{-1}(B_1, B_2)$, $\eta(x)$ is a odd and limited function and σ is a robust M-scale of $\varepsilon_{i,j}$.

2.3 BMM-ESTIMATOR

Similarly to how the BMM estimator for the time series was defined, the estimator BMM-2D is defined in $\mathbb{Z}^2$. The idea is to compute in a first stage a highly robust estimator of the error scale, and in a second stage this estimated scale is used to calculate an M-estimator of the parameters.

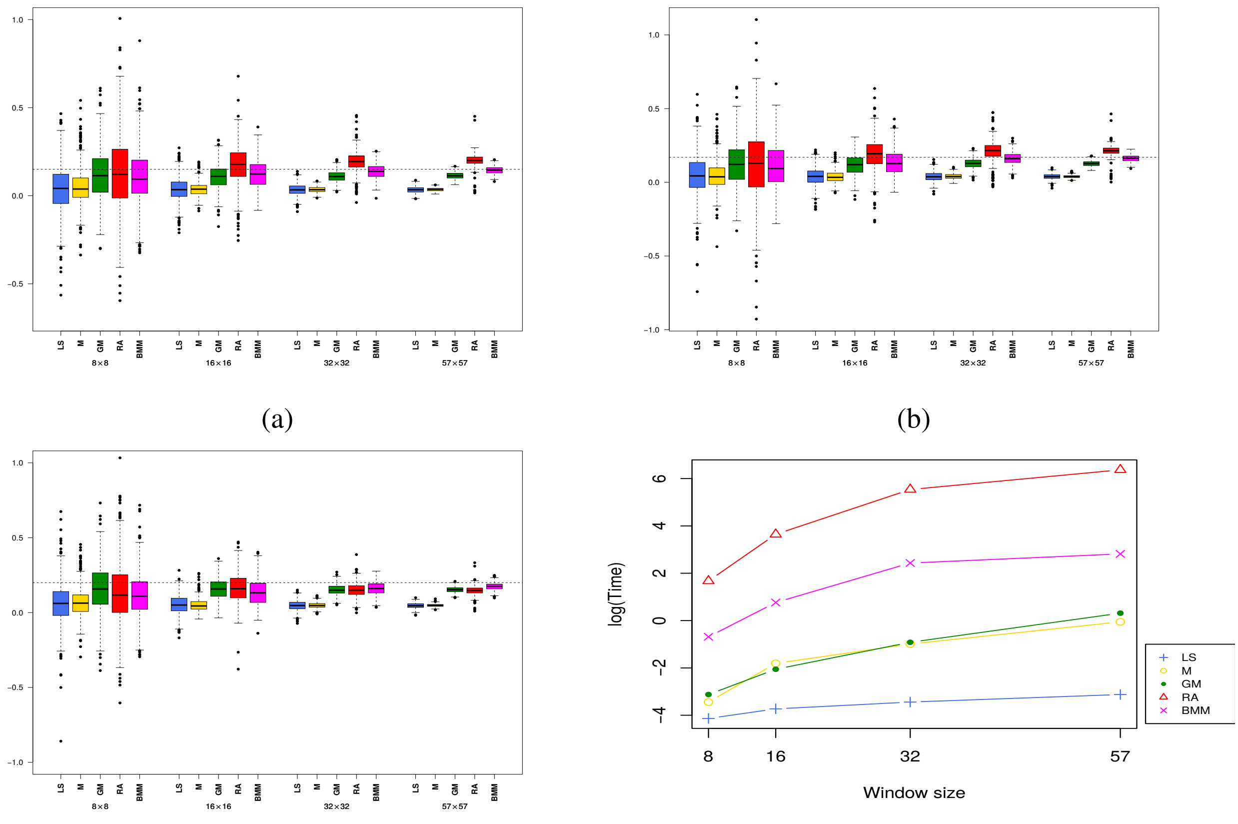
First Step: At this stage we obtain an estimator of σ. For this purpose we consider two estimators of σ, one using an AR-2D model and one using a BIP-AR 2D model, and choosing the smaller of them.
Then, our σ estimator is

$$s^*_{nm} = \min(s_{nm}, s^b_{nm})$$

Second Step: With the scale estimator obtained in the first step (s^*_{nm}) two M-estimators of the parameters are calculated, one using the residues of one AR-2D model and the other using the residues of a BIP-AR 2D model. The final estimator is the one that minimizes its corresponding objective function.

3 RESULTS AND DISCUSSION

The performances of the LS, M, GM, RA and BMM estimators were assessed in contaminated AR-2D processes. For this, Monte Carlo studies were performed simulating two-dimensional autoregressive processes as in (1) with parameters $\phi_1 = 0.15$, $\phi_2 = 0.17$ and $\phi_3 = 0.2$. For each observation window size (8×8, 16×16, 32×32, 57×57) were obtained 500 process simulations which were contaminated at 10% with a Gaussian process of $\mu = 0$ and $\sigma^2 = 50$. Then, the parameters of the model were estimated by LS, M, GM, RA and BMM. The Figures (1)-((a), (b), (c)) show the boxplots of the estimates. We can observe that in the size 32×32, the BMM estimates better than the others estimators in the sense that it is closer to the true value and has a small variance. For the others window size, the BMM-estimator remained close to the true value and maintained a small variance. It should be noted that the study without contamination was also carried out obtaining better results with the LS estimator, which was an expected result.
The computational routines were developed in the programming language R and carried out on a personal computer with a Pentium(R) Dual-Core 2.70GHz*2 processor. Figure (1)-(d) shows the logarithm of the time in seconds that it took to perform a single of these simulations with each one of the estimators and according to the window size used. It is observed that although the RA estimator is one of the major competitors of the BMM estimator due to the accuracy of its estimation and its asymptotic properties, it has a very high computational cost which makes it unattractive when applied to large image processing.



(a) (b) (c) (d)

Figure 1: Assume a AR-2D model of 3 parameters with additive contamination of $\sigma^2 = 50$. According to window size: (a) boxplot of $\phi_1 = 0.15$ estimate, (b) boxplot of $\phi_2 = 0.17$ estimate, (c) boxplot of $\phi_3 = 0.2$ estimate; (d) Logarithm of the estimation time (in seconds).

The analysis of contaminated images is of great interest in several areas of research. For example, the reconstruction of contaminated images is relevant in modeling of images ([2], [8]), and in general the reduction of the noise produced by interferences plays an important role in the literature ([3]). In Ojeda et al. ([7]) two algorithms were presented, the first produces an images local approximation by the use of unilateral AR-2D processes and the second is a segmentation algorithm. In the present work we will obtain reconstructions and segmentations of an image from a modification of the first algorithm that will consist of representing an image by block-fitting a unilateral AR-2D process with three parameters.
Figure 2 shows the image reconstruction ability by adjusting an AR-2D process in different window sizes and estimating the parameters with the BMM estimator. In the images from (b) to (e) the reconstructions obtained with windows of sizes 8×8, 16×16, 32×32 and 57×57 are observed. At first glance the reconstructions are good with any window size although if we analyze the similarity SSIM, CQ(1,1) and $\mathrm{CQ}_{\max}$ as can be observed in Table 1, as the window size increases, the similarity decreases which would indicate that the best fit is obtained with small size windows. This fact reflects the assumption that a two-dimensional autoregressive model is a local adjustment model. On the other hand, if we look at the images difference (second algorithm (f) to (i)), it is observed that (i) image highlights the edges more than the others which shows that when doing the reconstruction with window size 57×57 (e) a lot of information was left aside.

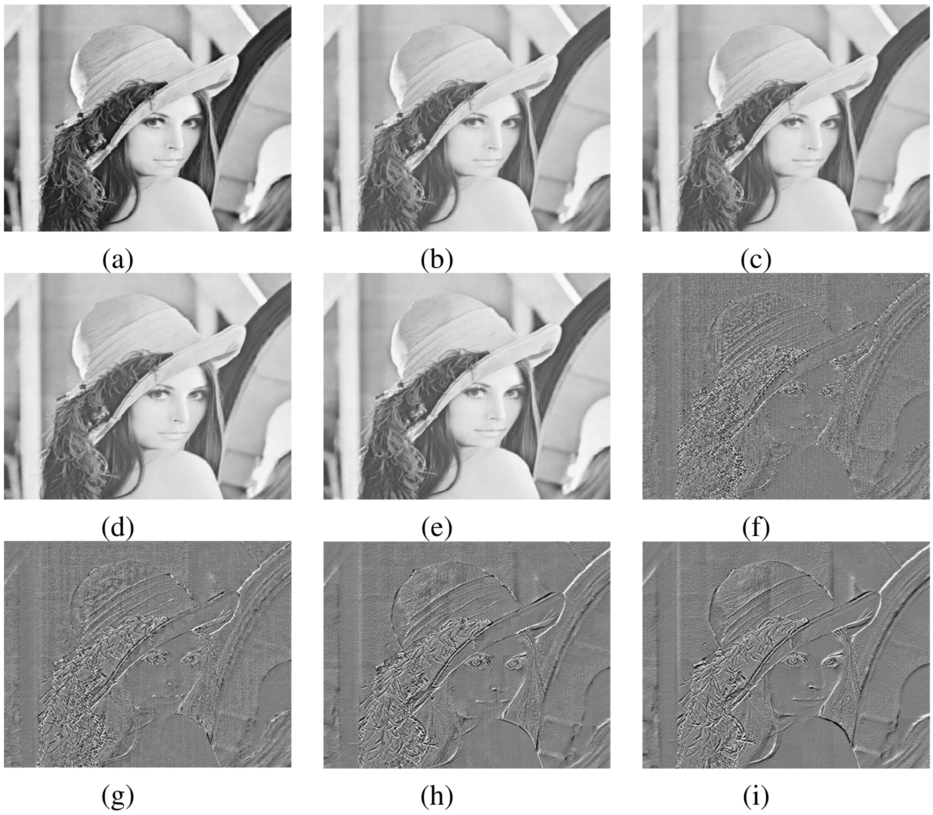


Figure 2: (a) Original image of Lena. From (b)-(e), the reconstructions based in BMM for windows 8×8, 16×16, 32×32 and 57×57 respectively. From (f)-(i), the differences of (a) with the reconstructions of (b) to (e) respectively.

Window size	SSIM	CQ(1,1)	$CQ_{\max}$
8×8	0.9948914	0.8582201	0.9706984
16×16	0.9827996	0.8309626	0.9544317
32×32	0.9779204	0.8151581	0.9462133
57×57	0.9762065	0.8073910	0.9423786

Table 1: Similarity between the original image and the reconstructions in Figure 2.

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