The Lattice of Congruences of a Finite Line Frame

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Abstract

Let $\mathbf{F} = \langle F, R\rangle$ be a finite Kripke frame. A *congruence* of $\mathbf{F}$ is a bisimulation of $\mathbf{F}$ that is also an equivalence relation on F. The set of all congruences of $\mathbf{F}$ is a lattice under the inclusion ordering. In this article we investigate this lattice in the case that $\mathbf{F}$ is a finite line frame. We give concrete descriptions of the join and meet of two congruences with a nontrivial upper bound. Through these descriptions we show that for every nontrivial congruence ρ, the interval $[\mathrm{Id}_{\mathrm{F}}, \rho]$ embeds into the lattice of divisors of a suitable positive integer. We also prove that any two congruences with a nontrivial upper bound permute.

1 Introduction

Let $\mathbf{F} = \langle F, R\rangle$ be a finite Kripke frame (i.e., directed graph). A *congruence* of $\mathbf{F}$ is a bisimulation of $\mathbf{F}$ that is also an equivalence relation on F. The set of all congruences of $\mathbf{F}$ is a lattice under the inclusion ordering. In this article we investigate congruence lattices of finite *line frames* (also called "lines" in the sequel). To be precise, a *line* is a finite frame $\mathbf{L} = \langle\{0, \dots, n\}, R\rangle$, where $x\ R\ y$ iff $|x - y| \leq 1$. We are able to give concrete descriptions of the join and meet of two congruences with a nontrivial upper bound. Through these descriptions we show that for every nontrivial congruence ρ, the interval $[\mathrm{Id}_{\mathrm{L}}, \rho]$ embeds in a natural way into the lattice of divisors of a suitable positive integer. We also prove that any two congruences with a nontrivial upper bound permute; that is their join is the composition.

The congruences of a line have an appealing geometrical nature, since a congruence can be thought of as a way of *folding* the line. It is also possible to give a geometrical representation of two congruences on a diagram resembling the *trajectory* of a particle traveling inside a rectangle and bouncing along its sides. These representations turn the problem of understanding the congruence lattice of a line into a surprisingly elementary geometrical puzzle.

1.1 Background

Our interest in congruences of frames arises from our study of equationally definable functions (called *algebraic functions* [3]) in modal algebras. Efforts to characterize this kind of functions for other classes of algebraic structures (e.g., Boolean algebras, distributive lattices, abelian groups, etc.) has been the focus of [3], [2] and [6]. This line of research is motivated by the understanding that many structural properties of algebras are tied to syntactical phenomena. Mal'cev conditions [1] are a premier example of this fact, and a variety of fundamental results in Universal Algebra are of this nature. Thus, studying the term-functions of the algebras in a given class is a powerful approach to understand structural properties. Term-functions are the most basic *definable functions*, and oftentimes characterizations require more complex syntactical counterparts. Algebraic functions are a natural generalization of term-functions; they share

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many of the basic properties of term-functions and every term-function is an algebraic function [3]. An interesting example of a result linking equationally definable functions to structural properties can be found in [5].

In [3] it is shown that the study of algebraic functions in the algebras of a variety $\mathcal{V}$ can be approached through the investigation of the subclasses of $\mathcal{V}$ axiomatizable by sentences of the form $\forall\bar{x}\exists_{=1}\bar{y}\,\alpha(\bar{x},\bar{y})$, where $\alpha(\bar{x},\bar{y})$ is a conjunction of term-equalities. We call these formulas *EFD-sentences* (EFD stands for equational function definition). In the case that $\mathcal{V}$ is a finitely generated discriminator variety (in particular, when $\mathcal{V}$ is a finitely generated variety of modal algebras [8]) the EFD-axiomatizable subclasses of $\mathcal{V}$ are in correspondence with the subclasses of simple members of $\mathcal{V}$ closed under isomorphisms, fixed-point subalgebras and intersection of subalgebras [4]. Thus, we set out to characterize the classes $\mathcal{C}$ of finite simple modal algebras satisfying the above closure conditions. This turned out to be a very challenging task. The key difficulty comes from the fact that there is no obvious description of the atoms of an intersection of two subalgebras of a modal algebra. Our tool of choice to tackle this problem is the duality linking finite modal algebras and Kripke frames [9]. The lattice of subuniverses of a finite modal algebra is dually isomorphic with the lattice of congruences of its dual frame. So understanding intersection of subalgebras amounts to the same thing as understanding the join of congruences of frames. Finding a simple description for the join of two congruences of an arbitrary finite frame appears to be a very hard (perhaps impossible) task. In this article we focus on line frames, and already for this special case we found it to be an interesting problem.

It is worth mentioning that we did not include here any of the (universal) algebraic applications of the results, as we plan to address them in a forthcoming article.

Outline In the next section we study the basic properties of congruences of lines and provide an arithmetical characterization for them. We also introduce their description as foldings; this representation proves to be very useful for getting insight and simplifying proofs. In particular, it will lead to the characterization of the order among congruences.

In Section 3 we address a special case of joins, namely, when one of the congruences has equivalence classes with at most two elements. This case is treated separately because it does not follow the general pattern that joins of congruences with bigger classes enjoy. This pattern is captured with the help of trajectories, which we introduce in Section 4.

Our main results are gathered in Section 5. Section 5.1 provides a complete catalog of the possible local configurations of pairs of congruences having nontrivial joins. This catalog will help in proving permutability in Section 5.2. Finally, we obtain a simple formula for calculating the quotient of a line by a nontrivial join in Section 5.4.

2 Congruences of Lines

2.1 Congruences of a frame

Let $\mathbf{F} = \langle F, R\rangle$ be a frame. Recall that a *bisimulation* of $\mathbf{F}$ is a binary relation $\theta \subseteq F \times F$ satisfying $\theta \circ R \subseteq R \circ \theta$ and $\theta^{-1} \circ R \subseteq R \circ \theta^{-1}$. We call a bisimulation of $\mathbf{F}$ that is also an equivalence relation on F a *congruence* of $\mathbf{F}$. We write $\operatorname{Con}\mathbf{F}$ to denote the set of congruences of $\mathbf{F}$. Since equivalence relations are symmetric, θ is a congruence of $\mathbf{F}$ if and only if

$$\big(\forall x, x', y \in F\big)\;\; x'\;\theta\; x \text{ and } x\; R\; y \text{ imply there is } y' \text{ such that } x'\; R\; y' \text{ and } y'\;\theta\; y.$$

Observe that when R is symmetric, the congruences of $\mathbf{F}$ are the equivalence relations θ on F that permute with R, i.e., such that $\theta \circ R = R \circ \theta$.

We say that an equivalence relation is *trivial* if it has exactly one equivalence class. The *quotient* of $\mathbf{F}$ by θ is the frame $\mathbf{F}/\theta \doteq \langle F/\theta, R_\theta\rangle$ where $x/\theta\; R_\theta\; y/\theta$ iff there are $x'\;\theta\; x$ and $y'\;\theta\; y$

such that $x' \; R \; y'$. For $x \in F$ we define $R[x] \doteq \{y \in F \mid x \; R \; y\}$ and $v(x) \doteq |R[x]|$. Here are some basic properties of congruences and quotients.

Lemma 1. *Let θ be a congruence of the frame $\mathbf{F} = \langle F, R \rangle$.*

1. *$x \; R \; y$ implies $x/\theta \; R_\theta \; y/\theta$.*
2. *If there is a R-path from x to y in $\mathbf{F}$, then there is one from x/θ to y/θ in $\mathbf{F}/\theta$. Thus if $\mathbf{F}$ is connected so is $\mathbf{F}/\theta$.*
3. *$R_\theta [x/\theta] = R[x]/\theta$.*
4. *$v(x/\theta) \leq v(x)$.*
5. *If $\mathbf{F}$ is connected and there is $x \in F$ such that $R[x] \subseteq x/\theta$* then θ is trivial.

In what follows we may omit the subscript θ in R_θ whenever there is no risk of confusion.

2.2 Congruences of a line

For an integer $n \geq 0$ let $\mathbf{L}_n \doteq \langle \{0, \ldots, n\}, R \rangle$ where $x \; R \; y$ iff $|x - y| \leq 1$. Note that $x \; R \; x$ for all x. We call such a frame a *line*. The remainder of this section is devoted to characterizing the congruences of $\mathbf{L}_n$.

Lemma 2. *Let θ be a congruence of $\mathbf{L}_n$.*

1. *$\mathbf{L}_n/\theta$ is a line.*
2. *If $a \; \theta \; b$ and $a < b$, then there is $x \in [a, b)$ such that $v(x/\theta) \leq 2$.*
3. *Suppose θ is nontrivial, and let $k = \min \{x \in [1, n] \mid v(x/\theta) = 2\}$. Then $\mathbf{L}_n/\theta$ is isomorphic with $\mathbf{L}_k$ and $0/\theta, \ldots, k/\theta$ are all the equivalence classes of θ.*

Proof. 1. Note that by Lemma 1 $\mathbf{L}_n/\theta$ must be connected and $v(x/\theta) \leq 3$ for all x. Furthermore, $v(0/\theta) \leq 2$ and thus the quotient has to be a line.
2. If $a = 0$ there is nothing to prove since $v(0/\theta) \leq 2$, so we may suppose $a > 0$. If $b = a + 1$, then $R[a/\theta] = R[a]/\theta = \{a - 1/\theta, a/\theta, a + 1/\theta\}$, but this is a contradiction since $a \; \theta \; a + 1$ and θ is nontrivial. If $b = a + 2$, then $R[a + 1/\theta] = R[a + 1]/\theta = \{a/\theta, a + 1/\theta, a + 2/\theta\}$, and it follows that $v(a + 1/\theta) \leq 2$. Assume $b - a > 2$. We proceed now by induction in b. The case $b = 1$ is vacuously true (since $a > 0$). Suppose $b > 1$ and observe that

$$b - 1/\theta \in R[b/\theta] = R[a/\theta] = R[a]/\theta = \{a - 1/\theta, a/\theta, a + 1/\theta\}.$$

If $b - 1/\theta \in \{a/\theta, a + 1/\theta\}$, we are done by inductive hypothesis (note that by our assumptions $a + 1 < b - 1$). So it only remains to deal with case $b - 1 \; \theta \; a - 1$. By inductive hypothesis there is $x \in [a - 1, b - 1)$ such that $v(x/\theta) \leq 2$. If $x = a - 1$ then $v(b - 1/\theta) = v(a - 1/\theta) \leq 2$. If on the other hand $x \neq a - 1$, then $x \in [a, b)$.
3. Observe that such a k must always exist. In fact, by 1 we know that $\mathbf{L}_n/\theta$ is a line, and since θ is not trivial this line has more than one point. One of the endpoints is $0/\theta$ so there must be a nonzero element whose equivalence class is the other endpoint. Also observe that as θ is not trivial we have $v(x/\theta) > 1$ for all x.

If there are $a, b \in [0, k]$ with $a \; \theta \; b$ and $a < b$, then by 2 there is $x \in [a, b)$ such that $v(x/\theta) = 2$, in contradiction with the minimality of k. Thus $0/\theta, \ldots, k/\theta$ are pairwise distinct and $(0/\theta) \; R \; (1/\theta) \; R \cdots R \; (k/\theta)$. Finally, as $\mathbf{L}_n/\theta$ is a line, there cannot be any more equivalence classes. $\square$

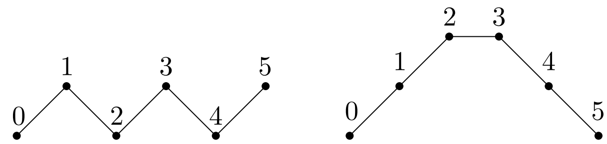

Figure 1: Relations $\langle 1\rangle$ and $\langle 2;2\rangle$ on $\mathbf{L}_5$

Given a congruence θ of $\mathbf{L}_n$ we define the *step* of θ to be the unique k such that $\mathbf{L}_n/\theta$ is isomorphic with $\mathbf{L}_k$, i.e., the step of θ is $|L_n/\theta| - 1$. Observe that 3 in the above lemma shows how to compute the step of a nontrivial congruence. A *rest* of θ is a pair in θ of the form $\langle r, r+1\rangle$. Here r is the *left part* and $r+1$ is the *right part* of the rest. We write θ *has a rest at* r to express that $\langle r, r+1\rangle$ is a rest of θ. The choice of terminology shall become clear once we introduce the geometrical representation of congruences as foldings of the line.

Next we introduce a family of equivalence relations on the set of integers. Given a (possibly empty) strictly increasing sequence of non-negative integers $\bar{r} = \langle r_1, \dots, r_m\rangle$ let $\Delta_{\bar{r}} : \mathbb{Z} \to \mathbb{Z}$ be defined by

$$\Delta_{\bar{r}}(x) \doteq |\{i \mid r_i < x\}|.$$

Observe that when $\bar{r}$ is the empty sequence then $\Delta_{\bar{r}}$ is constantly 0. We often drop the subscript in $\Delta_{\bar{r}}$ when $\bar{r}$ is clear from the context.

For a positive integer q let $\equiv_q$ denote the equivalence relation modulo q on the integers.

Definition 3. For an integer $k \geq 1$ and a (possibly empty) strictly increasing sequence of non-negative integers $\bar{r}$, let $\langle k;\bar{r}\rangle$ be the binary relation defined on the set of integers by

$$x \,\langle k;\bar{r}\rangle\, y \text{ iff } [x - \Delta_{\bar{r}}(x) \equiv_{2k} y - \Delta_{\bar{r}}(y)] \text{ or } [x - \Delta_{\bar{r}}(x) \equiv_{2k} -y + \Delta_{\bar{r}}(y)].$$

When $\bar{r}$ is the empty sequence we just write $\langle k\rangle$. Note that in this case we have $x \,\langle k\rangle\, y$ iff $x \equiv_{2k} \pm y$.

To avoid tiresome repetitions, from this point onward k shall always denote a positive integer and $\bar{r}$ a strictly increasing sequence of non-negative integers of length $|\bar{r}|$. We write $a \in \bar{r}$ to signify that a equals some entry of $\bar{r}$.

It is straightforward to check that $\langle k;\bar{r}\rangle$ is an equivalence relation on $\mathbb{Z}$. Thus, so is its restriction to $\{0, \dots, n\}$, which we shall also denote by $\langle k;\bar{r}\rangle$. To get a better understanding of these relations it is convenient to introduce a graphical representation for them. To make a picture of $\langle k;\bar{r}\rangle$ on $\mathbf{L}_n$ we draw $\mathbf{L}_n$ 'folded' in such a way that two points are at the same height if and only if they are $\langle k;\bar{r}\rangle$-related. Our convention is to place 0 (and thus every point in the same block) at the bottom level. Figure 1 shows two examples.

Next we show that every congruence of $\mathbf{L}_n$ is of this type.

Lemma 4. *Let θ be a nontrivial congruence of $\mathbf{L}_n$ with step k. Suppose $\{x \mid x \,\theta\, x+1\} = \{r_1, \dots, r_m\}$, where $m \geq 0$ and $r_1 < \ldots < r_m$. Then $\theta = \langle k;\bar{r}\rangle$.*

Proof. Let us write Δ for $\Delta_{\bar{r}}$. Our first observation is that point 3 of Lemma 2 implies $r_1 \geq k$, and hence

$$\Delta(j) = 0 \text{ for all } j \in \{0, \dots, k\}. \tag{1}$$

We prove by induction on x that θ and $\langle k;\bar{r}\rangle$ agree on the interval $[0, x]$. This is easily seen to be true for $x \leq k$ using 3 of Lemma 2, so assume $x > k$. Note that $x-1 \,\theta\, x$ iff $x-1 \,\langle k;\bar{r}\rangle\, x$, thus we may suppose

$$x - 1/\theta \neq x/\theta, \tag{2}$$

and so

$$\Delta(x-1) = \Delta(x). \tag{3}$$

Let $j \in [0, k]$ be such that $x - 1 \; \theta \; j$. By inductive hypothesis we have

$$x - 1 - \Delta(x-1) \equiv_{2k} j \text{ or } x - 1 - \Delta(x-1) \equiv_{2k} -j.$$

There are three cases to consider.

Case $j = 0$. We have $x - 1 - \Delta(x-1) \equiv_{2k} 0$, which by (3) implies $x - \Delta(x) \equiv_{2k} 1$, and this in combination with (1) yields $x \; \langle k; \bar{r} \rangle \; 1$. Also,

$$x/\theta \in R[x - 1/\theta] = R[0/\theta] = \{0/\theta, 1/\theta\},$$

so (2) says that $x \; \theta \; 1$.

Case $j = k$. Note that

$$x/\theta \in R[x - 1/\theta] = R[k/\theta] = \{k - 1/\theta, k/\theta\},$$

and so by (2) we have $x \; \theta \; k - 1$. Also, $x - 1 - \Delta(x-1) \equiv_{2k} -k$ and (3) imply $x - \Delta(x) \equiv_{2k} -(k-1)$, which in view of (1) says that $x \; \langle k; \bar{r} \rangle \; k - 1$.

Case $1 < j < k$. Observe that

$$\{x - 2/\theta, x - 1/\theta, x/\theta\} = R[x - 1/\theta] = R[j/\theta] = \{j - 1/\theta, j/\theta, j + 1/\theta\},$$

where the three elements in the right-hand side set are different. So,

$$\{x - 2/\theta, x/\theta\} = \{j - 1/\theta, j + 1/\theta\}, \tag{4}$$

and since $x - 2/\theta \neq x - 1/\theta$

$$\Delta(x-1) = \Delta(x-2).$$

Suppose first that

$$x - 1 - \Delta(x-1) \equiv_{2k} j.$$

Then,

$$\begin{aligned} x - 2 - \Delta(x-2) &\equiv_{2k} j - 1 \\ x - \Delta(x) &\equiv_{2k} j + 1, \end{aligned}$$

or equivalently

$$\begin{aligned} x - 2 \; &\langle k; \bar{r} \rangle \; j - 1 \\ x \; &\langle k; \bar{r} \rangle \; j + 1. \end{aligned}$$

So, by inductive hypothesis $x - 2 \; \theta \; j - 1$, and from (4) we obtain $x \; \theta \; j + 1$.

The case $x - 1 - \Delta(x-1) \equiv_{2k} -j$ is handled in the same way. □

Corollary 5. *A congruence of $\mathbf{L}_n$ is determined by its step and rests.*

The next lemma tells us which of the relations $\langle k; \bar{r} \rangle$ are congruences of $\mathbf{L}_n$. Note that when considering $\langle k; \bar{r} \rangle$ on $[0, n]$ we can safely assume $\bar{r} \subseteq [0, n-1]$, due to the fact that adding or removing entries of $\bar{r}$ greater than $n - 1$ does not change the values of $\Delta_{\bar{r}}$ on $[0, n]$.

Lemma 6. *Let $n \geq 3$, and suppose $r_1, \ldots, r_m = \bar{r} \subseteq [0, n-1]$ and $k \leq \frac{n}{2}$. Then $\langle k; \bar{r} \rangle$ is a nontrivial congruence of $\mathbf{L}_n$ if and only if the following conditions hold:*

1. *$r_1 \neq 0$, $r_m \neq n - 1$ and $r_{i+1} - r_i > 2$, for all $i \in \{1, \ldots, |\bar{r}| - 1\}$,*
2. *k divides $r_i - i + 1$, for all $i \in \{1, \ldots, |\bar{r}|\}$, and*

3. k divides $n - |\bar{r}|$.

Proof. Let us write θ for $\langle k; \bar{r}\rangle$. The lemma for $m = 0$ is an easy exercise and we leave it to the reader. Assume $m \geq 1$. We prove first that if conditions (1-3) hold, then θ is a congruence of $\mathbf{L}_n$. As we know that θ is an equivalence relation we only need to check that given $a, a', b \in [0, n]$ such that $a' \;\theta\; a \;R\; b$, there is b' satisfying $a' \;R\; b' \;\theta\; b$.

When $a = a'$ there is nothing to prove, so assume $a \neq a'$. Also, if $a \;\theta\; b$ we can take $b' = a'$, so we suppose

$$\langle a, b\rangle \notin \theta. \tag{5}$$

There are four cases to consider.

Case $a - \Delta(a) \equiv_{2k} a' - \Delta(a')$ and $b = a + 1$.

Observe that (5) implies $a \notin \bar{r}$. We consider first the sub-case $a' \in \bar{r}$. By 1 we know that $a' \neq 0$ and $a' - 1 \notin \bar{r}$, so $\Delta(a' - 1) = \Delta(a')$. By 2 we have that k divides $a' - \Delta(a')$, so $-a' + \Delta(a') \equiv_{2k} a' - \Delta(a')$. Hence,

$$-\left(a' - 1 - \Delta\left(a' - 1\right)\right) \equiv_{2k} a + 1 - \Delta(a + 1).$$

That is, $a' - 1 \;\theta\; a + 1$, and we can take $b' = a' - 1$. Observe that the same proof also works for the case $a' = n$ using 3 instead of 2. So the only sub-case left to address is $a' \notin \bar{r}$ and $a' < n$. Here it is easy to see that $b' = a' + 1$ works.

Case $a - \Delta(a) \equiv_{2k} a' - \Delta(a')$ and $b = a - 1$.

From (5) we obtain $a - 1 \notin \bar{r}$. If $a' = 0$ or $a' - 1 \in \bar{r}$ a reasoning analogous to the one used in the sub case $a' \in \bar{r}$ above shows that $a' + 1 \;\theta\; a - 1$, and so we can take $b' = a' + 1$. If $a' \neq 0$ and $a' - 1 \notin \bar{r}$, then it is straightforward to check that $b' = a' - 1$ does the job.

The two remaining cases are left to the reader.

Next suppose θ is a nontrivial congruence of $\mathbf{L}_n$. We prove that (1-3) hold. First observe that if 1 does not hold then there is $x \in [0, n]$ such that $R[x] \subseteq x/\theta$, and thus θ would be trivial by 5 of Lemma 1. To establish 2 and 3 we need to show first that $k \leq r_1$. For the sake of contradiction suppose $r_1 < k$. By Lemma 2, $v(r_1) = 2$. From the definition of $\langle k; \bar{r}\rangle$ it is easily checked that r_1 is the minimal element with such property. So by Lemma 2 we have that r_1 is the step of θ and $0/\theta, \ldots, r_1/\theta$ are all the θ-blocks. In particular $r_1 + 2 \;\theta\; j$, for some $j \in [0, r_1]$. So

$$r_1 + 1 = r_1 + 2 - \Delta(r_1 + 2) \equiv_{2k} \pm j.$$

That is, $2k$ divides either $r_1 + 1 - j$ or $r_1 + 1 + j$. Both cases easily yield a contradiction.

Now, if $k \leq r_1$ then k is the step of θ, and by Lemma 2 we know that $0/\theta, \ldots, k/\theta$ are all the equivalence classes of θ. Observe that $v(r_i/\theta) = 2$, and thus $r_i/\theta \in \{0/\theta, k/\theta\}$, for all $i \in \{1, \ldots, m\}$. Analogously $n/\theta \in \{0/\theta, k/\theta\}$. From here 2 and 3 are easily obtained. $\square$

Combining Lemmas 4 and 6 we obtain a complete description of the congruences of a line.

Theorem 7. *The congruences of $\mathbf{L}_n$ are the diagonal, $L_n \times L_n$, and all the relations $\langle k; \bar{r}\rangle$ such that:*

- $k \leq \frac{n}{2}$,
- $\bar{r} \subseteq [k, n - k]$,
- *k divides $r_i - i + 1$, for all $i \in \{1, \ldots, |\bar{r}|\}$, and*
- *k divides $n - |\bar{r}|$.*

We can rewrite the conditions in the theorem above to characterize the foldings of $\mathbf{L}_n$ that represent nontrivial congruences.

- The vertical segments have all the same length.
- The horizontal segments (rests) of the folding must be at the top and bottom levels.
- Consecutive rests are not allowed.
- Rests starting at 0 or ending at n are not allowed.
- n must be at the top or bottom level.

2.3 Basic properties of congruences

It turns out that for two congruences to have a nontrivial join both must have a highly regular and compatible structure. Thus a common theme throughout the paper is that most results apply to this case. In the current section we start to pin down the aforementioned regularity, and characterize the meet of two congruences with nontrivial join. The characterization of the join takes quite a bit more of work, and is handled in the subsequent sections.

For the remainder of the current section we fix a natural number n, and write θ and δ to denote congruences of $\mathbf{L}_n$.

We start out with a list of equivalent conditions for a congruence to be trivial.

Lemma 8. *Suppose $n \geq 2$. The following are equivalent:*

1. *θ is trivial.*
2. *There is x such that $x - 1 \; \theta \; x \; \theta \; x + 1$.*
3. *$0 \; \theta \; 1$.*
4. *$n - 1 \; \theta \; n$.*

Proof. Immediate by 5 of Lemma 1. □

We say that $e \in [0, n]$ is an *extreme* of θ if e/θ is one of the endpoints of $\mathbf{L}_n/\theta$. We denote the set of all extremes of θ by $\operatorname{ext}(\theta)$. Observe that if k is the step of θ we have

$$\operatorname{ext}(\theta) = 0/\theta \cup k/\theta.$$

Note that 0 and n are always extremes, and if r is part of a rest then $r \in \operatorname{ext}(\theta)$. Also note that the step of θ equals its first positive extreme.

Suppose $e \notin \{0, n\}$ is an extreme of θ that is not part of a rest. If we take two points, one on each side of e, that are at the same distance d from e they will be θ-related as long as d is small enough. For instance this is always true if d is not greater than the step of θ. In fact, it is evident from looking at the folding associated to θ that as we start moving away from e, equidistant points will be related as long as we don't hit a rest on one side that is missing from the other. So, the symmetry breaks for the first time when we encounter an asymmetrical rest or we fall off the line on one side and not the other. See Figure 2.

When there is a rest at e the situation is the same, only that the symmetry is with respect to $e + \frac{1}{2}$. These facts are summarized in a precise way below.

Lemma 9. *Let $e \notin \{0, n\}$ be an extreme of θ.*

1. *Suppose e is not part of a rest. Assume*

$$\{t \in \omega \mid e - t \text{ and } e + t \text{ are in } [0, n] \text{ and are not } \theta\text{-related}\} \neq \varnothing,$$

and let x be the least member of this set. Then $e + x - 1$ is an extreme of θ, and either

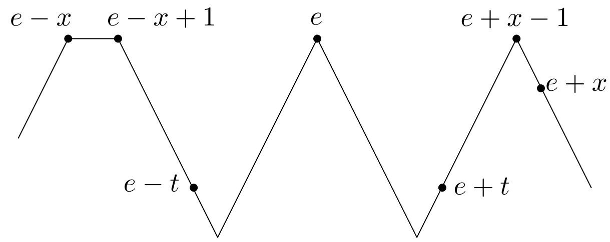

Figure 2: Moving away from an extreme

(a) *there is a rest at* $e-x$ *and* $e+x-1$ *is not part of a rest or*

(b) *there is a rest at* $e+x-1$ *and* $e-x$ *is not part of a rest.*

2. *Suppose there is a rest at* e. *Assume*

$$\{t \in \omega \mid e-t \text{ and } e+t+1 \text{ are in } [0,n] \text{ and are not } \theta\text{-related}\} \neq \varnothing,$$

and let x *be the least member of this set. Then* $e+x$ *is an extreme of* θ*, and either*

(a) *there is a rest at* $e-x$ *and* $e+x$ *is not part of a rest or*

(b) *there is a rest at* $e+x$ *and* $e-x$ *is not part of a rest.*

Next we show that when the join of two congruences is not trivial they behave exactly the same at common extremes.

Lemma 10. *Let* θ *and* δ *be congruences of* $\mathbf{L}_n$ *such that* $\theta \vee \delta$ *is not trivial, and let* $e \in \operatorname{ext}(\theta) \cap \operatorname{ext}(\delta)$. *Then* e *is the left (right) part of a rest of* θ *if and only if* e *is the left (right) part of a rest of* δ.

Proof. Suppose e is the left part of a rest of θ and is not part of a rest of δ. Then $e\ \theta\ e+1$ and $e-1\ \delta\ e+1$, so

$$e-1\ (\theta \vee \delta)\ e\ (\theta \vee \delta)\ e+1.$$

Thus $\theta \vee \delta$ is trivial by Lemma 8. The remaining cases are just as easy. $\square$

When applied to a pair of congruences with nontrivial join, the two lemmas above show that each of these congruences is determined by its behavior between any two consecutive common extremes that do not constitute a rest.

Lemma 11. *Assume* $\theta \vee \delta$ *is not trivial, and let* $e \in \operatorname{ext}(\theta) \cap \operatorname{ext}(\delta)$ *such that* $e \notin \{0, n\}$.

1. *Suppose* e *is not part of a rest of* θ*, and let* d *be the distance from* e *to the next greater element of* $\operatorname{ext}(\theta) \cap \operatorname{ext}(\delta)$. *Then* $e-d$ *is the greatest element of* $\operatorname{ext}(\theta) \cap \operatorname{ext}(\delta)$ *before* e *and*

$$e-t\ \theta\ e+t$$
$$e-t\ \delta\ e+t$$

for $t = 0, \ldots, d$.

2. *Suppose* θ *has a rest at* e*, and let* d *be the distance from* $e+1$ *to the next greater element of* $\operatorname{ext}(\theta) \cap \operatorname{ext}(\delta)$. *Then* $e-d$ *is the greatest element of* $\operatorname{ext}(\theta) \cap \operatorname{ext}(\delta)$ *before* e *and*

$$e-t\ \theta\ e+t+1$$
$$e-t\ \delta\ e+t+1$$

for $t = 0, \ldots, d$.

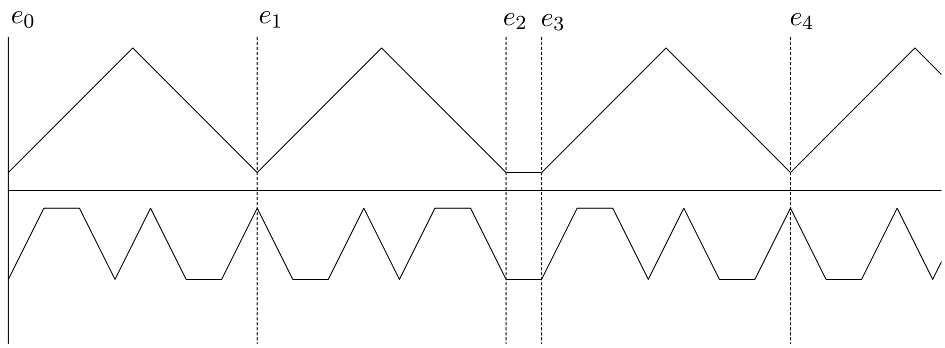

Figure 3: Periodicity

Proof. 1. For the sake of contradiction suppose there is $t \in [1, d]$ such that $e - t$ and $e + t$ are not $(\theta \cap \delta)$-related. Let x be the smallest such t. Note that $e \geq x$, because otherwise $0 = e - e\ (\theta \cap \delta)\ e + e$, and as $0 \in \text{ext}(\theta) \cap \text{ext}(\delta)$ it would follow that $2e$ is a common extreme satisfying $e < 2e < e + d$.

So $e - x \geq 0$ and $\langle e - x, e + x \rangle$ is either not in θ or not δ. We may suppose without loss that $\langle e - x, e + x \rangle$ is not in θ (the other case is symmetrical since by Lemma 10 we have that e is not part of a rest of δ). Now by 1 of Lemma 9 we know that $e + x - 1$ is an extreme of θ and either (a) or (b) of 1 in that lemma hold. Assume (a) holds, i.e., there is a rest of θ at $e - x$ and $e + x - 1$ is not part of a rest of θ. This says in particular that

$$e + x - 2\ \theta\ e + x.$$

We argue that $\langle e - x, e + x \rangle$ is not in δ. Suppose this is not the case, then

$$e + x\ \delta\ e - x\ \theta\ e - x + 1\ \theta\ e + x - 1,$$

which in combination with the previous display implies that $e + x - 2$, $e + x - 1$ and $e + x$ are all $(\theta \vee \delta)$-related, making $\theta \vee \delta$ trivial by Lemma 8. Thus $e - x$ and $e + x$ cannot be δ-related. So, applying 1 of Lemma 9 to δ it follows in particular that $e + x - 1$ is an extreme of δ, and hence

$$e + x - 1 \in \text{ext}(\theta) \cap \text{ext}(\delta).$$

Note that as θ and δ are not trivial and e is not part of a rest of either congruence we have $e - 1\ (\theta \cap \delta)\ e + 1$. Thus $x > 1$ and so

$$e < e + x - 1 < e + d.$$

But then $e + x - 1$ cannot be a common extreme by our choice of d. A contradiction.

A similar argument works in the case that point (b) of 1 in Lemma 9 holds.

2. This proof is the same as for 1 adjusting the formulas for the rest at e and applying 2 of Lemma 9 instead of 1. □

Let us take closer look at what have just proved.

Remark 12. Take an enumeration $0 = e_0 < e_1 < \cdots < e_N = n$ of $\text{ext}(\theta) \cap \text{ext}(\delta)$. If we draw the folding that represents θ, Lemma 11 says that the part from e_1 to e_2 is the reflection of the part from 0 to e_1 (assuming there is no rest at e_1). Now the part from e_2 to e_3 is a reflection of the previous part and so on. Thus θ (and δ too of course) is determined (up to rests at common extremes) by its restriction to $[0, e_1]$. Figure 3 illustrates this fact.

2.4 The meet of two congruences

Lemma 11 allows us to characterize the meet of two congruences with nontrivial join. But first we need to point out a few easy facts.

Lemma 13. *Let θ, δ and ρ be congruences of $\mathbf{L}_n$.*

1. *If $\rho \subseteq \theta$, then $\operatorname{ext}(\rho) \subseteq \operatorname{ext}(\theta)$.*
2. *If $\rho \subsetneq \theta$, then step of ρ is strictly greater than the step of θ.*
3. *If $\rho \subseteq \theta \cap \delta \neq L \times L$, then the step of ρ is greater than or equal to*

$$e_1 \doteq \min\{e \in \operatorname{ext}(\theta) \cap \operatorname{ext}(\delta) \mid e > 0\}.$$

4. *If $n = \min\{e \in \operatorname{ext}(\theta) \cap \operatorname{ext}(\delta) \mid e > 0\}$ and $\theta \cap \delta \neq L \times L$ then $\theta \wedge \delta = \mathrm{Id}$.*

Proof. 1. Let $e \in \operatorname{ext}(\rho)$. If $e \in \{0, n\}$ we are done, so assume $0 < e < n$. If e is part of a rest of ρ, then $r\ \rho\ r+1$ or $r\ \rho\ r-1$. So r must also be part of a rest of θ and thus $r \in \operatorname{ext}(\theta)$. If e is not part of a rest of ρ then $e-1\ \rho\ e+1$, so $e-1\ \theta\ e+1$ and e must be an extreme of θ.
2. Recall that the step of a congruence is the size of its quotient minus one.
3. Let q be the step of ρ. Since q is an extreme of ρ, by 1 we know that $q \in \operatorname{ext}(\theta) \cap \operatorname{ext}(\delta)$, and as $\rho \neq L \times L$ we have $q > 0$.
4. By 3 any lower bound must have step at least n, and thus the only lower bound is Id. ☐

Theorem 14. *Let $\theta = \langle k; \bar{r}\rangle$ and $\delta = \langle l; \bar{s}\rangle$ be such that their join is not trivial. Then,*

$$\theta \wedge \delta = \langle e_1; \bar{r} \cap \bar{s}\rangle,$$

where e_1 is the first positive element in $\operatorname{ext}(\theta) \cap \operatorname{ext}(\delta)$.

Proof. Let $0 = e_0 < e_1 < \cdots < e_N = n$ be the common extremes of θ and δ. Our first step is to prove that:

(i) if $e_j \notin \bar{r} \cap \bar{s}$ then, $e_{j+1} - e_j = e_1$, for $j = 0, \ldots, N-1$.

We prove it by induction in j. When $j = 0$ it is certainly true. Suppose $j > 0$; there are two cases. Assume first that $e_{j-1} \notin \bar{r} \cap \bar{s}$. Then e_j is not part of a rest of θ, so by 1 of Lemma 11 we have

$$e_{j+1} - e_j = e_j - e_{j-1},$$

and $e_j - e_{j-1} = e_1$ by inductive hypothesis. Suppose next that $e_{j-1} \in \bar{r} \cap \bar{s}$. Then, by 2 of Lemma 11 it follows that

$$e_{j+1} - e_j = e_{j-1} - e_{j-2},$$

and the right-hand side equals e_1 by inductive hypothesis.

An easy consequence of (i) is:

(ii) e_1 divides $e_j - \Delta_{\bar{r} \cap \bar{s}}(e_j)$, for $j = 0, \ldots, N$.

Let us write γ for $\langle e_1; \bar{r} \cap \bar{s}\rangle$. Now we check that:

(iii) γ is a congruence of $\mathbf{L}_n$.

In fact, the first two conditions in Theorem 7 obviously hold, and the last two follow from (ii). Our next step is to prove that γ is a lower bound of θ and δ.

(iv) Let $x \in [0, n]$ and $y \in [0, e_1]$. If $x\ \gamma\ y$, then $x\ (\theta \cap \delta)\ y$.

We proceed by induction in x. If $x \leq e_1$ it clearly holds as in this case $x = y$. Suppose $x > e_1$, and let e_j, e_{j+1} be consecutive common extremes such that $e_j < x \leq e_{j+1}$. We have three cases to consider. Assume first that neither e_{j-1} nor e_j are in $\bar{r} \cap \bar{s}$. Then it is easy to see from the definition of γ that

$$e_j - t \; \gamma \; e_j + t, \text{ for } t = 0, \ldots, e_{j+1} - e_j.$$

Also, by 1 of Lemma 11, the same holds for $\theta \cap \delta$, that is

$$e_j - t \; (\theta \cap \delta) \; e_j + t, \text{ for } t = 0, \ldots, e_{j+1} - e_j.$$

In particular for $t = x - e_j$ we have

$$2e_j - x \; (\gamma \cap \theta \cap \delta) \; x,$$

and so by inductive hypothesis we know that $2e_j - x \; (\theta \cap \delta) \; y$. Thus $x \; (\theta \cap \delta) \; y$.

In the case that $e_{j-1} \in \bar{r} \cap \bar{s}$ and $e_j \notin \bar{r} \cap \bar{s}$ the proof above works with minor adjustments. Finally, suppose $e_j \in \bar{r} \cap \bar{s}$. Then $x = e_{j+1}$, and

$$e_j \; (\gamma \cap \theta \cap \delta) \; e_{j+1}.$$

Our inductive hypothesis yields $e_j \; (\theta \cap \delta) \; y$, and hence $x \; (\theta \cap \delta) \; y$.

Observe that $0/\gamma, \ldots, e_1/\gamma$ are all the γ-blocks, and thus (iv) immediately produces:

(v) $\gamma \subseteq \theta \cap \delta$.

It only remains to see that there cannot be a lower bound greater than γ. Fix a congruence $\rho \subseteq \theta \cap \delta$, and let q be the step of ρ. By 3 of Lemma 13 we have that $q \geq e_1$, and by 2 of the same lemma ρ cannot strictly contain γ. So we proved that γ is a maximal lower bound, but as the congruences of $\mathbf{L}_n$ form a lattice γ must be the meet. $\square$

The *frequency* of a congruence $\theta = \langle k; \bar{r} \rangle$ of $\mathbf{L}_n$ is

$$\mathsf{f}_\theta \doteq \frac{n - |\bar{r}|}{k}.$$

When thinking of θ as a folding, f_θ is the number of upward and downward slopes. Thus, another way to obtain the frequency of θ is through counting extremes; this yields the formula

$$\mathsf{f}_\theta = |\mathrm{ext}\,(\theta)| - |\bar{r}| - 1. \tag{6}$$

It is worth noting that given the frequency of θ we can recover its step and number of rests. The step is the quotient of the integer division of n by f_θ, and the number of rests is the remainder.

Given $a \in [0, n]$ and θ a congruence of $\mathbf{L}_n$ let θ_a be the restriction of θ to the interval $[0, a]$. That is,

$$\theta_a \doteq \theta \cap ([0, a] \times [0, a]).$$

It is easily checked using Theorem 7 that if a is an extreme of θ that is not the right part of a rest then θ_a is a congruence of $\mathbf{L}_a$.

We conclude this section with some facts concerning frequencies needed in the sequel.

Lemma 15. *Let $\theta = \langle k; \bar{r} \rangle$ and $\delta = \langle l; \bar{s} \rangle$ be such that their join is not trivial and let e_1 be the first positive element in $\mathrm{ext}(\theta) \cap \mathrm{ext}(\delta)$. Then:*

1. $\mathsf{f}_{\theta \wedge \delta} = |\mathrm{ext}\,(\theta) \cap \mathrm{ext}\,(\delta)| - |\bar{r} \cap \bar{s}| - 1.$

2. $\mathsf{f}_\theta = \mathsf{f}_{\theta_{e_1}} \mathsf{f}_{\theta \wedge \delta}.$

3. $(\theta \vee \delta)_{e_1} = \theta_{e_1} \vee \delta_{e_1}.$

4. $\mathsf{f}_{\theta \vee \delta} = \mathsf{f}_{\theta_{e_1} \vee \delta_{e_1}} \mathsf{f}_{\theta \wedge \delta}.$

Proof. 1. This is immediate from Theorem 14.
2. Remark 12 says that the folding given by θ is composed of successive reflections of the folding given by θ_{e_1}. As each of these reflections runs between consecutive common extremes that do not constitute a common rest, the number of symmetrical pieces making up θ is

$$|\mathrm{ext}\,(\theta) \cap \mathrm{ext}\,(\delta)| - |\bar{r} \cap \bar{s}| - 1.$$

Each of these pieces contributes the same amount of upward and downward slopes, namely $\mathsf{f}_{\theta_{e_1}}$. Thus, f_θ is the number in the display above times $\mathsf{f}_{\theta_{e_1}}$, and the desired formula follows from 1.
3. We prove the nontrivial inclusion, that is from left to right. Suppose $\langle a, b\rangle \in (\theta \vee \delta)_{e_1}$. Then there are $x_1, \dots, x_m \in [0, n]$ such that

$$a \;\theta\; x_1 \;\delta\; x_2 \;\theta \cdots \theta\; x_m \;\delta\; b.$$

Now, by Theorem 14 e_1 is the step of $\theta \wedge \delta$, and so $0/\theta \wedge \delta, \dots, e_1/\theta \wedge \delta$ are all the $\theta \wedge \delta$-blocks. In particular, there are $x'_1, \dots, x'_m \in [0, e_1]$ such that $x_j \;(\theta \vee \delta)\; x'_j$ for $j = 1, \dots, m$. So we have

$$a \;\theta\; x'_1 \;\delta\; x'_2 \;\theta \cdots \theta\; x'_m \;\delta\; b,$$

which implies $\langle a, b\rangle \in \theta_{e_1} \vee \delta_{e_1}$.
4. Note that $\theta \wedge \delta$ and $\theta \vee \delta$ have a nontrivial join. Also, as $\theta \wedge \delta \leq \theta \vee \delta$, the first positive common extreme of these two congruences is the step of $\theta \wedge \delta$, which equals e_1 by Theorem 14. So 2 produces

$$\mathsf{f}_{\theta\vee\delta} = \mathsf{f}_{(\theta\vee\delta)_{e_1}} \mathsf{f}_{\theta\wedge\delta},$$

and we apply 3 to yield the equality we set out to prove. ☐

2.5 Characterization of the order of $\mathrm{Con}\,\mathbf{L}$

Our next result is a characterization of the order relation in the lattice of congruences of a line. It is also the first step in understanding when two congruences have a nontrivial join.

To achieve this goal, we provide an alternative characterization of the periodicity alluded to in Remark 12. As it was seen in Theorem 14, the "skeleton" that holds both congruences in Figure 3 is actually their meet: the points at which the structure of both θ and δ is symmetric are the extremes of $\gamma = \theta \wedge \delta$.

Take the case of θ, the congruence of the lower part of the figure. What we just stated amounts to say that the extremes of γ are extremes of θ and the structure of rests is copied from one slope of γ to the next. So we say that the rests of a congruence θ are *compatible* with γ if for every $r \notin \mathrm{ext}(\gamma)$ that is part of rest of θ, and $r' \;\gamma\; r$, then r' is part of a rest of θ.

Theorem 16. *Assume $\theta = \langle k; \bar{r}\rangle$ and $\gamma = \langle e; \bar{t}\rangle$ are nontrivial congruences of a line $\mathbf{L}$. Then $\gamma \leq \theta$ if and only if*

1. *$e \in \mathrm{ext}(\theta)$,*
2. *$\bar{t} \subseteq \bar{r}$, and*
3. *the rests of θ are compatible with γ.*

Proof. ($\Rightarrow$) Assume $\gamma \subseteq \theta \neq L \times L$; then $\gamma = \gamma \wedge \theta$. We may then apply Theorem 14 and similar arguments to show all three conditions hold.

($\Leftarrow$) We will see that for all $x \in [0, e]$, the γ-class of x is included in its θ-class. Let $y > e$ such that $x \;\gamma\; y$. Since $\Delta_{\bar{t}}(x) = 0$, there exists a $d > 0$ such that

$$y - \Delta_{\bar{t}}(y) = \pm x + 2d \cdot e. \tag{7}$$

We need the following:

Claim. $\Delta_{\bar{r}}(y) = 2d \cdot \Delta_{\bar{r}}(e) + \Delta_{\bar{t}}(y) \pm \Delta_{\bar{r}}(x)$.

We prove the Claim in the case that the "$\pm$" is a "$-$". Our first step is to show that compatibility (Item 3) implies that the rests of θ in the interval $[0, e)$ determine the remaining rests.

Assume $s \notin \text{ext}(\gamma)$, and let r_s be the unique element of $[0, e)$ such that $s\ \langle e; \bar{t}\rangle\ r_s$; by definition of $\langle e; \bar{t}\rangle$ there exists b such that either

(I) $s - \Delta_{\bar{t}}(s) = r_s + 2b \cdot e$, or

(II) $s - \Delta_{\bar{t}}(s) = -r_s + 2b \cdot e$

hold. We claim that

$$s \in \bar{r} \iff [r_s \in \bar{r} \text{ in case (I) or } r_s - 1 \in \bar{r} \text{ in case (II)}]. \tag{8}$$

For ($\Rightarrow$), assume by way of contradiction that there exists $s \in \bar{r}$ falsifying (8), and take the one with minimal r_s. Assume that s satisfies (I); hence $r_s \notin \bar{r}$. Since by compatibility r_s must be part of a rest, we conclude that $r_s - 1 \in \bar{r}$, and in particular $r_s - 1 > 0$. By subtracting 1 from Equation (I), and considering that $\Delta_{\bar{t}}(s) = \Delta_{\bar{t}}(s-1)$ (since $s - 1$ cannot be a rest since γ is nontrivial), we obtain

$$s - 1 - \Delta_{\bar{t}}(s-1) = r_s - 1 + 2b \cdot e,$$

and hence $s - 1\ \langle e; \bar{t}\rangle\ r_s - 1$. By compatibility $s - 1$ must be part of a rest, therefore $s - 2 \in \bar{r}$. Now a similar reasoning yields $r_s - 2 \notin \bar{r}$ and $r_s - 2$ is part of rest, contradicting our choice of s as the counterexample with the minimal r_s. Case (II) and the ($\Leftarrow$) direction are similar.

We conclude in particular that the number of θ-rests between two consecutive elements of $\text{ext}(\gamma)$ not related by γ is constant and equals $\Delta_{\bar{r}}(e)$. Then, taking f as the greatest extreme of γ such that $f \leq y$, we easily have

$$\Delta_{\bar{r}}(f) = (2d - 1) \cdot \Delta_{\bar{r}}(e) + \Delta_{\bar{t}}(f) = (2d - 1) \cdot \Delta_{\bar{r}}(e) + \Delta_{\bar{t}}(y),$$

where the last equality holds by definition of f and item 2. Now $\Delta_{\bar{r}}(y)$, the number of θ-rests before y, is equal to the number of rests before f plus the number of rests between f and y.

$$\Delta_{\bar{r}}(y) = (2d - 1) \cdot \Delta_{\bar{r}}(e) + \Delta_{\bar{t}}(y) + |\{\theta\text{-rests between } f \text{ and } y\}|.$$

By using compatibility one more time we realize that the last summand equals the number of θ-rests between x and e, that is $\Delta_{\bar{r}}(e) - \Delta_{\bar{r}}(x)$. So we finally obtain the Claim.

The case where the "$\pm$" is a "$+$" is handled very similarly; in this case, $\Delta_{\bar{r}}(f) = 2d \cdot \Delta_{\bar{r}}(e) + \Delta_{\bar{t}}(y)$ and $|\{\theta\text{-rests between } f \text{ and } y\}| = \Delta_{\bar{r}}(x)$. This finishes the proof of the Claim.

We show next that $x\ \langle k; \bar{r}\rangle\ y$, i.e.,

$$y - \Delta_{\bar{r}}(y)\ \langle k\rangle\ x - \Delta_{\bar{r}}(x) \tag{9}$$

Let us work from the left hand side; substitute $\Delta_{\bar{r}}(y)$ according to the Claim:

$$\begin{aligned} y - \Delta_{\bar{r}}(y) &= y - 2d \cdot \Delta_{\bar{r}}(e) - \Delta_{\bar{t}}(y) \mp \Delta_{\bar{r}}(x) \\ &= \pm x \mp \Delta_{\bar{r}}(x) + 2d \cdot e - 2d \cdot \Delta_{\bar{r}}(e) && \text{by Eq. (7)} \\ &= \pm(x - \Delta_{\bar{r}}(x)) + 2d \cdot (e - \Delta_{\bar{r}}(e)). \end{aligned}$$

But now, since $e \in \text{ext}(\theta)$, $e - \Delta_{\bar{r}}(e)$ is congruent to 0 or k modulo $2k$, hence it is a multiple of k. Thus we have proved (9).

□

3 The case of Frequency Two

We now obtain the main results of the paper restricted to the case when one of the congruences has frequency equal to 2. We need one more concept to handle this case.

We say that a congruence $\langle k;\bar{r}\rangle$ of $\mathbf{L}_n$ is *mirrored* provided that for all r

$$r \in \bar{r} \text{ if and only if } n - r - 1 \in \bar{r}.$$

It is easy to see that a congruence is mirrored if and only if the folding given by it is symmetric with respect to its middle point.

Observe that for a mirrored $\langle k;\bar{r}\rangle$ we have, for all a

$$\Delta_{\bar{r}}(a) + \Delta_{\bar{r}}(n-a) = |\bar{r}|. \tag{10}$$

By applying this to different b and c and equating, we conclude that

$$\Delta_{\bar{r}}(b) - \Delta_{\bar{r}}(c) = \Delta_{\bar{r}}(n-c) - \Delta_{\bar{r}}(n-b). \tag{11}$$

We use this concept to study joins $\theta \vee \delta$ in which $\mathsf{f}_\delta = 2$. It is immediate that there are two cases for this; let $l = \lfloor \frac{n}{2} \rfloor$. If n is even, then $\delta = \langle l \rangle$, and if n is odd, then $\delta = \langle l; l\rangle$. In either case we have that $a \;\delta\; b$ if and only if $a = b$ or $a + b = n$.

Lemma 17. *Assume* $\mathsf{f}_\delta = 2$.

1. *If $\theta = \langle k;\bar{r}\rangle$ is mirrored, θ and δ permute.*

2. *If θ is not mirrored, then $\theta \vee \delta$ is trivial.*

Proof. For item 1, assume $a \;\delta\; b \;\langle k;\bar{r}\rangle\; c$ with $a \neq b$. Hence $a = n-b$. Let $x \doteq n-c$. Immediately $c \;\delta\; x$; it is enough to see that $x \;\langle k;\bar{r}\rangle\; a$. By definition this is $x - \Delta_{\bar{r}}(x) \;\langle k\rangle\; a - \Delta_{\bar{r}}(a)$, i.e.,

$$n - c - \Delta_{\bar{r}}(n-c) \;\langle k\rangle\; n - b - \Delta_{\bar{r}}(n-b). \tag{12}$$

We have two cases, according to the definition of $b \;\langle k;\bar{r}\rangle\; c$. First consider that $c - \Delta_{\bar{r}}(c) \equiv_{2k} b - \Delta_{\bar{r}}(b)$. We operate as follows:

$$\begin{aligned}
c - \Delta_{\bar{r}}(c) \equiv_{2k} b - \Delta_{\bar{r}}(b) \iff & \\
c + (\Delta_{\bar{r}}(b) - \Delta_{\bar{r}}(c)) &\equiv_{2k} b \\
c + (\Delta_{\bar{r}}(n-c) - \Delta_{\bar{r}}(n-b)) &\equiv_{2k} b && \text{by Eq. (11)} \\
-b - \Delta_{\bar{r}}(n-b) &\equiv_{2k} -c - \Delta_{\bar{r}}(n-c) \\
n - b - \Delta_{\bar{r}}(n-b) &\equiv_{2k} n - c - \Delta_{\bar{r}}(n-c),
\end{aligned}$$

and this implies (12).

The other case is when $c - \Delta_{\bar{r}}(c) \equiv_{2k} -b + \Delta_{\bar{r}}(b)$. We have

$$\begin{aligned}
c - \Delta_{\bar{r}}(c) \equiv_{2k} -b + \Delta_{\bar{r}}(b) \iff & \\
-\Delta_{\bar{r}}(b) - \Delta_{\bar{r}}(c) &\equiv_{2k} -c - b \\
-\Delta_{\bar{r}}(b) - \Delta_{\bar{r}}(c) &\equiv_{2k} n - c + n - b - 2|\bar{r}| && \text{since } k \mid n - |\bar{r}| \\
|\bar{r}| - \Delta_{\bar{r}}(b) + |\bar{r}| - \Delta_{\bar{r}}(c) &\equiv_{2k} n - c + n - b \\
\Delta_{\bar{r}}(n-b) + \Delta_{\bar{r}}(n-c) &\equiv_{2k} n - c + n - b && \text{by Eq. (10)} \\
-(n - b - \Delta_{\bar{r}}(n-b)) &\equiv_{2k} n - c - \Delta_{\bar{r}}(n-c),
\end{aligned}$$

and this also implies (12).

Now we turn to the proof of item 2. Suppose that θ is not mirrored. Let r be the least element of L such that the equivalence $r \in \bar{r} \iff n-r-1 \in \bar{r}$ is false. Assume first that for this r, $r = r_j \in \bar{r}$ but $n-r-1 \notin \bar{r}$.

Since $r \in \bar{r}$, $0 < r < n-1$, and hence $0 \leq n-r-2, n-r \leq n$. By the observations prior to Lemma 17 we know that $n-r \; \delta \; r$ and $n-r-1 \; \delta \; r+1$, and since $r \in \bar{r}$ implies $r \; \theta \; r+1$, we conclude that

$$n-r \; \delta \; r \; \theta \; r+1 \; \delta \; n-r-1,$$

We claim that either $n-r-1 \; \theta \; n-r+1$ or there is a rest at $n-r$. This would show $\theta \vee \delta$ trivial by Lemma 8 since $n-r-1$, $n-r$ and $n-r+1$ are related by this congruence, and we are done.

To see the claim, suppose there is no a rest at $n-r$. Then we have $\Delta_{\bar{r}}(n-r-1) = \Delta_{\bar{r}}(n-r+1) = m-j+1$ where $m = |\bar{r}|$, by the minimality of $r = r_j$ and because $n-r-1 \notin \bar{r}$. By expanding the definition of $n-r-1 \; \langle k; \bar{r} \rangle \; n-r+1$, it suffices to prove

$$n-r-1-\Delta_{\bar{r}}(n-r-1) \equiv_{2k} -(n-r+1)+\Delta_{\bar{r}}(n-r+1).$$

In our context this is equivalent to $n-r-1+(n-r+1)-2(m-j+1) \equiv_{2k} 0$ and then to $k \mid n-r_j-m+j-1$. But this follows immediately from Theorem 7.

The case in which $r \notin \bar{r}$ and $n-r-1 \in \bar{r}$ for the minimal r is entirely analogous. $\square$

Corollary 18. *Assume that $\theta, \delta \in \operatorname{Con} \mathbf{L}_n$ and $\mathsf{f}_\delta = 2$. If $\theta \vee \delta$ is nontrivial then θ and δ permute.*

4 Trajectories

We introduce next a practical representation of pairs of congruences that will be of great help in understanding the join.

Notation. In the sequel, $\mathbf{L} = \langle L, R \rangle = \langle \{0, \ldots, n\}, R \rangle$ will denote a line (with $n \geq 1$) and $\theta = \langle k; \bar{r} \rangle$ and $\delta = \langle l; \bar{s} \rangle$ will be nontrivial congruences of $\mathbf{L}$ with $k \leq l$.

Consider a rectangle of base l and height k, that for convenience can be regarded as sitting on the Euclidean plane and having the origin and points of coordinates $(0,k)$, (l,k), and $(l,0)$ as vertices. Now we represent each element x of $\mathbf{L}$ as a point $P(x)$ with coordinates $\big(P_1(x), P_2(x)\big) = (\min x/\delta, \min x/\theta)$, and for each $0 \leq x < n$, we connect the points $P(x)$ and $P(x+1)$ with a line segment.

The resulting configuration of points and segments is called the *trajectory diagram of* $\theta \vee \delta$. By construction, for any $0 \leq x, y \leq n$, $x \; (\theta \vee \delta) \; y$ if and only if there exists a sequence $x_0, \ldots, x_j$ of elements of $\mathbf{L}$ such that $x_0 = x$, $x_j = y$, and for each $0 \leq i < j$, $P(x_i)$ and $P(x_{i+1})$ have the same abscissa or the same ordinate. The uses of trajectory diagrams in this paper will be restricted to congruences having $\operatorname{ext}(\theta) \cap \operatorname{ext}(\delta) = \{0, n\}$, although they have been a key tool in understanding the general case.

It may be considered that the diagram pictures the trajectory of a particle that starting at the origin $P(0)$ moves through the line $\mathbf{L}$, bouncing on the horizontal borders of the rectangle at elements e of $\operatorname{ext}(\theta) \setminus \{0, n\}$ and on vertical borders at elements of $\operatorname{ext}(\delta) \setminus \{0, n\}$. There are two types of *bounces*, depending whether e is part of a rest or not. The part of the trajectory that lies in the interior of the rectangle consists of line segments with slope ± 1. We will see that trajectories picturing nontrivial joins do not have "overlapping" segments, i.e., self-intersections consist in isolated points. It is clear that such intersections occur only at points with (half-) integral coordinates. We call *crossings* such intersections in the interior of the rectangle.

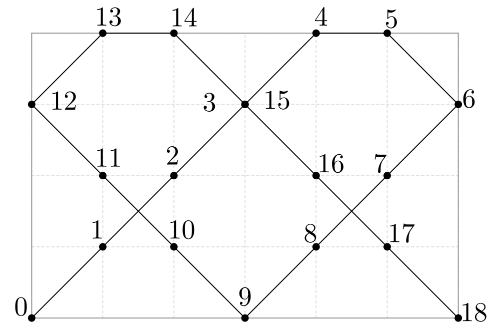

Figure 4: Trajectory diagram of $\langle 4; 4, 13\rangle \vee \langle 6\rangle$ on $\mathbf{L}_{18}$.

When a crossing occurs at a point Q with integral coordinates, then there must exist two elements of the line a and d such that $Q = Q(a) = Q(d)$, and their immediate successors and predecessors satisfy either

$$d \pm 1 \;\theta\; a - 1 \;\delta\; d \mp 1 \;\theta\; a + 1 \;\delta\; d \pm 1 \tag{13}$$

or

$$d \pm 1 \;\delta\; a - 1 \;\theta\; d \mp 1 \;\delta\; a + 1 \;\theta\; d \pm 1. \tag{14}$$

Similarly, when a crossing occurs at a point P with half-integral coordinates, there must be elements b, c satisfying either

$$b \;\delta\; c \;\theta\; b \pm 1 \;\delta\; c \pm 1 \;\theta\; b$$

or

$$b \;\theta\; c \;\delta\; b \pm 1 \;\theta\; c \pm 1 \;\delta\; b$$

where $|P_i - P_i(b)| = |P_i - P_i(c)| = \frac{1}{2}$ for $i = 1, 2$.

We will usually speak in geometrical terms when referring to diagrams, as in *there is rest at the bounce at* k and the like.

Lemma 19. *Suppose that in the trajectory diagram of $\theta \vee \delta$ one of the following happens:*

- *There are two crossings at points $P = (P_1, P_2)$ and $Q = (Q_1, Q_2)$ such that $|P_i - Q_i| = \frac{1}{2}$ for some $i = 1, 2$,*
- *there is a crossing at P such that there is $x \in \operatorname{ext}(\theta) \cup \operatorname{ext}(\delta)$ which is not part of a rest and $|P_i(x) - P_i| = \frac{1}{2}$ for some $i = 1, 2$, or*
- *there is a crossing at P such that there is $x \in L$ which is part of a rest of either θ or δ and $P_i(x) = P_i$ for some $i = 1, 2$.*

Then the join is trivial.

More intuitively: If the trajectory diagram of $\theta \vee \delta$ has two crossings with one coordinate differing in $\frac{1}{2}$, the join is trivial. And the same applies to a crossing with one coordinate differing in $\frac{1}{2}$ from "the center" of a bounce. An example of the first situation is depicted in Figure 5.

Proof. We only prove the first item. Without loss of generality, assume that Q has integral coordinates. Consider the case when $|P_2 - Q_2| = \frac{1}{2}$. By the observation prior to the statement of the Lemma, let $a, d \in L$ satisfying one of the equations (13) or (14). We might also find $b, c \in L$ such that $|P_i - P_i(b)| = |P_i - P_i(c)| = \frac{1}{2}$ for $i = 1, 2$, $b \;\theta\; a$ and $b \;\delta\; c$. There are two possibilities for the relative position of the two crossings:

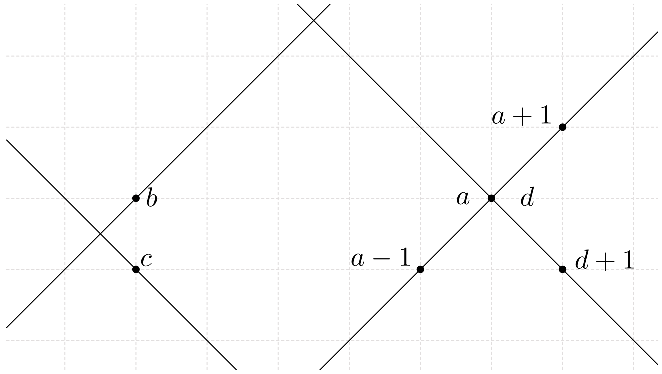

Figure 5: Two crossings with a $\frac{1}{2}$ difference.

1. $c \;\theta\; a-1 \;\delta\; d+\epsilon$ (where $\epsilon = \pm 1$): In this sub-case, we have

$$d+\epsilon \;\theta\; a-1 \;\delta\; d-\epsilon \;\theta\; a+1 \;\delta\; d+\epsilon.$$

   Hence we have the following chain of relations

$$a \;\theta\; b \;\delta\; c \;\theta\; a-1 \;\delta\; d+\epsilon \;\delta\; a+1,$$

   which witnesses that $a-1, a, a+1$ all belong to the same $\theta \vee \delta$-class, hence the join is trivial.

2. $c \;\theta\; a+1 \;\delta\; d+\epsilon$ for some $\epsilon = \pm 1$: We deduce

$$d+\epsilon \;\delta\; a-1 \;\theta\; d-\epsilon \;\delta\; a+1 \;\theta\; d+\epsilon$$

   and obtain

$$a \;\theta\; b \;\delta\; c \;\theta\; a+1 \;\delta\; d+\epsilon \;\delta\; a-1,$$

   reaching triviality once again.

The case where $|P_1 - Q_1| = \frac{1}{2}$ is completely analogous. □

Lemma 20. *Assume that the trajectory of $\theta \vee \delta$ has two bounces at x and y on the same side of the diagram, such that:*

1. *y neither lies at a corner nor is part of a rest,*
2. *x is part of a rest, and*
3. *the distance between the bounces is less than k.*

Then the join $\theta \vee \delta$ is trivial.

Proof. We consider first the case of bounces on a horizontal border. Without loss of generality, we may assume that y is pictured to the left of x, and that x is the part of the rest farthest from y (i.e., $\{x, x'\}$ constitute a rest and $P_1(x) = P_1(x') + 1$).

Let a be the closest point to y in the line having $P_1(a) = P_1(x)$. If the bounces are indeed adjacent (distance 0), then y and a are consecutive elements of the line (see Figure 6). Let c be the other point on the line at distance 1 from y; hence $P(c) = (P_1(y) - 1, 1)$. We have $y \;\theta\; x \;\delta\; a \;\theta\; c$, hence the consecutive points c, y, and a are $(\theta \vee \delta)$-related, hence $\theta \vee \delta$ is trivial.

If the distance is positive, we may assume without loss of generality that the situation at hand is the one pictured in Figure 7. The points b, b', b'' are the closest points to x such that

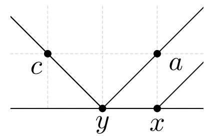

Figure 6: Adjacent bounces.

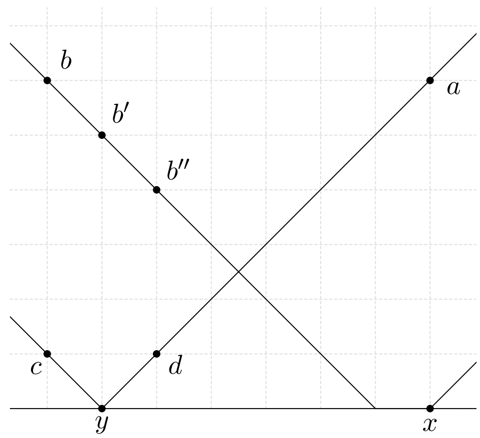

Figure 7: Different bounces closer than k imply triviality.

$P(b) = (P_1(y)-1, P_2(a))$, $P(b') = (P_1(y), P_2(a)-1)$, and $P(b'') = (P_1(y)+1, P_2(a)-2)$. These are well defined since $P(y)$ is not at a corner and $P_2(a) \geq 2$. We obtain

$$a\ \theta\ b\ \delta\ c\ \theta\ d\ \delta\ b'' \quad \text{and} \quad a\ \delta\ x\ \theta\ y\ \delta\ b',$$

hence the join is trivial. All of the reasoning took place inside the rectangle having vertices a, b and x, having height $\overline{ax} \leq k$.

For the case of the vertical border, one should only be careful with the case of the distance being exactly $k-1$. But then the bounce at y lies on a corner, and the result holds vacuously. $\square$

Corollary 21. *If the join $\theta \vee \delta$ is nontrivial, the bounces on the interior of each vertical border of its trajectory diagram are of the same type.*

We work under the hypothesis that $\theta \vee \delta \neq L \times L$ up to the end of the present section.

Lemma 22. *Assume that the trajectory of $\theta \vee \delta$ bounces at y on the interior of the left side of the diagram, and let x and z be the bounces next to y on the top side and bottom side, respectively. Then the bounce at y is of the same type as the one at z.*

Proof. Assume without loss of generality that $x < y < z$. By Lemma 20, we have that x is part of a rest if and only if k is a rest. Let w the last element of the line before x such that $P_1(w) = k$. We may proceed to perform a case analysis.

Assume k is not a rest (see the left diagram in Figure 8). If both y and z are (not) parts of a rest, it can be shown that the element $w + 2k + 1$ ($w + 2k$) is related to w by both congruences. Hence they are pictured in the same point $P(w)$ and correspond to a crossing of the trajectory, right under $P(k)$.

Now if y is part of a rest and z is not (upper thin dashed line in Figure 8), $w\ \theta\ w + 2k$ and $w\ \delta\ w + 2k + 1$; this can be seen analytically noticing that $\Delta_{\bar{r}}(w) = \Delta_{\bar{r}}(w + 2k)$ and $\Delta_{\bar{s}}(w)+1 = \Delta_{\bar{s}}(w+2k+1)$. This implies that there is a crossing at coordinates $P(w)+(-\frac{1}{2}, \frac{1}{2}) =$

$(k-\frac{1}{2}, P_2(w)+\frac{1}{2})$, which differs from the bounce at k by $\frac{1}{2}$ in the abscissa. Hence the join is trivial by Lemma 19.

The remaining cases follow analogously.

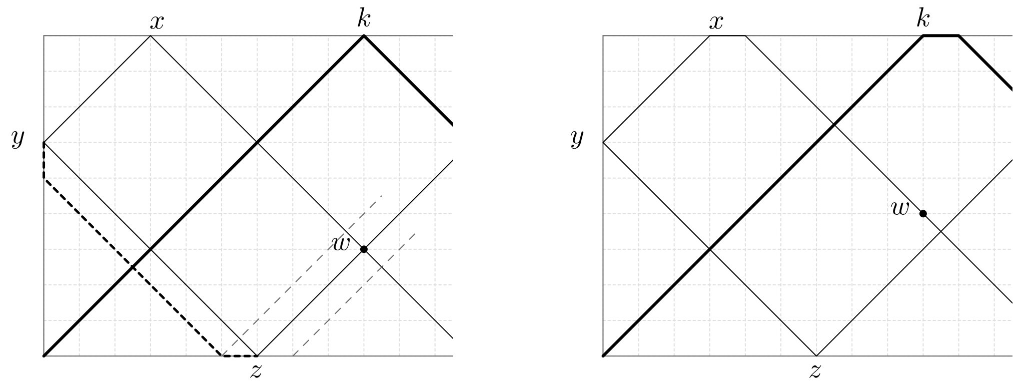

Figure 8: Correlation of bounces near the origin.

□

4.1 Bounces in Trajectory Diagrams

We will now prove that under the assumption of $\mathsf{f}_\theta, \mathsf{f}_\delta \geq 3$, bounces in each side of the trajectory diagram must be of the same type, extending the previous lemmas.

By Corollary 21 we only have to consider 4 cases; they are as follows:

1. k is a not a rest of θ.
   a. No rests on the left border.
   b. There are rests on the left border.
2. k is a rest of θ.
   a. No rests on the left border.
   b. There are rests on the left border.

Lemma 23. *If* $\operatorname{ext}(\theta) \cap \operatorname{ext}(\delta) = \{0, n\}$ *and* $\mathsf{f}_\delta \geq 3$, *there exists* $x \in 0/\delta$ *such that* $x \notin \operatorname{ext}(\theta)$.

Proof. Let $x \doteq \min(0/\delta \setminus \{0\})$. Note that $x \neq n$, otherwise we would have $\mathsf{f}_\delta = 2$. By definition, $x \in \operatorname{ext}(\delta)$. If $x \in \operatorname{ext}(\theta)$, we would have that $x \in \operatorname{ext}(\theta) \cap \operatorname{ext}(\delta) = \{0, n\}$, an absurdity. □

Lemma 24. *If* $\mathsf{f}_\theta, \mathsf{f}_\delta \geq 3$, *bounces in each side of the trajectory diagram of* $\theta \vee \delta$ *must be of the same type.*

Proof. We will work on case 2a, i.e., k is a rest of θ and there are no rests on the left border. The reader will easily note that the arguments for the other cases are completely similar.

By Lemma 23, there must be at least one bounce on the interior of the left side. Every such bounce determines unique bounces on the upper and lower side with abscissas smaller than k. Enumerate the distances between the upper bounces to the left of k as $d_1, \ldots, d_\alpha$ (hence $\alpha \geq 1$), and let d_0 be the distance from the left border to the leftmost bounce. The first $\alpha + 1$ upper bounces are of the same type by Lemma 20.

Note that the distances on the left and lower sides of the rectangle are as labeled since every segment in its interior has slope ± 1. Hence, by Lemma 22, the first α lower bounces are not

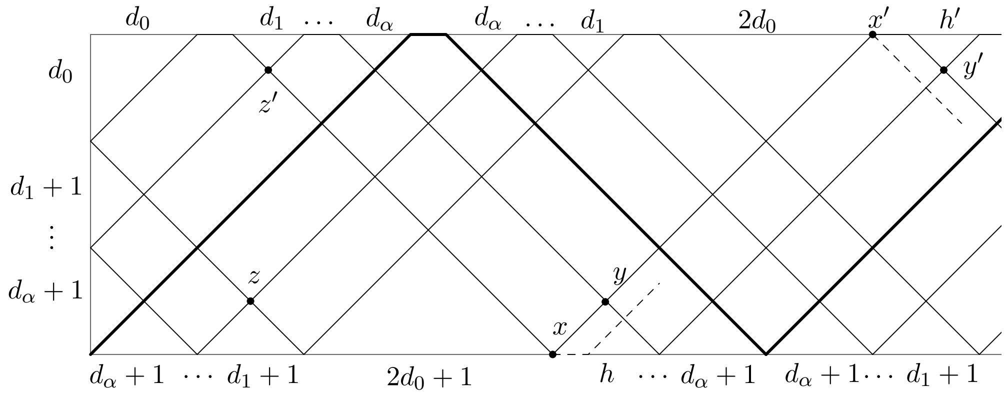

Figure 9: Trajectory diagram for case 2a.

rests. If there are no further upper bounces, we are done; otherwise, the next upper bounce (immediately to the right of k) must also be a rest, since $d_\alpha \leq k-2$.

The bounce labeled as x cannot be a rest (see Figure 9). If it were, the distance h would equal d_1 and then the heights of crossings at y and z would differ in $\frac{1}{2}$. This contradicts non-triviality of the join by Lemma 19. Hence $h = d_1+1$; moreover, since all segments of the trajectory in the interior of the rectangle have slope ± 1, this allows us to conclude that the distances between the subsequent lower bounces repeat the pattern of the first α.

A similar argument enables us to conclude that the bounce at x' in Figure 9 *must be* a rest; otherwise the distance h' would equal $h = d_1+1$ and then the crossing labeled as y' would be $\frac{d_1+1}{2}$ below the upper border of the diagram, but the crossing at z' is $\frac{1}{2}$ above that, contradicting nontriviality by Lemma 19.

Observe that by the previous reasoning we can conclude that all the distances between two bounces in the diagram are either less than k and hence they are of the same type by Lemma 20; or else they are equal to $2d_0+1$ (for lower bounces) or $2d_0$ (for upper bounces), and in this case we are in a position where Lemma 19 can be applied. Therefore, by repeating the previous arguments we obtain the result. □

The previous argument can be made more formal (although painstakingly more cumbersome) by considering an inductive argument, which we proceed to sketch. It can be seen that in the diagram for a nontrivial $\theta \vee \delta$, if there is (not) a rest at k, every time the trajectory crosses from right to left the vertical segment at abscissa $k+\frac{1}{2}$ (k) and bounces in the left border, it crosses this segment in the same point in its way back. Hence if we disregard all the part of the diagram with abscissas less than $k+1$ (less than k) if there is (not) a rest at k, what remains is also a trajectory diagram (of eventually a different type, e.g., chopping a diagram in case 2a at $k+1$ will lead to a diagram in case 1b flipped vertically). Hence this provides a way to pass to a smaller line.

5 Main Results

5.1 The catalog of nontrivial joins

In this section we will give a complete description of the possible shapes of trajectories picturing nontrivial joins.

We will say that a congruence λ is even (resp. odd) if f_λ is even (resp. odd). It is immediate to see that if θ is even (odd), the trajectory of $\theta \vee \delta$ finishes at one of the lower (upper) corners of the diagram: that is, $P(n)$ is equal to either $(0,0)$ or $(l,0)$ (resp., $(0,k)$ or (l,k)). Likewise, if δ is even (odd), the trajectory of $\theta \vee \delta$ finishes at one of the left (right) corners of the diagram.

We will focus in the present subsection in the case $\text{ext}(\theta) \cap \text{ext}(\delta) = \{0,n\}$. Assume that the join $\theta \vee \delta$ is nontrivial and that $\mathsf{f}_\theta, \mathsf{f}_\delta \geq 3$. In view of Lemma 24, we can classify the congruence $\theta = \langle k; \bar{r} \rangle$ according to it having no rests, having rests only on the top (i.e., $r_i = (2k+1)i - k - 1$ for each $i \leq |\bar{r}|$), having rests only on the bottom (i.e., $r_i = (2k+1)i - 1$ for each $i \leq |\bar{r}|$), or having rests everywhere. The same holds for δ.

Lemma 25. *Assume that $\theta \vee \delta$ is nontrivial, $\text{ext}(\theta) \cap \text{ext}(\delta) = \{0,n\}$, and $\mathsf{f}_\theta, \mathsf{f}_\delta \geq 3$.*

1. *θ even and δ odd imply [θ has rests on bottom $\iff$ δ has rests on top].*
2. *θ odd and δ odd imply [θ has rests on top $\iff$ δ has rests on top].*
3. *θ odd and δ even imply [θ has rests on top $\iff$ δ has rests on bottom].*
4. *For every θ and δ, [θ has rests on bottom $\iff$ δ has rests on bottom].*

Proof. For 1, consider the trajectory diagram of $\theta \vee \delta$. We may apply the proof of Lemma 22 but now looking at the lower right corner, that is, where the trajectory finishes. We conclude that bounces on the right side are of the same type as those on the bottom side. By Lemma 24, we conclude that θ has rests on bottom if and only if δ has rests on top. The next two items follow by considering the upper right and upper left corners, respectively.

The final item is simply the consequence of Lemma 22 under the light of Lemma 24. $\square$

Theorem 26 (The catalog)**.** *Assume that $\theta \vee \delta \neq L \times L$ and $\text{ext}(\theta) \cap \text{ext}(\delta) = \{0,n\}$, where $\theta = \langle k; \bar{r} \rangle$ and $\delta = \langle l; \bar{s} \rangle$ with $k \leq l$. Then θ and δ satisfy one of the following conditions:*

1. *$\mathsf{f}_\delta = 1$ (i.e., δ is the identity).*
2. *$\mathsf{f}_\delta = 2$ and θ is mirrored.*
3. *$\mathsf{f}_\delta \geq 3$ and one of the following holds:*
   - *(a) None of them have a rest,*
   - *(b) θ is even with rests on top and δ is odd without rests or conversely,*
   - *(c) both are odd with rests on top,*
   - *(d) both have the same parity with rests on the bottom,*
   - *(e) θ is even with rests on the bottom and δ is odd with rests everywhere or conversely, or*
   - *(f) both congruences have rests everywhere.*

Proof. We start from the second item; we obtain our conclusion by the contrapositive to Lemma 17(2).

For item 3, since $k \leq l$ we have that $\mathsf{f}_\theta \geq \mathsf{f}_\delta \geq 3$, and we are under the hypothesis of Lemma 25. A straightforward but dull case analysis shows that some of the sub-cases applies. $\square$

We underline that each case included in the list before is possible. An enumeration of examples follows.

Example 27. 1. $\mathsf{f}_\delta = 1$: Trivially, an arbitrary nontrivial θ would do.

2. $\mathsf{f}_\delta = 2$ and θ is mirrored: $\langle 2; 2, 9 \rangle \vee \langle 6 \rangle$ for $n = 12$.
3. For $\mathsf{f}_\delta \geq 3$:

(a) None of them have a rest: this is always nontrivial. The reader may find this to be an easy exercise.

(b) θ is even with rests on top and δ is odd without rests or conversely: $\langle 4;4,13\rangle \vee \langle 6\rangle$ for $n = 18$ and $\langle 6\rangle \vee \langle 7;7,22\rangle$ for $n = 30$.

(c) Both are odd with rests on top: $\langle 4;4,13\rangle \vee \langle 7;7\rangle$ for $n = 22$.

(d) Both have the same parity with rests on the bottom: $\langle 4;8,17\rangle \vee \langle 7;14\rangle$ for $n = 22$.

(e) θ is even with rests on the bottom and δ is odd with rests everywhere or conversely: $\langle 7;14,29\rangle \vee \langle 8;8,17,26,35\rangle$ for $n = 44$ and $\langle 5;5,11,17,23\rangle \vee \langle 7;14\rangle$ for $n = 29$.

(f) Both congruences have rests everywhere: $\langle 7;7,15,23,31\rangle \vee \langle 9;9,19,29\rangle$ for $n = 39$.

$\theta = \langle k;\bar{r}\rangle$	
No rests	$\equiv_{2k} \cup \{\langle x,y\rangle \mid x+y \equiv_{2k} 0\}$
Rests on top	$\equiv_{2k+1} \cup \{\langle x,y\rangle \mid x+y \equiv_{2k+1} 0\}$
Rests on bottom	$\equiv_{2k+1} \cup \{\langle x,y\rangle \mid x+y+1 \equiv_{2k+1} 0\}$
Rests everywhere	$\equiv_{2k+2} \cup \{\langle x,y\rangle \mid x+y+1 \equiv_{2k+2} 0\}$

Table 1: Simplified expression of congruences in the catalog, case $\mathsf{f}_\theta, \mathsf{f}_\delta \geq 3$.

5.2 Permutability

We now proceed to show that for any two congruences with a nontrivial upper bound, their join is the composition.

It is well known that the relations $\equiv_s$ and $\equiv_t$ permute on any set of integers that contains $\{0,\dots,\mathrm{lcm}(s,t)\}$. Moreover, this is witnessed by the fact that every (abelian) group has a Mal'cev term $p(x,y,z) = x - y + z$ [10]. We will show that every pair of congruences θ and δ having $\mathsf{f}_\theta, \mathsf{f}_\delta \geq 3$ and nontrivial $\theta \vee \delta$ decompose essentially into the union of a modular relation $x \equiv_s y$ and one of the form $x+y \equiv_t 0$ or $x+y+1 \equiv_t 0$, and indeed there exist variants of the Mal'cev term showing that all of these relations pairwise permute.

Lemma 28. *Let s and t be positive integers. The pairs of relations*

1. $\equiv_s$ *and* $x+y \equiv_t 0$,
2. $x+y \equiv_s 0$ *and* $x+y \equiv_t 0$,
3. $\equiv_s$ *and* $x+y+1 \equiv_t 0$,
4. $x+y+1 \equiv_s 0$ *and* $x+y+1 \equiv_t 0$, *and*
5. $x+y \equiv_s 0$ *and* $x+y+1 \equiv_t 0$

permute on $\{0,\dots,\mathrm{lcm}(s,t)\}$.

Proof. We show the first item. Assume $a \equiv_s b$ and $b+c \equiv_t 0$ where a, b and c are pairwise distinct. Let $x \doteq c+b-a$; immediately, we have $c \equiv_s c+(b-a) = x$. Also, $x+a = c+b \equiv_t 0$.

It is clear that $-\mathrm{lcm}(s,t) < x < 2\,\mathrm{lcm}(s,t)$; we may obtain a solution in the intended range by adding $\pm\,\mathrm{lcm}(s,t)$.

The argument for, e.g., item 2 uses $x \doteq -(a+b+c)$. The rest are very similar. □

Corollary 29. *Every pair of congruences appearing in Table 1 (regardless of their join) permute.*

Proof. Let θ and δ be a pair of congruences in the table. These are of the form $\vartheta \cup \varpi$, where each of ϑ and ϖ are one of the relations $x \equiv_s y$, $x + y \equiv_t 0$, or $x + y + 1 \equiv_t 0$ for suitable s and t. It is easy to check that for every three relations φ, ϑ and ϖ on a set such that φ permutes with the other two, then φ permutes with $\vartheta \cup \varpi$. But now we may apply Lemma 28 to see that the components ϑ and ϖ of each congruence pairwise permute, and hence θ and δ permute by the previous observation. $\square$

Theorem 30. *Assume that $\theta \vee \delta \neq L \times L$. Then $\theta \circ \delta = \delta \circ \theta = \theta \vee \delta$.*

Proof. We begin by considering the case $\mathrm{ext}(\theta) \cap \mathrm{ext}(\delta) = \{0, n\}$; we divide it into three subcases, whether f_δ equals 1 or 2, or $\mathsf{f}_\delta \geq 3$.

If $\mathsf{f}_\delta = 1$, then $\delta = \mathrm{Id}_L$ and hence $\theta \circ \delta = \delta \circ \theta = \theta$ regardless of any other assumption on θ.

If $\mathsf{f}_\delta = 2$, since $\theta \vee \delta \neq L \times L$ we infer that θ is mirrored and hence both congruences permute by Lemma 17.

Finally, we consider the case where $\mathsf{f}_\delta \geq 3$. According to the catalog of nontrivial joins, θ and δ appear in Table 1 and hence Corollary 29 implies that they permute.

Now we wrap up the proof for the general case. Assume $x \; \theta \; y \; \delta \; z$. By Theorem 14 we may find x', y' and z' between 0 and e_1 such that $x \; \theta \cap \delta \; x'$, $y \; \theta \cap \delta \; y'$ and $z \; \theta \cap \delta \; z'$. The restrictions θ', δ' to $[0, e_1]$ of θ and δ, respectively, are congruences on L_{e_1} with $\mathrm{ext}(\theta') \cap \mathrm{ext}(\delta') = \{0, e_1\}$; therefore they permute. Find $w \in [0, e_1]$ such that $x' \; \delta' \; w \; \theta' \; z'$. We immediately conclude that $x \; \delta \; w \; \theta \; z$ by construction of x' and z', and we have our result. $\square$

5.3 Criterion for nontriviality

We now proceed to give a complete criterion to decide if the join of two congruences $\theta = \langle k; \bar{r} \rangle$ and $\delta = \langle l; \bar{s} \rangle$ is not trivial. It can be regarded as a converse to Theorem 14.

As a first ingredient, we must enumerate the elements of $\mathrm{ext}(\theta) \cap \mathrm{ext}(\delta) = \{e_i \mid 0 \leq i < N\}$.

Theorem 31. *Assume θ and δ are nontrivial congruences of a line $\mathbf{L}$. Then $\theta \vee \delta \neq L \times L$ if and only if*

1. *for all i, e_i is part of a θ-rest if and only if it is part of a δ-rest,*
2. *$\gamma \doteq \langle e_1; \bar{r} \cap \bar{s} \rangle$ is congruence of $\mathbf{L}$,*
3. *the restrictions of θ and δ to $\{0, \ldots, e_1\}$ are a pair of congruences appearing in the catalog,*
4. *the rests of θ are compatible with γ, and*
5. *the rests of δ are compatible with γ.*

Proof. ($\Leftarrow$) Assume all the conditions hold for θ and δ as in the hypothesis, and by way of contradiction assume $\theta \vee \delta = L \times L$. We first solve the problem for the case where $\mathrm{ext}(\theta) \cap \mathrm{ext}(\delta) = \{0, n\}$. By the third condition (the only not trivial in this case), both congruences appear in the catalog and then they permute by Corollary 29; hence $\delta \circ \theta = L \times L$. Since $0 \; (\delta \circ \theta) \; k$, there exist $e \in L$ such that $0 \; \delta \; e \; \theta \; k$; we have $e \in \mathrm{ext}(\theta) \cap \mathrm{ext}(\delta)$ by definition. Now consider $l \; (\delta \circ \theta) \; e$; then the e' satisfying $l \; \delta \; e' \; \theta \; e$ is also a common extreme. Both e and e' are not null since $0 \not\theta \; k$, and $e \neq e'$ since $0 \not\delta \; l$. This contradicts $\mathrm{ext}(\theta) \cap \mathrm{ext}(\delta) = \{0, n\}$.

Next, we reduce the general case to the previous one by using the other conditions. It is immediate by Theorem 16 that $\gamma \subseteq \theta, \delta$. Now the arguments in Lemma 15(3) will show that $\theta_{e_1} \vee \delta_{e_1} = (\theta \vee \delta)_{e_1} = [0, e_1] \times [0, e_1]$; but θ_{e_1} and δ_{e_1} are congruences on $\mathbf{L}_{e_1}$ having $\mathrm{ext}(\theta_{e_1}) \cap \mathrm{ext}(\delta_{e_1}) = \{0, e_1\}$, a contradiction.

($\Rightarrow$) Assume that the join is not trivial. The first condition follows from Lemma 10. The second and the last two are consequences of the characterization of the order (Theorem 16) and the meet (Theorem 14). Finally, Theorem 26 yields the third condition. $\square$

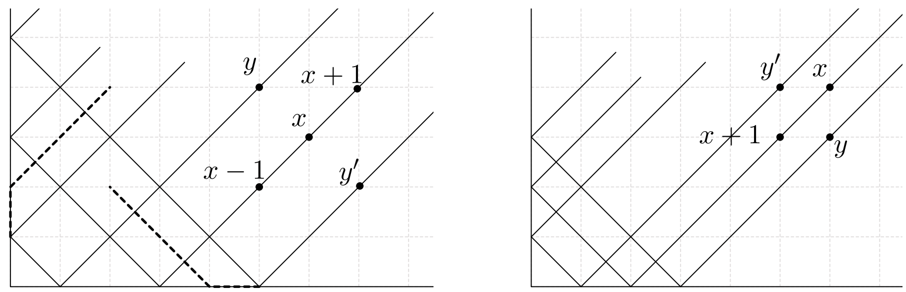

Figure 10: Fragments at horizontal distance 2 and 1, respectively.

5.4 Computing the join

In this section we will compute the step of the join of two congruences; that is, given $\theta, \delta \in \operatorname{Con} \mathbf{L}$, we will calculate the cardinality of $\mathbf{L}/\theta \vee \delta$.

In the following lemmas before Theorem 40, we work under the assumption that $\operatorname{ext}(\theta) \cap \operatorname{ext}(\delta) = \{0, n\}$ and $\theta \vee \delta \neq L \times L$. By the results of Section 5.2, this implies that θ and δ permute.

Lemma 32. *Assume that θ or δ has at least one rest. Then for all $x \in \operatorname{ext}(\theta \vee \delta)$, either*

1. *$x \in \operatorname{ext}(\theta) \cup \operatorname{ext}(\delta)$ or*
2. *x is at a distance strictly less than 1 from a crossing.*

Moreover, if i (h) denotes the number of crossings at (half-) integral coordinates, we have

$$|\operatorname{ext}(\theta \vee \delta)| = |\operatorname{ext}(\theta)| + |\operatorname{ext}(\delta)| + 2i + 4h - 2.$$

Proof. It is clear that $\operatorname{ext}(\theta) \cup \operatorname{ext}(\delta) \subseteq \operatorname{ext}(\theta \vee \delta)$, and that any $x \in L$ at a distance smaller than 1 from a crossing belongs to $\operatorname{ext}(\theta \vee \delta)$. So it remains to be checked that every $x \in \operatorname{ext}(\theta \vee \delta) \setminus (\operatorname{ext}(\theta) \cup \operatorname{ext}(\delta))$ is at distance less than one from a crossing. Take such an x. Since $x \notin \operatorname{ext}(\theta) \cup \operatorname{ext}(\delta)$, x is in the interior of the trajectory diagram. There are two cases, depending on whether x is part of a rest, or not.

Assume that x is not part of a rest of $\theta \vee \delta$. Hence $x - 1\ \theta \vee \delta\ x + 1$. By permutability, there must exist y, y' such that $x - 1\ \theta\ y\ \delta\ x + 1$ and $x - 1\ \delta\ y'\ \theta\ x + 1$. If $y' = y \pm 2$, the segment joining y and y' crosses at x the one determined by $x - 1$ and $x + 1$, and we are done. Otherwise, if x does not belong to the line segment $\overline{yy'}$, each of them must lie in parallel lines at horizontal distance 2; see Figure 10. It can be easily seen by considering the bounces of these three fragments of the trajectory that either $\theta \vee \delta$ is trivial or none of them has any rest. To finish this case, observe that this kind of crossing contributes with 2 extremes of $\theta \vee \delta$ (x and $y \pm 1$).

The case for x being part of a rest of $\theta \vee \delta$ is very similar. For instance, if $x - 1$ is a rest, then $x - 1\ \theta \vee \delta\ x$, and by permutability, there must exist y, y' such that $x - 1\ \theta\ y\ \delta\ x$ and $x - 1\ \delta\ y'\ \theta\ x$. It is easily seen that for nontrivial $\theta \vee \delta$ we must have $y' = y \pm 1$, and then this kind of crossing contributes with 4 in the count of $\operatorname{ext}(\theta \vee \delta)$ (namely, $x - 1$, x', y, y'). ☐

For the purpose of the next proofs, let r_θ denote the number of rests of θ and let $c_{\theta,\delta}$ denote the number of crossings in the trajectory diagram of $\theta \vee \delta$. From Equation (6) we have that

$$r_\theta = |\operatorname{ext}(\theta)| - \mathsf{f}_\theta - 1.$$

Corollary 33. *If any of θ, δ has at least one rest, we have $\mathsf{f}_{\theta \vee \delta} = \mathsf{f}_\theta + \mathsf{f}_\delta - 1 + 2 \cdot c_{\theta,\delta}$.*

Proof. By the proof of Lemma 32, rests of $\theta \vee \delta$ correspond to rests of either θ or δ, or to crossings at half-integral coordinates, the latter contributing with two rests each. We keep the notation of the previous Lemma (h, i for the number of crossings at half-integral, integral coordinates, resp.). So we have

$$\begin{aligned}
\mathsf{f}_{\theta\vee\delta} &= |\mathrm{ext}(\theta \vee \delta)| - 1 - r_{\theta\vee\delta} \\
\mathsf{f}_{\theta\vee\delta} &= |\mathrm{ext}(\theta \vee \delta)| - 1 - \bigl(r_\theta + r_\delta + 2h\bigr) \\
&= |\mathrm{ext}(\theta \vee \delta)| - 1 - \bigl(|\mathrm{ext}(\theta)| - \mathsf{f}_\theta - 1 + |\mathrm{ext}(\delta)| - \mathsf{f}_\delta - 1 + 2h\bigr) \\
&= |\mathrm{ext}(\theta)| + |\mathrm{ext}(\delta)| + 2i + 4h - 2 - 1 - |\mathrm{ext}(\theta)| + \mathsf{f}_\theta + 1 - |\mathrm{ext}(\delta)| + \mathsf{f}_\delta + 1 - 2h \\
&= 2i + 2h - 1 + \mathsf{f}_\theta + \mathsf{f}_\delta \\
&= \mathsf{f}_\theta + \mathsf{f}_\delta - 1 + 2 \cdot c_{\theta,\delta}.
\end{aligned}$$

□

Lemma 34. *One of* f_θ, f_δ *must be odd.*

Proof. We show that if both f_θ and f_δ are even, $\mathrm{ext}(\theta) \cap \mathrm{ext}(\delta) \neq \{0, n\}$.

First observe that if both frequencies are even, then neither $k\ \theta\ n$ nor $l\ \delta\ n$ hold. For $\theta = \langle k; \bar{r}\rangle$, this happens because

$$n - \Delta_{\bar{r}}(n) = n - |\bar{r}| = \mathsf{f}_\theta \cdot k \equiv_{2k} 0 = 0 - \Delta_{\bar{r}}(0),$$

and hence $0\ \theta\ n$. But by definition of k, $k\ \theta\ 0$ is not possible; hence $k\ \not\theta\ n$. The same works for δ.

We proceed by considering some cases. If $0\ (\theta \vee \delta)\ k$, by permutability there must exist x such that $0\ \delta\ x\ \theta\ k$; this x belongs to $\mathrm{ext}(\theta) \cap \mathrm{ext}(\delta) \setminus \{0, n\}$, because $k\ \not\theta\ 0$ and $k\ \not\theta\ n$. If $0\ (\theta \vee \delta)\ l$ we arrive at the same consequence *mutatis mutandis*.

Otherwise, assume $(0, k), (0, l) \notin \theta \vee \delta$. Since $l \in \mathrm{ext}(\delta) \subseteq \mathrm{ext}(\theta \vee \delta)$ (Lemma 13.1), we have $l\ (\theta \vee \delta)\ k$. We may then find some x such that $l\ \delta\ x\ \theta\ k$ holds. But again $x \in \mathrm{ext}(\theta) \cap \mathrm{ext}(\delta) \setminus \{0, n\}$ and we are done. □

Lemma 35. *There exist no distinct* $x, y, y' \in L$ *such that* y *and* y' *are consecutive,* $x\ \theta \cap \delta\ y$ *and* $x + 1\ \theta \cap \delta\ y'$.

Proof. By way of contradiction, assume there exist such x, y, y' with x minimal with this property.

We first check $x \neq 0$; otherwise, since $y\ \theta \cap \delta\ x$, then $y \in \mathrm{ext}(\theta) \cap \mathrm{ext}(\delta) = \{0, n\}$. Therefore, $y = n$, and we have

$$0 \equiv_{2k} n - |\bar{r}| = k \cdot \mathsf{f}_\theta,$$

and hence $2 \mid \mathsf{f}_\theta$. Similarly, $2 \mid \mathsf{f}_\delta$, but this contradicts Lemma 34.

We will show that x must be a common extreme of both θ and δ.

We first consider the case where $y' = y + 1$, and perform an analysis of the θ-relationships. Since $x\ \theta\ y$, $x + 1\ \theta\ y + 1$ we have four options, according to the signs below:

$$x - \Delta_{\bar{r}}(x) \equiv_{2k} \pm(y - \Delta_{\bar{r}}(y)) \tag{15}$$
$$x + 1 - \Delta_{\bar{r}}(x + 1) \equiv_{2k} \pm(y + 1 - \Delta_{\bar{r}}(y + 1)), \tag{16}$$

which we denote by $(++)$, $(+-)$, $(-+)$, and $(--)$. By minimality of x, we have

$$x - 1 - \Delta_{\bar{r}}(x - 1) \not\equiv_{2k} y - 1 - \Delta_{\bar{r}}(y - 1). \tag{17}$$

Cases $(++)$, $(+-)$, and $(-+)$ can be treated uniformly; in each of them, there is a $d \in \{0, 1\}$ such that

$$x + d - \Delta_{\bar{r}}(x + d) \equiv_{2k} y + d - \Delta_{\bar{r}}(y + d). \tag{18}$$

Add 1 to each side of (17) and subtract from (18):

$$d + \Delta_{\bar{r}}(x-1) - \Delta_{\bar{r}}(x+d) \not\equiv_{2k} d + \Delta_{\bar{r}}(y-1) - \Delta_{\bar{r}}(y+d)$$

and then

$$\Delta_{\bar{r}}(x-1) - \Delta_{\bar{r}}(x+d) \not\equiv_{2k} \Delta_{\bar{r}}(y-1) - \Delta_{\bar{r}}(y+d).$$

Since θ is not trivial, only one of $x-1, x$ can be a rest; the same happens with y. And then each side of the inequality is either 1 or 0. So at least one of x, y is part of rest, but then both are, because $x \,\theta\, y$. Hence $x \in \operatorname{ext}(\theta)$.

For case $(--)$, we subtract the corresponding version of (16) from (15):

$$-\Delta_{\bar{r}}(x) - 1 + \Delta_{\bar{r}}(x+1) \equiv_{2k} \Delta_{\bar{r}}(y) + 1 - \Delta_{\bar{r}}(y+1)$$

and then

$$\big(\Delta_{\bar{r}}(x+1) - \Delta_{\bar{r}}(x)\big) + \big(\Delta_{\bar{r}}(y+1) - \Delta_{\bar{r}}(y)\big) \equiv_{2k} 2.$$

Since both differences are 1 or 0, the only way that this holds is that both x, y are rests (hence extremes) or $k = 1$ (in which case *every* element is a extreme).

We conclude that in each of the four cases, $x \in \operatorname{ext}(\theta)$. The same analysis applies to δ-relationships, hence $x \in \operatorname{ext}(\theta) \cap \operatorname{ext}(\delta)$.

Now consider the case where $y' = y - 1$. As before, the θ-relationships are

$$x - \Delta_{\bar{r}}(x) \equiv_{2k} \pm(y - \Delta_{\bar{r}}(y)) \tag{19}$$

$$x + 1 - \Delta_{\bar{r}}(x+1) \equiv_{2k} \pm(y - 1 - \Delta_{\bar{r}}(y-1)), \tag{20}$$

and

$$x - 1 - \Delta_{\bar{r}}(x-1) \not\equiv_{2k} y + 1 - \Delta_{\bar{r}}(y+1) \tag{21}$$

$$x - 1 - \Delta_{\bar{r}}(x-1) \not\equiv_{2k} -y - 1 + \Delta_{\bar{r}}(y+1) \tag{22}$$

by minimality of x.

The corresponding $(++)$ case is similar to $(--)$ above: By subtracting (19) from (20) we conclude that either x, y are parts of rests or $k = 1$, and then $x \in \operatorname{ext}(\theta)$.

For the remaining cases, observe that if some of $x, y, x-1, y-1$ is a rest, we have $x \in \operatorname{ext}(\theta)$ as before. Hence we will work under this assumptions:

$$\begin{aligned} \Delta_{\bar{r}}(x-1) &= \Delta_{\bar{r}}(x) = \Delta_{\bar{r}}(x+1) \\ \Delta_{\bar{r}}(y-1) &= \Delta_{\bar{r}}(y) = \Delta_{\bar{r}}(y+1). \end{aligned} \tag{23}$$

Cases $(-+)$ and $(--)$ are impossible: Subtract (22) from (19),

$$x - \Delta_{\bar{r}}(x) - x + 1 + \Delta_{\bar{r}}(x-1) \not\equiv_{2k} -y + \Delta_{\bar{r}}(y) + y + 1 - \Delta_{\bar{r}}(y+1),$$

and applying (23) we get $1 \not\equiv_{2k} 1$, a contradiction.

Case $(+-)$: Using (23) we may add (19) and (20) and obtain:

$$2(x - \Delta_{\bar{r}}(x)) \equiv_{2k} 0,$$

and then $x - \Delta_{\bar{r}}(x) \equiv_{2k} k$ or 0, and hence $x \in \operatorname{ext}(\theta)$.

As before, the same analysis applies to δ, hence $x \in \operatorname{ext}(\theta) \cap \operatorname{ext}(\delta)$. $\square$

Lemma 36. *For all θ and δ, $c_{\theta,\delta} = \frac{1}{2}(\mathsf{f}_\theta - 1)(\mathsf{f}_\delta - 1)$.*

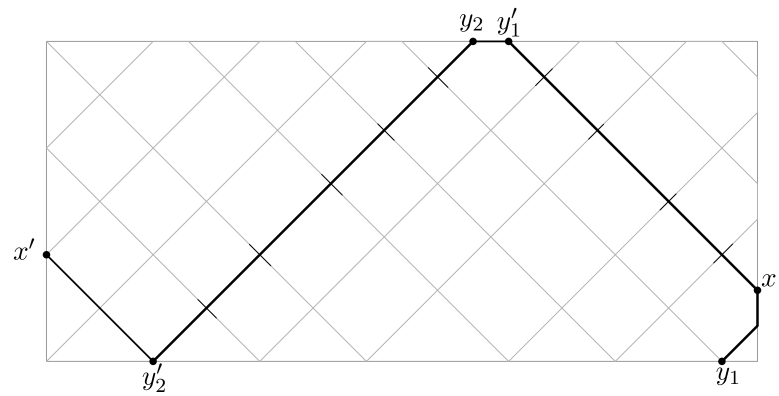

Figure 11: A trip $\overline{xx'}$ and pieces $\overline{y_1y'_1}$, $\overline{y_2y'_2}$ of resp. first and second type.

Proof. Given two consecutive elements $x < x'$ of $\mathrm{ext}(\delta)$ such that x is not a rest (hence $x'-x = l$), call *trip* the fragment of the line $\mathbf{L}$ going from x to x'. Thus a trip corresponds to a polygonal piece of the trajectory of $\theta \vee \delta$ going across the whole diagram from left to right or viceversa.

The crucial observation is that for nontrivial $\theta \vee \delta$, Lemma 35 implies that trips do not overlap and therefore intersect each other at finitely many points. This allows us to use a counting argument to calculate the number of crossings. Given x, x' as above and consecutive $y, y' \in \mathrm{ext}(\theta)$ such that

- $y < y'$,
- none of them is between x and x', and
- y not a rest (hence $y' - y = k$),

the trip going from x to x' crosses the polygonal piece $\overline{yy'}$ of the trajectory exactly once. Hence $\overline{yy'}$ is crossed by all the trips that do not contain y nor y'. We may consider two types of pieces $\overline{yy'}$: Those such that y, y' belong to different trips (i.e., there is $y < x < y'$ with $x \in \mathrm{ext}(\delta)$) or those such that y, y' belong to the same trip (see Figure 11). There are $\mathsf{f}_\delta - 1$ pieces of the first type, and each of them is crossed $\mathsf{f}_\delta - 2$ times.

On the other hand, the total number pieces is equal to f_θ, hence the number pieces of the second type is $\mathsf{f}_\theta - (\mathsf{f}_\delta - 1)$, and each of them is crossed $\mathsf{f}_\delta - 1$ times.

In this calculation, we have counted each crossing twice, so we finally obtain the total number of crossings as

$$\frac{1}{2}\big[(\mathsf{f}_\delta - 1)(\mathsf{f}_\delta - 2) + \big(\mathsf{f}_\theta - (\mathsf{f}_\delta - 1)\big)(\mathsf{f}_\delta - 1)\big] = \frac{1}{2}(\mathsf{f}_\theta - 1)(\mathsf{f}_\delta - 1).$$

□

Theorem 37. *Assume $\theta \vee \delta \neq L \times L$ and $\mathrm{ext}(\theta) \cap \mathrm{ext}(\delta) = \{0, n\}$. Then $\mathsf{f}_{\theta\vee\delta} = \mathsf{f}_\theta \cdot \mathsf{f}_\delta$.*

Proof. Assume first that at least one of the congruences has a rest. By Corollary 33, we obtain

$$\mathsf{f}_{\theta\vee\delta} = \mathsf{f}_\theta + \mathsf{f}_\delta - 1 + 2c_{\theta,\delta}.$$

Using Lemma 36,

$$\mathsf{f}_\theta + \mathsf{f}_\delta - 1 + (\mathsf{f}_\theta - 1)(\mathsf{f}_\delta - 1) = \mathsf{f}_\theta \cdot \mathsf{f}_\delta,$$

and we are done.

Now we turn to the case without rests. That is, assume $\theta = \langle k \rangle$ and $\delta = \langle l \rangle$. Note that k and l divide n by Theorem 7. Also observe that $\mathrm{ext}(\theta) = \{t \cdot k : t \in [0, \frac{n}{k}]\}$ and analogously for δ. In particular, $\mathrm{lcm}(k,l) \in \mathrm{ext}(\theta) \cap \mathrm{ext}(\delta)$ and hence $n = \mathrm{lcm}(k,l)$. So,

$$\mathsf{f}_\theta = \frac{n}{k} = \frac{\mathrm{lcm}(k,l)}{k}, \qquad \text{and similarly} \qquad \mathsf{f}_\delta = \frac{\mathrm{lcm}(k,l)}{l}.$$

It is easy to see now that $\gcd(\mathsf{f}_\theta, \mathsf{f}_\delta) = 1$. On the other hand, by Lemma 15.(2), we know that both f_θ and f_δ divide $\mathsf{f}_{\theta\vee\delta}$, therefore $\mathsf{f}_\theta \cdot \mathsf{f}_\delta \mid \mathsf{f}_{\theta\vee\delta}$. Finally, $\langle \gcd(k,l) \rangle$ is an upper bound of $\{\theta, \delta\}$ in $\mathrm{Con}\,\mathbf{L}$, having frequency $\mathsf{f}_\theta \cdot \mathsf{f}_\delta$; therefore we obtain $\mathsf{f}_{\theta\vee\delta} = \mathsf{f}_\theta \cdot \mathsf{f}_\delta$. □

Corollary 38. *Assume that for $\theta = \langle k; \bar{r} \rangle$, $\delta = \langle l; \bar{s} \rangle$, $\theta \vee \delta \neq L \times L$ and $\mathrm{ext}(\theta) \cap \mathrm{ext}(\delta) = \{0, n\}$. Then the cardinality of $\mathbf{L}/\theta \vee \delta$ is*

$$\left\lfloor \frac{nkl}{(n - |\bar{r}|)(n - |\bar{s}|)} \right\rfloor.$$

Proof. Immediate since $\mathsf{f}_\theta = \frac{n - |\bar{r}|}{k}$ (analogously with f_δ) and the step of the congruence $\theta \vee \delta$ is equal to the integral division of n by $\mathsf{f}_{\theta\vee\delta}$. □

We need now one further result in order to lift the assumption of $\mathrm{ext}(\delta) \cap \mathrm{ext}(\delta) = \{0, n\}$. Given congruences θ, δ in the catalog of nontrivial joins, it is useful to be able to recover the length n of the line given the frequency f_δ and the step l of δ. In Table 2 we gather the values of $n = n(l, \mathsf{f}_\delta)$ in terms of the corresponding parameters l and f_δ.

	$n = n(l, \mathsf{f}_\delta)$	
	f_δ even	f_δ odd
No rests	$l\,\mathsf{f}_\delta$	
Rests on top	$(2l+1)\frac{\mathsf{f}_\delta}{2}$	$\frac{1}{2}[(2l+1)\mathsf{f}_\delta - 1]$
Rests on bottom	$(2l+1)\frac{\mathsf{f}_\delta}{2} - 1$	$\frac{1}{2}[(2l+1)\mathsf{f}_\delta - 1]$
Rests everywhere	$(l+1)\mathsf{f}_\delta - 1$	

Table 2: Values of n in terms of step and frequency (case $\mathsf{f}_\delta \geq 3$).

Note that actually, for $0 \leq i \leq \mathsf{f}_\delta$, the values $n(l, i)$ belong to $\mathrm{ext}(\delta)$ (using the formula according to the parity of i). These values will appear in the proof of the following result.

Lemma 39. *Assume that $\mathrm{ext}(\theta) \cap \mathrm{ext}(\delta) = \{0, n\}$ and $\theta \vee \delta \neq L \times L$. Then $\gcd(\mathsf{f}_\theta, \mathsf{f}_\delta) = 1$.*

Proof. The main argument will be to show that if f_θ and f_δ have a common divisor, then θ and δ have a common non null extreme before n. In the following we will make silent use of Lemma 34.

If $\mathsf{f}_\delta = 1$, the result follows. If $\mathsf{f}_\delta = 2$, by the assumption $\mathrm{ext}(\theta) \cap \mathrm{ext}(\delta) = \{0, n\}$ we conclude that $2 \nmid \mathsf{f}_\theta$, and this case is also ready.

Next, assume that $\mathsf{f}_\theta, \mathsf{f}_\delta \geq 3$. We have to analyze all cases of the catalog of nontrivial joins. We first consider the case in which θ has no rests.

1. *δ has no rests.* Every common, non null extreme of θ and δ is of the form $x = k\,a = l\,b$ for some $0 < a \leq \mathsf{f}_\delta$, $0 < b \leq \mathsf{f}_\theta$. Hence, if $d > 0$ is a common divisor of f_θ and f_δ, we may choose a and b such that $\mathsf{f}_\theta = d \cdot a$ and $\mathsf{f}_\delta = d \cdot b$, hence $k \cdot a = l \cdot b$ is a common non null extreme of θ and δ, hence it must be n. We conclude $d = 1$.

2. *δ has rests on top.* There are two sub-cases, according to whether $2 \mid \mathsf{f}_\delta$ or not. If f_δ is even, f_θ must be odd. We have $k\,\mathsf{f}_\theta = (2l+1)\frac{\mathsf{f}_\delta}{2}$, and thus $2k\,\mathsf{f}_\theta = (2l+1)\mathsf{f}_\delta$. We may use the same reasoning in item 1 to conclude that $\gcd(\mathsf{f}_\theta, \mathsf{f}_\delta) = 1$.

There remain four more cases, when both congruences have at least one rest.

a. *Both congruences have rests on top.* In this case both congruences must have odd frequencies. We have $n = \frac{1}{2}[(2k+1)\mathsf{f}_\theta - 1] = \frac{1}{2}[(2l+1)\mathsf{f}_\delta - 1]$, hence $(2k+1)\mathsf{f}_\theta = (2l+1)\mathsf{f}_\delta$. We may then reason as in item 1.

b. *Both congruences have rests on bottom.* As in the previous case, both frequencies are odd. Since the values of n are exactly the same as in the previous item, we have immediately the conclusion.

c. *θ has rests on bottom, and δ has rests everywhere.* In this case, f_θ must be even. We have $(2k+1)\frac{\mathsf{f}_\theta}{2} - 1 = (l+1)\mathsf{f}_\delta - 1$. Hence we obtain $(2k+1)\mathsf{f}_\theta = 2(l+1)\mathsf{f}_\delta$, and we may reason as in item 1.

d. *Both congruences have rests everywhere.* Here we equate $(k+1)\mathsf{f}_\theta - 1 = (l+1)\mathsf{f}_\delta - 1$ and by adding 1 we are led to the same case as item 1. $\square$

Next we obtain a simple formula for the frequency of $\theta \vee \delta$ in terms of the original frequencies, thereby enabling us to calculate the step of a nontrivial join in general.

Theorem 40. *Let θ and δ be congruences of $\mathbf{L}_n$ with nontrivial join. Then, $\mathsf{f}_{\theta\vee\delta} = \operatorname{lcm}(\mathsf{f}_\theta, \mathsf{f}_\delta)$ and $\mathsf{f}_{\theta\wedge\delta} = \gcd(\mathsf{f}_\theta, \mathsf{f}_\delta)$.*

Proof. Let e_1 be the first positive element in $\operatorname{ext}(\theta) \cap \operatorname{ext}(\delta)$. Observe that θ_{e_1} and δ_{e_1} are congruences of $\mathbf{L}_{e_1}$, and $\theta_{e_1} \vee \delta_{e_1}$ is nontrivial (because otherwise $\langle 0, 1\rangle \in \theta_{e_1} \vee \delta_{e_1} \subseteq \theta \vee \delta$). So Theorem 37 yields

$$\mathsf{f}_{\theta_{e_1}\vee\delta_{e_1}} = \mathsf{f}_{\theta_{e_1}}\mathsf{f}_{\delta_{e_1}},$$

and Lemma 39 says that

$$\gcd\left(\mathsf{f}_{\theta_{e_1}}, \mathsf{f}_{\delta_{e_1}}\right) = 1.$$

Recall from Lemma 15 that

$$\begin{aligned}
\mathsf{f}_{\theta\vee\delta} &= \mathsf{f}_{\theta_{e_1}\vee\delta_{e_1}}\mathsf{f}_{\theta\wedge\delta}\\
\mathsf{f}_\theta &= \mathsf{f}_{\theta_{e_1}}\mathsf{f}_{\theta\wedge\delta}\\
\mathsf{f}_\delta &= \mathsf{f}_{\delta_{e_1}}\mathsf{f}_{\theta\wedge\delta}.
\end{aligned}$$

Now both desired formulas follow easily:

$$\begin{aligned}
\operatorname{lcm}(\mathsf{f}_\theta, \mathsf{f}_\delta) &= \operatorname{lcm}\left(\mathsf{f}_{\theta_{e_1}}\mathsf{f}_{\theta\wedge\delta}, \mathsf{f}_{\delta_{e_1}}\mathsf{f}_{\theta\wedge\delta}\right)\\
&= \operatorname{lcm}\left(\mathsf{f}_{\theta_{e_1}}, \mathsf{f}_{\delta_{e_1}}\right)\mathsf{f}_{\theta\wedge\delta}\\
&= \mathsf{f}_{\theta_{e_1}}\mathsf{f}_{\delta_{e_1}}\mathsf{f}_{\theta\wedge\delta}\\
&= \mathsf{f}_{\theta_{e_1}\vee\delta_{e_1}}\mathsf{f}_{\theta\wedge\delta}\\
&= \mathsf{f}_{\theta\vee\delta},
\end{aligned}$$

and

$$\begin{aligned}
\gcd(\mathsf{f}_\theta, \mathsf{f}_\delta) &= \gcd\left(\mathsf{f}_{\theta_{e_1}}\mathsf{f}_{\theta\wedge\delta}, \mathsf{f}_{\delta_{e_1}}\mathsf{f}_{\theta\wedge\delta}\right)\\
&= \gcd\left(\mathsf{f}_{\theta_{e_1}}, \mathsf{f}_{\delta_{e_1}}\right)\mathsf{f}_{\theta\wedge\delta}\\
&= \mathsf{f}_{\theta\wedge\delta}.
\end{aligned}$$

$\square$

Corollary 41. *Assume that* $\theta = \langle k; \bar{r}\rangle$, $\delta = \langle l; \bar{s}\rangle$, *and* $\theta \vee \delta \neq L \times L$. *Then the cardinality of* $\mathbf{L}/\theta \vee \delta$ *is the integral division of* n *by*

$$\operatorname{lcm}\left(\frac{n - |\bar{r}|}{k}, \frac{n - |\bar{s}|}{l}\right).$$

We finish by making some observations concerning the class of lattices of the form $\operatorname{Con}\mathbf{L}_n$. It is not difficult to see that for prime p, the non distributive lattice M_{p-2} embeds into $\operatorname{Con}\mathbf{L}_p$. Also, $\operatorname{Con}\mathbf{L}_9$ has a sublattice isomorphic to N_5: It is generated by the congruences $\langle 4; 4\rangle$, $\langle 2; 4\rangle$, and $\langle 1\rangle$. As a consequence, the class of congruence lattices of lines does not satisfy the modular law. But on the other hand, every proper ideal of $\operatorname{Con}\mathbf{L}_n$ must be modular indeed, since it is a lattice of permuting relations. Even more is true:

Theorem 42. *Let* $\rho \neq L \times L$ *be a congruence of* $\mathbf{L}_n$, *and let*

$$(\rho] \doteq \{\vartheta \subseteq \rho \mid \vartheta \text{ is a congruence of } \mathbf{L}_n\}.$$

Then the map

$$\vartheta \mapsto \mathsf{f}_\vartheta$$

is a lattice embedding from $\langle(\rho], \wedge, \vee, \mathrm{Id}_{L_n}, \rho\rangle$ *into* $\left\langle\{\text{positive divisors of } \mathsf{f}_\rho\}, \gcd, \operatorname{lcm}, 1, \mathsf{f}_\rho\right\rangle$. *In particular* $\langle(\rho], \wedge, \vee\rangle$ *is distributive.*

Proof. In view of Theorem 40 it only remains to be shown that the map is injective. Assume $\mathsf{f}_\theta = \mathsf{f}_\delta$. Then θ and δ have the same step; say $\theta = \langle k; \bar{r}\rangle$ and $\delta = \langle k; \bar{s}\rangle$. For the sake of contradiction suppose $\bar{r} \neq \bar{s}$, and let i be the first index such that $r_i \neq s_i$. If $r_i < s_i$, then r_i is a common extreme, and Lemma 10 says that $r_i \in \bar{s}$, a contradiction. Of course the case $s_i < r_i$ is symmetric, and thus $\bar{r} = \bar{s}$. ☐

6 Conclusions & Further Work

As mentioned in the introduction, the starting point for this article is the study of equationally definable functions in the variety of modal algebras. A critical step in this study is understanding the lattice of subalgebras of a modal algebra, which is dually isomorphic to the lattice of congruences of the dual frame of the algebra. We were able to obtain a very good description of this lattice for line frames. Congruences on line frames are represented as a tuple of integer parameters. Employing this representation we were able to determine their order relation and describe how their meets and joins are computed, showing that the latter coincide with the relational composition whenever the result is not trivial. The study of the lattice operations on congruences required extensive combinatorial reasoning, which we approached geometrically by using trajectory diagrams — the essential tool behind the results of the paper. We obtained explicit formulas for the cardinality of the quotient of a line by the join of two congruences, by using the said parameters. Finally, we provided a description of the general structural of congruence lattices of line frames by proving that they are composed of lattices of divisors of integer numbers, with a new top element attached (in particular, these lattices are not modular in general, but every proper decreasing subset is distributive).

The continuation of this work will be to apply the results in this paper to obtain a characterization of algebraic functions in subvarieties of modal algebras generated by algebras whose duals are finite line frames.

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