Stock returns and their distribution: an empirical assessment of the US and Argentina's stock market for the period 2002/18

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1. Introduction

The objective of this work is to present a set of analytical tools to characterize the nature of the distribution of monthly returns of the stocks that compose the Merval index in the period 2002-2018, and at the same time compare the results for the argentine market with those of the US stock market, where the same analysis will be performed for most of the 30 equities that comprise the Dow Jones Industrial Index. In a first instance, a set of univariate normality tests will be resorted to in order to determine if the distribution of monthly returns is in fact normal, which include the Jarque - Bera and D’ Agostino K squared tests. The coefficients of skewness and kurtosis will be estimated to better gauge the distribution of returns. Afterwards, multivariate normality tests will be performed, particularly in concern with the third and fourth moments of equities’ return distributions. Additionally, a Generalized Method of Moments (GMM) based test will be used, since it allows for contemporaneous correlation between securities and hence it accounts for its effect on skewness and kurtosis.

Finally, an estimation of the most relevant parameters of the CAPM model and an analysis of its residuals will be performed, ending with a Markov-Switching model that will be applied to stock returns with the goal of measuring the probability of regime switching (changes in mean and variance) in the period of analysis for returns in both the Argentinian and US stock market.

The results for the Argentinian stock market for the period 2002-2018 reveal that the returns of the most important stocks that trade in the market do not come from a multivariate normal distribution. The normality tests performed on univariate time series of the largest market cap and most traded equities that composed the Merval index in the period reject the normality assumption. At the same time, a change of regime (in variance) in returns seem to have occurred in 2008, when several disruptive events like the financial crisis, the conflict between the federal government with farmers and the nationalization of pension funds took place. Besides the results for the full sample, most of the tests are performed for 2 subsamples: for the period October 2002 – April 2008 and January 2009 – December 2018. Results are similar, with the exception that more univariate time series seem to come from a normal distribution, although multivariate normality is rejected for both subsamples.

Results for US equities over the same period are similar. Most univariate time series reject the normality assumption for the whole October 2002 – December 2018 period, and of course multivariate normality is rejected. There’s also a change of regime in the variance of returns in 2008, although for US stocks seems to have occurred in January 2008. The full sample is then subdivided in 2 subsamples, one between October 2002 and December 2007, and the other one between January 2009 and December 2018. Much in-line with Argentina’s equities, multivariate normality is rejected, although more univariate time in both subsample series show normality.

Finally, for both sets of returns the CAPM parameters are estimated. Residuals of the market model are then used to test for normality, and skewness and kurtosis are computed. As is to be expected, since the model is derived from returns that do not come from a multivariate normal distribution, multivariate normality tests for residuals reject the null hypothesis of normality. Skewness and kurtosis values for residuals show almost identical patterns to assets' returns, and a similar conclusion holds for univariate normality tests.

The normality assumption is crucial in the derivation of the most important results of Portfolio Theory and CAPM, thus empirical tests to check for normality of returns must be performed in order to derive empirical insights from the application of these models to financial data. If normality does not hold, alternative distributions (like Student's t or Pareto) should be resorted to with the objective of modelling returns' time series. Hence CAPM and Portfolio Theory are still valid models, although their implications must be adapted to allow for alternative returns' distributions.

Another important insight of the present work is to shed light on the presence of considerable regime changes in assets returns. Normality may hold for monthly returns in a certain period but seems implausible in the long run. Stock markets experience periods of considerable turmoil were variance skyrockets and returns vary wildly on a daily and monthly basis. Therefore the need to use alternative distributions for asset returns over long periods, that assign a higher probability to events that receive almost no weight under a normal distribution.

2. Data

The data for all the returns (both of the Argentinian and US stock market) was extracted from Yahoo Finance. The monthly close price for the period October 2002 – December 2018 was used. The choice for October 2002 as the starting date stems from the fact that Argentina's stock market experienced considerable turmoil at the beginning of 2002, in the context of a large economic crisis, with most macroeconomic variables experiencing excessive volatility. A change of regime occurred in this period, thus the choice of October as the starting date.

Returns were obtained using the formula: $R_t = \ln(P_t) - \ln(P_{t-1})$, with P_t representing the current day's close price and P_{t-1} the previous day's close price.

A similar assessment applies to US stocks. Assets traded in the US stock market experienced considerable volatility between mid-2001 to mid-2002, spurred by the 9/11 terrorist attacks, a recession that started in late 2000 and the burst of the dot com bubble. Therefore, October 2002 was chosen as the starting date.

For some of Argentina's stocks included in the sample, some data is missing, since these assets started trading in the market at a date after October 2002. It's the case of Tenaris (TS), for which the price series starts in December 2002, Ledesma (LEDE) -that started trading in January 2003- and Pampa Energía (PAMP), whose price series starts in January 2004. Because of this fact, the latter is excluded from the multivariate normality tests, since it doesn't have enough observations for the first subsample.

As mentioned previously, the choice of the periods for the subsamples was made on the grounds of the presence of structural breaks in mean and variance of returns. This change of regime can be appreciated both through the application of a Markov Switching process to the data and by testing univariate normality for time series including an additional month of 2008 (like May 2008 for Argentina and January 2008 for the US), with several returns series rejecting normality only if this additional month is included.

All tests were performed using Python, except for Mardia's multivariate tests, for which R was resorted to.

3. Univariate Normality

We start computing skewness and kurtosis for all the equities for both Argentina and the US for the whole sample. As is well known, the sample skewness (S_a) and the sample kurtosis (K_a) - third and fourth moment respectively- are computed as follows:

$$S_a = (\frac{1}{T})\sum_{i=1}^{T}(R_{it} - M_{Ri})^3$$

$$K_a = (\frac{1}{T})\sum_{i=1}^{T}(R_{it} - M_{Ri})^4$$

Where: R_{it} is asset's i return in month t, M_{Ri} is asset's i average monthly return and T is the sample's size.

Simultaneously, normality tests are performed to gauge if the sample skewness and kurtosis come from a normal distribution. The following statistics are constructed:

$$A_s = \frac{1}{T}\frac{\sum_{i=1}^{T}(R_{it} - M_{Ri})^3}{S_{R_i}^2} = \frac{S_a}{S_{R_i}^2} \sim N(0, \frac{6}{T})$$

$$K_s = \frac{1}{T}\frac{\sum_{i=1}^{T}(R_{it} - M_{Ri})^4}{S_{R_i}^4} = \frac{S_a}{S_{R_i}^4} \sim N(3, \frac{24}{T})$$

Where: S_{Ri} is the monthly return's sample standard deviation.

Results are summarized in the following table, first for Argentina's stocks and then for US stocks:

H_0: Skewness/Kurtosis comes from a normal distribution

H_1: Skewness/Kurtosis does not come from a normal distribution

Significance: if the null hypothesis is not rejected for skewness nor kurtosis, then significance = 0 (the distribution cannot be significantly considered different from normal).

Table 1 – Skewness and Kurtosis tests for Argentina's market (full sample)

	Skewness	p-value	Kurtosis	p-value	Significance
BMA	-1.318	0.000	9.122	0.000	1
CEPU	-0.089	0.306	2.781	0.267	0
COME	0.121	0.754	1.656	0.000	1
CRES	-0.386	0.014	1.303	0.000	1
FRAN	-0.193	0.136	1.629	0.000	1

GGAL	-0.813	0.000	5.872	1.000	1
INDU	1.457	1.000	10.788	0.000	1
IRSA	-0.280	0.055	1.600	0.000	1
LEDE	0.379	0.985	0.288	0.000	1
MIRG	-0.174	0.162	3.831	0.991	0
MOLI	0.676	1.000	1.539	0.000	1
PAMP	3.482	1.000	24.094	0.000	1
TECO2	-0.281	0.055	3.647	0.967	0
TEF	-0.210	0.117	0.355	0.000	1
TGSU2	0.117	0.747	1.079	0.000	1
TRAN	0.332	0.971	2.147	0.008	1
TS	-0.347	0.024	1.079	0.000	1
TXAR	0.155	0.811	0.824	0.000	1
YPFD	-0.872	0.000	4.619	1.000	1

Table 2 – Skewness and Kurtosis tests for the US market (full sample)

	Skewness	**p-value**	**Kurtosis**	**p-value**	**Significance**
AAPL	-0.730	0.000	2.490	0.069	1
AXP	1.032	1.000	13.661	0.000	1
BA	-0.549	0.001	1.151	0.000	1
CAT	-0.794	0.000	3.988	0.998	1
CSCO	-0.204	0.117	1.119	0.000	1
CVX	-0.278	0.053	0.343	0.000	1
DIS	-0.514	0.001	1.163	0.000	1
GE	-0.581	0.000	3.221	0.740	1
HD	-0.460	0.004	0.705	0.000	1
IBM	-0.571	0.000	4.549	1.000	1
INTC	-0.795	0.000	3.127	0.644	1
JNJ	-0.549	0.001	1.619	0.000	1
JPM	-0.791	0.000	2.566	0.103	1
KO	-0.474	0.003	1.560	0.000	1
MCD	-1.010	0.000	5.391	1.000	1
MRK	-0.664	0.000	2.095	0.004	1
MSFT	-0.076	0.329	0.741	0.000	1
PFE	-0.352	0.020	0.411	0.000	1
PG	-0.357	0.019	0.369	0.000	1
T	-0.334	0.026	1.804	0.000	1
TRV	-0.293	0.044	1.424	0.000	1
UNH	-1.525	0.000	5.975	0.000	1
UTX	-0.457	0.004	0.323	0.000	1
VZ	0.327	0.971	3.504	0.929	0
WMT	-0.246	0.076	0.756	0.000	1

As can be observed from the tables above, only a small fraction of the returns sets for each market seems to come from a normal distribution. Considering a level of confidence of 0.05, for the Argentinian market, just 3 of the 19 stocks (15.7%) have monthly returns that exhibit skewness and kurtosis consistent with a normal distribution, while this is true for just 1 of the total 25 companies included for the US market.

To further assess univariate normality, 2 of the most renowned normality tests, namely Jarque-Bera, and D'Agostino K^2 tests, are performed for the whole sample for both US and Argentinian stocks. The Jarque-Bera (JB) test includes the estimated Kurtosis and Skewness for the samples, according to the computed statistics

The statistic for the Jarque-Bera test is the following one:

$$\frac{T{S_a}^2}{6}+\frac{T(K_s-3)^2}{24} \quad \sim \chi_2^2$$

Table 3 - Jarque-Bera test for Argentina's market (full sample)

	Statistic	p-value	Est. Skewness	Est. Kurtosis	Significance
BMA	728.784	0.000	-1.318	12.122	1
CEPU	62.795	0.000	-0.089	5.781	1
COME	22.652	0.000	0.121	4.656	1
CRES	18.533	0.000	-0.386	4.303	1
FRAN	22.651	0.000	-0.193	4.629	1
GGAL	300.060	0.000	-0.813	8.872	1
INDU	1009.411	0.000	1.457	13.788	1
IRSA	23.245	0.000	-0.280	4.600	1
LEDE	5.269	0.072	0.379	3.288	0
MIRG	119.614	0.000	-0.174	6.831	1
MOLI	33.945	0.000	0.676	4.539	1
PAMP	4717.705	0.000	3.482	27.094	1
TECO2	110.081	0.000	-0.281	6.647	1
TEF	2.438	0.295	-0.210	3.355	0
TGSU2	9.849	0.007	0.117	4.079	1
TRAN	40.846	0.000	0.332	5.147	1
TS	13.229	0.001	-0.347	4.079	1
TXAR	6.269	0.044	0.155	3.824	1
YPFD	197.005	0.000	-0.872	7.619	1

Table 4 - Jarque-Bera test for the US market (full sample)

	Statistic	p-value	Est. Skewness	Est. Kurtosis	Significance
AAPL	73.330	0.000	-0.706	5.661	1

AXP	1832.530	0.000	1.138	17.884	1
BA	30.030	0.000	-0.615	4.484	1
CAT	177.905	0.000	-0.886	7.344	1
CSCO	4.003	0.135	-0.264	3.466	0
CVX	1.818	0.403	-0.208	3.227	0
DIS	20.851	0.000	-0.538	4.192	1
GE	115.645	0.000	-0.585	6.597	1
HD	4.009	0.135	-0.274	3.442	0
IBM	121.482	0.000	-0.919	6.414	1
INTC	16.252	0.000	-0.433	4.123	1
JNJ	14.452	0.001	-0.378	4.103	1
JPM	32.854	0.000	-0.510	4.739	1
KO	32.251	0.000	-0.492	4.738	1
MCD	8.014	0.018	-0.048	3.991	1
MRK	66.640	0.000	-0.785	5.405	1
MSFT	5.551	0.062	-0.111	3.798	0
PFE	6.562	0.038	-0.363	3.532	1
PG	5.425	0.066	-0.377	3.318	0
T	12.796	0.002	-0.525	3.694	1
TRV	4.473	0.107	-0.004	3.744	0
UNH	386.308	0.000	-1.578	9.151	1
UTX	6.132	0.047	-0.420	3.230	1
VZ	2.984	0.225	-0.289	2.815	0
WMT	9.469	0.009	-0.242	3.968	1

These tests corroborate the results previously shown for skewness and kurtosis of returns: only a very small subset of stocks seem to come from a normal distribution when the whole period is considered. For Argentina's market, just 2 assets (10.5%) do not reject the null hypothesis of normality, while for the US market this is true for 7 equities (28% of the total).
As a check for robustness, D'Agostino K^2 test is performed. This test uses the following statistics for skewness and kurtosis:

$$g_3 = \frac{(\frac{1}{T})\sum_{i=1}^{T}(R_{it} - M_{Ri})^3}{(\left(\frac{1}{T}\right)\sum_{i=1}^{T}(R_{it} - M_{Ri})^2)^{3/2}}$$

$$g_3 = \frac{(\frac{1}{T})\sum_{i=1}^{T}(R_{it} - M_{Ri})^3}{(\left(\frac{1}{T}\right)\sum_{i=1}^{T}(R_{it} - M_{Ri})^2)^{3/2}}$$

With the statistic: $K^2 = {g_3}^2 + {g_4}^2 \sim \chi_2^2$

Table 5 - D'Agostino K^2 test for Argentina's stocks (full sample)

Statistic	**p-value**	**Significance**

BMA	81.752	0.000	1
CEPU	17.747	0.000	1
COME	10.589	0.005	1
CRES	12.440	0.002	1
FRAN	11.167	0.004	1
GGAL	51.091	0.000	1
INDU	91.710	0.000	1
IRSA	12.330	0.002	1
LEDE	5.657	0.059	0
MIRG	24.467	0.000	1
MOLI	22.653	0.000	1
PAMP	173.684	0.000	1
TECO2	25.098	0.000	1
TEF	2.833	0.243	0
TGSU2	6.477	0.039	1
TRAN	17.088	0.000	1
TS	9.933	0.007	1
TXAR	5.060	0.080	0
YPFD	47.758	0.000	1

Table 6 - D'Agostino K^2 test for US stocks (full sample)

	Statistic	**p-value**	**Significance**
AAPL	31.124	0.000	1
AXP	87.799	0.000	1
BA	20.226	0.000	1
CAT	46.972	0.000	1
CSCO	4.288	0.117	0
CVX	2.209	0.331	0
DIS	15.769	0.000	1
GE	32.579	0.000	1
HD	4.335	0.114	0
IBM	43.320	0.000	1
INTC	12.336	0.002	1
JNJ	10.837	0.004	1
JPM	18.788	0.000	1
KO	18.273	0.000	1
MCD	5.471	0.065	0
MRK	32.291	0.000	1
MSFT	4.486	0.106	0
PFE	6.676	0.036	1
PG	5.807	0.055	0

T	11.923	0.003	1
TRV	3.696	0.158	0
UNH	82.047	0.000	1
UTX	6.434	0.040	1
VZ	2.928	0.231	0
WMT	7.210	0.027	1

As can be gauged from the results above, although slightly more assets in both markets seem to come from a normal distribution, this test confirms the result of JB that most equities do not exhibit return time series consistent with a normal distribution.

Inside this large sample of over 15 years, 2 subsamples are distinguishable: one that stems from the beginning of the period to early 2008, and another one that starts in January 2009 and extends until the end of the sample in December 2018. The Jarque – Bera test is performed for both subsamples for Argentina and US data. For the Argentinian market, the first subsample spans from October 2002 to April 2008, while for the US starts in October 2002 and ends in December 2007. The second subsample for the Argentinian market spans from January 2009 to December 2018, and the same is true for US returns. The tables are shown below.

Table 7 - Jarque-Bera test for Argentina's market (2002-2008)

	Statistic	p-value	Est. Skewness	Est. Kurtosis	Significance
BMA	0.579	0.749	-0.198	3.230	0
CEPU	32.186	0.000	-0.058	6.419	1
COME	5.470	0.065	0.425	4.125	0
CRES	0.180	0.914	-0.081	2.802	0
FRAN	1.307	0.520	0.315	3.280	0
GGAL	4.816	0.090	0.636	3.366	0
INDU	0.826	0.662	-0.152	2.545	0
IRSA	1.822	0.402	-0.110	3.784	0
LEDE	11.121	0.004	0.763	4.357	1
MIRG	1.052	0.591	0.308	3.051	0
MOLI	1.293	0.524	0.276	3.406	0
PAMP	441.627	0.000	3.201	15.761	1
TECO2	1.318	0.517	0.209	3.552	0
TEF	29.082	0.000	-0.831	5.795	1
TGSU2	3.806	0.149	0.461	3.730	0
TRAN	0.141	0.932	0.032	2.783	0
TS	1.759	0.415	0.354	3.384	0
TXAR	2.686	0.261	0.474	3.277	0
YPFD	14.131	0.001	-0.029	5.266	1

Table 8 - Jarque-Bera test for Argentina's market (2009-2018)

	Statistic	p-value	Est. Skewness	Est. Kurtosis	Significance
BMA	0.838	0.658	0.204	3.043	0
CEPU	17.952	0.000	-0.115	4.889	1
COME	23.705	0.000	-0.032	5.186	1
CRES	1.017	0.601	-0.194	3.233	0
FRAN	0.565	0.754	0.151	2.851	0
GGAL	0.322	0.851	0.127	2.989	0
INDU	386.069	0.000	1.474	11.317	1
IRSA	1.801	0.406	0.056	3.592	0
LEDE	1.964	0.375	0.313	2.931	0
MIRG	33.668	0.000	0.518	5.391	1
MOLI	25.951	0.000	0.756	4.716	1
PAMP	5.335	0.069	0.514	3.144	0
TECO2	15.194	0.001	0.669	4.128	1
TEF	0.536	0.765	-0.121	2.778	0
TGSU2	3.541	0.170	0.146	3.793	0
TRAN	11.688	0.003	0.353	4.364	1
TS	0.335	0.846	-0.128	3.043	0
TXAR	3.052	0.217	0.339	3.396	0
YPFD	81.918	0.000	-0.866	6.677	1

As can be observed in the tables above, when these 2 subperiods are considered, there's a considerably larger number of returns whose time series seem to come from a normal distribution. For the first subsample, 11 of the 19 stocks do not reject the null hypothesis of the JB test (57.9% of the total), and the same results hold for the second subsample. This shows that a regime change occurred in 2008, particularly in what concerns to volatility.

Table 9 - Jarque-Bera test for the US market (2002-2007)

	Statistic	p-value	Est. Skewness	Est. Kurtosis	Significance
AAPL	13.300	0.001	-0.682	4.791	1
AXP	7.740	0.021	-0.535	4.343	1
BA	1.523	0.467	-0.062	2.249	0
CAT	1.027	0.598	0.311	2.928	0
CSCO	0.004	0.998	-0.015	3.029	0
CVX	4.062	0.131	-0.327	4.058	0
DIS	7.031	0.030	-0.548	4.215	1
GE	3.488	0.175	0.271	4.017	0
HD	0.147	0.929	0.118	3.009	0
IBM	11.005	0.004	-0.582	4.684	1
INTC	4.827	0.090	-0.445	4.023	0
JNJ	1.162	0.559	0.307	2.744	0
JPM	15.665	0.000	0.344	5.344	1

KO	8.875	0.012	-0.372	4.682	1
MCD	0.816	0.665	-0.162	3.454	0
MRK	23.039	0.000	-0.913	5.333	1
MSFT	9.572	0.008	0.137	4.890	1
PFE	0.204	0.903	-0.022	2.724	0
PG	1.886	0.389	-0.262	3.666	0
T	0.829	0.661	-0.193	3.409	0
TRV	0.844	0.656	-0.193	2.585	0
UNH	2.155	0.340	-0.448	3.131	0
UTX	1.164	0.559	0.214	2.491	0
VZ	1.838	0.399	-0.418	2.963	0
WMT	1.365	0.505	0.250	2.480	0

Table 10 - Jarque-Bera test for the US market (2009-2018)

	Statistic	p-value	Est. Skewness	Est. Kurtosis	Significance
AAPL	1.658	0.437	-0.280	3.148	0
AXP	1967.690	0.000	2.553	22.256	1
BA	0.504	0.777	-0.109	3.233	0
CAT	5.168	0.075	0.201	3.939	0
CSCO	2.175	0.337	-0.249	3.436	0
CVX	0.327	0.849	0.039	2.755	0
DIS	3.112	0.211	-0.182	3.704	0
GE	18.694	0.000	0.141	4.921	1
HD	12.510	0.002	-0.527	4.189	1
IBM	99.606	0.000	-0.807	7.181	1
INTC	0.060	0.970	0.023	3.100	0
JNJ	4.112	0.128	-0.242	3.771	0
JPM	14.087	0.001	-0.457	4.416	1
KO	0.207	0.902	-0.048	3.180	0
MCD	0.714	0.700	0.103	3.319	0
MRK	0.910	0.634	-0.072	3.404	0
MSFT	1.282	0.527	-0.142	3.422	0
PFE	1.718	0.424	0.060	2.424	0
PG	0.801	0.670	-0.152	2.737	0
T	5.428	0.066	-0.504	3.277	0
TRV	5.348	0.069	0.424	3.598	0
UNH	3.831	0.147	-0.431	3.168	0
UTX	1.666	0.435	-0.285	2.900	0
VZ	0.868	0.648	-0.164	2.741	0
WMT	6.516	0.038	-0.194	4.079	1

A similar pattern holds for US stocks, where for the first subsample, 17 of the 25 stocks (68%) do not reject the null hypothesis of univariate normality of their returns, and an even larger number - 19 of 25 or 76%- seem to come from a normal distribution for the second subsample. As is the case with Argentinian returns, a regime change seems to have occurred in 2008, in the midst of the financial crisis that started that year. Although there is one difference: the threshold month for US stocks seems to be January 2008, while for Argentina is April 2008. That month of 2008 marked the beginning of a conflict between the federal government and local farmers. Was this idiosyncratic event more important than the turmoil in international financial markets for Argentina? Further research should be conducted on this topic.

4. Multivariate Normality

The most important theories in finance like CAPM and Portfolio Theory rely on the multivariate normality assumption of returns. If returns come from such distribution, marginal densities should be univariate normal, which as shown above, is not the case for most equities in both Argentina's and the US market for the 2002 – 2018 period. In this section, 2 important multivariate normality tests are performed. The first one is known as the Mardia test, developed by Mardia in 1970, while the other one is a multivariate test that is based in the Generalized Method of Moments (GMM) developed by Lars Hansen in 1982.

4.1. Mardia Test

The Mardia multivariate normality test develops a sample measure for multivariate skewness and kurtosis, and gauges whether these estimators are consistent with skewness and kurtosis that come from a multivariate normal distribution. The sample measures for skewness and kurtosis (b1, b2) are given by:

$$\bar{R} = \frac{1}{T}\sum_{i=1}^{T} R_i$$

$$S = \frac{1}{T}\sum_{i=1}^{T}(R_i - \bar{R})(R_i - \bar{R})'$$

$$b_1 = \frac{1}{T^2}\sum_{i=1}^{T}\sum_{j=1}^{t}\left[(R_i - \bar{R})'^{S^{-1}}\left(R_j - \bar{R}\right)\right]^3$$

$$b_2 = \frac{1}{T}\sum_{i=1}^{T}\left[(R_i - \bar{R})'^{S^{-1}}(R_i - \bar{R})\right]^2$$

The following are the test statistics for skewness and kurtosis:

$$z_1 = \frac{N}{6} b_1^2$$

$$z_2 = \frac{b_2 - \frac{N-1}{N+1}p(p+2)}{\sqrt{\frac{8}{N}p(p+2)}}$$

Z1 is distributed chi-squared with p(p+1)(p+2)/6 degrees of freedom, while Z2 is distributed normal (0,1).

Results for the whole sample for Argentina's and the US stock market are represented below, where test for multivariate normality skewness, kurtosis and for the distribution are performed:

Table 11 – Mardia Test for the full sample

Market	Mardia skewness p-value	Mardia kurtosis p-value	Mardia multivariate normality p-value
Argentina	0.000	0.000	0.000
US	0.000	0.000	0.000

Table 12 – Mardia Test for the first subsample

Market	Mardia skewness p-value	Mardia kurtosis p-value	Mardia multivariate normality p-value
Argentina	0.001	0.006	0.001
US	0.000	0.14	0.000

Table 13 – Mardia Test for the second subsample

Market	Mardia skewness p-value	Mardia kurtosis p-value	Mardia multivariate normality p-value
Argentina	0.000	0.000	0.000
US	0.000	0.000	0.000

Multivariate normality as well as multivariate skewness and kurtosis that come from a normal distribution are rejected for the whole sample. The same is true for both subsamples in both markets, except for the multivariate kurtosis test for the first subsample in the US.

4.2. Multivariate GMM-based test

When a set of returns arises from a multivariate normal distribution, marginal densities should be univariate normal. However, univariate normality tests and most multivariate tests as well do not account for the effect that contemporaneous correlations have on returns' distributions. The multivariate normal distribution imposes restrictions on the marginal and joint moments of the multivariate time series in terms of a relatively small number of parameters: the means, variances, and cross correlation. Therefore, univariate tests can be misleading: given the correlation across assets, univariate statistics will in general be correlated. This correlation suggests the need for a joint test across the asset returns being analyzed.

It is for this reason that Richardson and Smith developed in 1993 a Generalized Method of Moments (GMM) test. This test makes use of the first 5 moments of the joint distribution of returns.

As is well known from the GMM model, given a series of returns that belong to a certain multivariate distribution F, if a certain condition is imposed on the moments of this distribution F (which is characterized by a set of parameters θ), then the following orthogonality conditions must hold:

$$E[h(R_t, \theta)] = \vec{0}$$

Thus, if the series for returns converge in distribution to a certain function F, given a particular assumption on its moments (for example, if F is assumed multivariate normal), then $E[h(R_t, \theta)]$ converges in mean square to 0. The equivalent condition for sample moments is:

$$\lim_{t \to \infty} g_t(\theta) = \lim_{T \to \infty} \frac{1}{T} \sum_{i=1}^{T} h(R_i, \theta) = 0$$

The GMM method allows to estimate the parameters vector θ by solving the following set of conditions (optimization problem):

$$\hat{\theta} = argmin\ g_T(\theta)' W g_T(\theta)$$

If the returns' time series are stationary (and the third and fourth moments are finite), the asymptotic distribution of the estimated coefficients is the following:

$$\sqrt{T}(\hat{\theta} - \theta) \rightarrow N[0, (D_0' \, S_0^{-1} \, D_0)^{-1}]$$

And hence, when restrictions are placed

$$J_T = T g_T(\hat{\theta})' S_0^{-1} g_T(\hat{\theta}) \rightarrow \chi^2_{R-M}$$

Where: $D_0 = E[\frac{\partial h(R_t,\theta)}{\partial \theta}]$ and $S_0 = E[h(R_t, \theta) h(R_t, \theta)']$.

The sample estimators of the D_0 and S_0 matrices are D_T and S_T, which are computed using sample observations. It should be noticed that there are R parameters and M restrictions.

In the particular case (without loss of generality) of just two returns that are assumed to arise from a bivariate normal distribution, it is possible to compute a vector of 5 parameters $(\mu_i, \mu_j, \sigma_i^2, \sigma_j^2, \rho_{ij})$ from which higher order moments and cross moments can be estimated. Specifically, the interest resides in computing the sample values of cross-kurtosis and cross-skewness. Although the marginal distribution of a set of returns that come from a multivariate normal distribution must be normal as well, the rejection of a univariate normality test is not a sufficient condition to discard the possibility that these returns are distributed multivariate normal, since the effect of cross correlations is not accounted for in univariate tests. Kurtosis and skewness of returns will have a high correlation for a set of assets with returns that have a high correlation coefficient. To account for this effect, cross-skewness and cross-kurtosis are computed for both US and Argentina stocks, for the whole sample. For the sake of simplicity, cross-skewness and cross-kurtosis is computed taking one asset as a benchmark (BMA for Argentina and AAPL for the US).
From the resolution of the optimization problem that involves the $g_T(\theta)$ function, we have the following measures of cross-skewness and cross-kurtosis:

$$S_{ij} \equiv \frac{\frac{1}{T}\sum_{t=1}^{T}(R_{it}-\hat{\mu}_i)^2(R_{jt}-\hat{\mu}_j)}{\left[\frac{1}{T}\sum_{t=1}^{T}(R_{it}-\hat{\mu}_i)^2\right]\left[\frac{1}{T}\sum_{t=1}^{T}(R_{jt}-\hat{\mu}_j)^2\right]^{1/2}}$$

$$K_{ij} \equiv \frac{\frac{1}{T}\sum_{t=1}^{T}(R_{it}-\hat{\mu}_i)^2(R_{jt}-\hat{\mu}_j)^2}{\left[\frac{1}{T}\sum_{t=1}^{T}(R_{it}-\hat{\mu}_i)^2\right]\left[\frac{1}{T}\sum_{t=1}^{T}(R_{jt}-\hat{\mu}_j)^2\right]} - (1+2\hat{\rho}_{ij}^2)$$

Where $(S_{ij},\ K_{ij})$ symbolize cross-skewness and cross-kurtosis, respectively.

Thus, for 2 elements of the cross-skewness vector, the following is true for the asymptotic distribution:

$$\sqrt{T}\begin{pmatrix} S_{ij} \\ S_{kl} \end{pmatrix} \overset{asy}{\sim} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4\rho_{ij}^2+2 & 2\rho_{ik}^2\rho_{jl}+4\rho_{ik}\rho_{il}\rho_{jk} \\ 2\rho_{ik}^2\rho_{jl}+4\rho_{ik}\rho_{il}\rho_{jk} & 4\rho_{kl}^2+2 \end{pmatrix}\right)$$

Similarly, for 2 elements of the cross-kurtosis vector:

$$\sqrt{T}\begin{pmatrix} K_{ij} \\ K_{kl} \end{pmatrix} \overset{asy}{\sim} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix},\right.$$
$$\left.\begin{pmatrix} 4\rho_{ij}^4+16\rho_{ij}^2+4 & 4\rho_{ik}^2\rho_{jl}^2+16\rho_{ik}\rho_{il}\rho_{jk}\rho_{jl}+4\rho_{il}^2\rho_{jk}^2 \\ 4\rho_{ik}^2\rho_{jl}^2+16\rho_{ik}\rho_{il}\rho_{jk}\rho_{jl}+4\rho_{il}^2\rho_{jk}^2 & 4\rho_{kl}^4+16\rho_{kl}^2+4 \end{pmatrix}\right)$$

For the sake of simplicity, one asset will be taken as benchmark, and both cross-skewness and cross-kurtosis will be computed with respect to this asset (BMA for Argentina and AAPL for the US)
Hence the univariate cross-skewness limiting distribution for each of the remaining equities looks like this:

$$\sqrt{T}\,S_{ij} \sim N(0, 4\rho_{ij}^2+2)$$

Similarly, for the univariate cross-kurtosis:

$$\sqrt{T}\,K_{ij} \sim N(0, 4\rho_{ij}^4+16\rho_{ij}^2+2)$$

Finally, the Wald test for the joint restrictions that all cross-skewness sample measures equal 0 ($S_{ij}=0\ \forall\ j$), and the equivalent for the sample cross-kurtosis measures ($K_{ij}=0\ \forall\ j$) adopts the form:

$$TS'[V(S)]S \sim \chi_R^2$$

$$TK'[V(K)]K \sim \chi_R^2$$

Where: T is the number of observations, S is the vector of cross-skewness measures for all the assets, V(S) is the covariance matrix of the cross-skewness vector, K is the vector of cross-kurtosis measures for all the assets and V(K) is the covariance matrix of the cross-kurtosis vector. R is the number of restrictions, which is exactly equal to the number of assets under consideration.

The results for individual cross-skewness and cross-kurtosis, and the respective Wald tests for joint restrictions, are summarized in the tables below.

Table 14 – Coskewness and Cokurtosis for Argentina's market (full sample)

	Cross Skewness	p-value	Cross Kurtosis	p-value	Significance
CEPU	-0.724	0.305	-5.414	0.003	1
COME	-4.804	0.001	0.785	0.630	1
CRES	-11.867	0.000	41.633	0.000	1
FRAN	-12.057	0.000	49.396	0.000	1
GGAL	-16.682	0.000	97.279	0.000	1
INDU	-7.389	0.000	20.302	1.000	1
IRSA	-12.716	0.000	40.932	0.000	1
LEDE	-4.320	0.003	1.580	0.736	1
MIRG	-14.054	0.000	47.126	0.000	1
MOLI	-0.814	0.300	-4.058	0.045	1
TECO2	-14.561	0.000	69.756	0.000	1
TEF	-4.418	0.002	-4.544	0.025	1
TGSU2	-8.941	0.000	24.028	1.000	1
TRAN	-10.310	0.000	19.974	1.000	1
TS	-12.002	0.000	35.048	0.000	1
TXAR	-10.801	0.000	26.020	0.000	1
YPFD	1.620	0.841	-3.267	0.102	0
χ^2_{17}		0.000		0.000	0

Table 15 – Coskewness and Cokurtosis for the US market (full sample)

	Cross Skewness	p-value	Cross Kurtosis	p-value	Significance
AXP	-1.558	0.164	-1.721	0.245	0
BA	-4.668	0.001	-3.979	0.047	1
CAT	-1.533	0.158	-5.883	0.006	1
CSCO	-2.783	0.038	-9.042	0.000	1
CVX	-2.738	0.040	-3.845	0.056	1
DIS	-2.474	0.057	-5.476	0.012	1
GE	-3.091	0.021	-5.713	0.007	1
HD	1.103	0.772	-2.672	0.110	0
IBM	-1.369	0.187	-7.987	0.000	1
INTC	-7.490	0.000	9.804	0.000	1
JNJ	-1.680	0.122	-3.825	0.033	1

JPM	4.138	0.998	3.874	0.967	0
KO	-0.223	0.440	-5.065	0.009	1
MCD	-1.656	0.138	-2.858	0.106	0
MRK	-8.265	0.000	16.140	0.000	1
MSFT	-3.269	0.018	-7.468	0.001	1
PFE	-0.760	0.296	-1.810	0.183	0
PG	-3.840	0.004	3.878	0.967	1
T	-5.424	0.000	2.851	0.904	1
TRV	-1.562	0.150	-2.982	0.093	0
UNH	-6.806	0.000	9.840	0.000	1
UTX	-3.824	0.007	-5.550	0.011	1
VZ	-4.768	0.001	4.658	0.986	1
WMT	0.883	0.734	-2.359	0.119	0
χ^2_{24}		0.000		0.000	0

Both univariate and multivariate tests of cross-skewness and cross-kurtosis reject the normality assumption, therefore the evidence of normality of returns is weak even when contemporaneous cross-correlations are considered. The conclusions are similar when the subsample periods are analyzed (results are not shown).

5. CAPM Estimation and Residual Analysis

The coefficients of the CAPM model are estimated for the full sample for both the US and Argentina's market. One of the assumptions of the model is that returns come from a multivariate normal distribution. Therefore, the normality tests will be applied to the residuals of the model.
As is well known, CAPM postulates the following equation as the benchmark for pricing assets:

$$R_i = R_f \ + \beta_i \ (R_m - R_f)$$

Where: R_i refers to the asset's return, R_f symbolizes the risk-free return and R_m represents the market portfolio's return. The beta coefficient is the key element of the model, and it measures the variability of the asset return with respect to the market's portfolio, thus representing a measure of systematic risk. If this coefficient is greater than one, then the stock return experiences a larger variation than the market portfolio, hence a larger return is expected to compensate for this higher risk.

To test the model against the available data, the following equation is estimated:

$$R_i - R_f = \ \alpha \ + \beta_i \ (R_m - R_f)$$

According to the model's implications, the value of alpha should not be significantly different from 0: the excess return from an equity must only depend on the relationship between this stock's return and the market premium, expressed by the beta coefficient. Only systematic risk is important in an asset's valuation for this model.

For Argentina, the 30-day BADLAR time deposits rate is used as the risk-free rate, while the MERVAL index is considered the market's portfolio. For the US, the one-month treasury rate represents the risk-free rate, while the benchmark Dow Jones index is used as the market's portfolio (the Russell 3000 index was tested as the market's portfolio as a check for robustness, and the results where almost identical).

Table 16 – CAPM coefficients for Argentina's market (full sample)

	Constant	Std. Error	t-value	p-value	Beta	Std. Error	t-value	p-value
BMA	0.005	0.007	0.631	0.268	0.968	0.061	15.964	0.000
CEPU	0.015	0.111	0.131	0.448	1.506	0.900	1.672	0.056
COME	-0.007	0.012	-0.590	0.281	0.977	0.097	10.047	0.000
CRES	-0.002	0.008	-0.252	0.402	0.864	0.062	13.993	0.000
FRAN	0.002	0.007	0.242	0.406	1.151	0.059	19.676	0.000
GGAL	0.013	0.008	1.735	0.050	1.086	0.061	17.867	0.000
INDU	-0.008	0.016	-0.519	0.305	0.887	0.126	7.025	0.000
IRSA	0.003	0.008	0.340	0.369	0.993	0.067	14.864	0.000
LEDE	-0.015	0.008	-1.945	0.034	0.919	0.064	14.429	0.000
MIRG	0.007	0.013	0.547	0.296	0.900	0.109	8.241	0.000
MOLI	-0.015	0.010	-1.524	0.072	0.960	0.082	11.745	0.000
TECO2	-0.001	0.007	-0.097	0.462	0.842	0.058	14.434	0.000
TEF	-0.011	0.008	-1.353	0.096	0.714	0.067	10.582	0.000
TGSU2	0.009	0.008	1.039	0.156	0.942	0.068	13.927	0.000
TRAN	0.007	0.013	0.553	0.293	1.190	0.102	11.705	0.000
TS	-0.004	0.009	-0.463	0.324	0.819	0.074	11.108	0.000
TXAR	-0.004	0.008	-0.473	0.321	0.937	0.065	14.374	0.000
YPFD	-0.014	0.010	-1.418	0.087	0.893	0.077	11.561	0.000

Table 17 – CAPM coefficients for the US market (full sample)

	Constant	Std. Error	t-value	p-value	Beta	Std. Error	t-value	p-value
AAPL	0.021	0.006	3.310	0.001	1.054	0.154	6.825	0.000
AXP	0.004	0.004	0.854	0.201	1.550	0.106	14.637	0.000
BA	0.009	0.004	2.287	0.015	1.318	0.097	13.581	0.000
CAT	0.007	0.005	1.552	0.067	1.619	0.115	14.133	0.000
CSCO	0.002	0.004	0.443	0.331	1.190	0.103	11.601	0.000
CVX	0.000	0.003	0.047	0.481	0.906	0.080	11.285	0.000
DIS	0.005	0.003	1.589	0.062	1.155	0.073	15.735	0.000
GE	-0.008	0.005	-1.773	0.044	1.342	0.111	12.084	0.000
HD	0.006	0.004	1.505	0.072	1.035	0.092	11.304	0.000
IBM	-0.004	0.003	-1.094	0.142	0.891	0.083	10.716	0.000
INTC	0.000	0.004	0.101	0.460	1.096	0.107	10.198	0.000
JNJ	-0.003	0.002	-1.240	0.113	0.692	0.060	11.619	0.000

JPM	0.004	0.004	0.956	0.174	1.344	0.102	13.229	0.000
KO	-0.003	0.003	-1.264	0.109	0.666	0.066	10.107	0.000
MCD	0.005	0.003	1.752	0.046	0.633	0.070	9.006	0.000
MRK	-0.006	0.005	-1.381	0.090	0.681	0.111	6.125	0.000
MSFT	0.002	0.004	0.427	0.336	0.969	0.094	10.314	0.000
PFE	-0.005	0.003	-1.567	0.065	0.829	0.081	10.277	0.000
PG	-0.004	0.003	-1.548	0.067	0.576	0.067	8.621	0.000
T	-0.007	0.003	-2.017	0.027	0.664	0.083	8.005	0.000
TRV	0.001	0.003	0.251	0.402	0.928	0.067	13.753	0.000
UNH	0.007	0.005	1.531	0.069	0.983	0.113	8.708	0.000
UTX	0.002	0.002	0.701	0.245	1.125	0.060	18.830	0.000
VZ	-0.006	0.003	-1.641	0.057	0.641	0.084	7.653	0.000
WMT	-0.005	0.003	-1.551	0.067	0.491	0.085	5.815	0.000

Finally, the residual analysis is performed for both markets. As expected, normality of residuals is rejected for most equities.

Table 18 – CAPM residuals for Argentina's market (full sample)

	Skewness	p-value	Kurtosis	p-value	Significance
BMA	0.278	0.892	0.322	0.000	1
CEPU	0.018	0.531	1.961	0.010	1
COME	-0.240	0.143	4.767	0.000	1
CRES	0.013	0.523	0.111	0.000	1
FRAN	-0.156	0.243	-0.145	0.000	1
GGAL	0.153	0.752	0.194	0.000	1
INDU	1.148	1.000	9.836	0.000	1
IRSA	0.049	0.587	0.104	0.000	1
LEDE	0.644	0.998	0.328	0.000	1
MIRG	1.136	0.000	4.633	0.000	1
MOLI	0.894	0.000	3.304	0.751	0
TECO2	1.046	0.000	5.047	0.000	1
TEF	-0.391	0.041	0.622	0.000	1
TGSU2	-0.460	0.020	1.330	0.000	1
TRAN	0.208	0.823	1.312	0.000	1
TS	-0.068	0.381	-0.178	0.000	1
TXAR	0.217	0.833	0.267	0.000	1
YPFD	-0.856	0.000	3.519	0.876	1

Table 19 – CAPM residuals for the US market (full sample)

	Skewness	p-value	Kurtosis	p-value	Significance
AAPL	-0.535	0.001	2.421	0.051	1
AXP	2.557	0.000	25.400	0.000	1
BA	0.022	0.549	0.363	0.000	1

CAT	-0.256	0.074	0.637	0.000	1
CSCO	0.038	0.584	0.669	0.000	1
CVX	-0.196	0.133	0.526	0.000	1
DIS	0.292	0.951	0.391	0.000	1
GE	-0.842	0.000	4.291	0.000	1
HD	-0.401	0.012	2.285	0.022	1
IBM	-0.483	0.003	3.682	0.973	1
INTC	-0.027	0.440	-0.046	0.000	1
JNJ	0.015	0.534	-0.329	0.000	1
JPM	0.424	0.992	3.393	0.867	0
KO	-0.213	0.114	0.591	0.000	1
MCD	0.229	0.902	0.788	0.000	1
MRK	-0.988	0.000	2.795	0.281	1
MSFT	0.268	0.936	1.741	0.000	1
PFE	-0.127	0.236	-0.069	0.000	1
PG	-0.080	0.324	-0.225	0.000	1
T	-0.356	0.022	0.857	0.000	1
TRV	0.202	0.873	-0.076	0.000	1
UNH	-0.866	0.000	3.986	0.997	1
UTX	0.092	0.698	0.387	0.000	1
VZ	-0.074	0.338	-0.041	0.000	1
WMT	-0.114	0.259	0.962	0.000	1

6. Markov-Switching Model

From inspection of the previous results for univariate normality tests, it seems there are 2 different regimes for asset returns, that is, equities' returns in both markets belong to one of two distributions that follow a latent process S_t that in each point in time is adopts one of two states. Hence assets' distributions behave according to:

$$Y_t \sim \begin{cases} N(\mu, \sigma_0) \\ N(\mu, \sigma_1) \end{cases}$$

Therefore, equities' returns arise from a normal distribution the variance of which switches between two values according to the process S_t, that follows a first order ergodic Markov Chain. This means that the probability for regime 0 to occur at time t depends solely on the regime at time t − 1. We denote these transition probabilities by: $P_r(S_t = i) = P_r(S_t = i/S_{t-1} = j)$. Applying a Markov-Switching model, it is possible to estimate not only the variances of both regimes, but the probability of switching from one regime to the other. The model is applied to the market premium of each market.

Market	LV	p-value	HV	p-value	$\boldsymbol{P_r(0 \to 0)}$	$\boldsymbol{P_r(1 \to 0)}$
Argentina	$\sigma_0^2 = 0.014$	0.000	$\sigma_1^2 = 0.078$	0.001	0.982	0.062
US	$\sigma_0^2 = 0.001$	0.000	$\sigma_1^2 = 0.003$	0.001	0.986	0.034

Where: LV and HV stand for Low Variance and High Variance regimes, respectively.

7. Conclusion

An empirical assessment of stock returns' time series for the period 2002-2018 for both Argentina and the US casts doubts on the normality assumption of assets' distributions usually employed in benchmark asset pricing models as CAPM and Portfolio Theory. Very few assets (taken from a set considered representative of the market as a whole) pass the univariate normality tests for the full sample period, and multivariate normality is rejected for different tests across markets. However, when the sample is partitioned into two subsamples, the percentage of assets whose distributions fit a normal increases considerably, suggesting that the normality assumption shouldn't be discarded, but adapted to a different framework: there are 2 regimes for stock returns, each characterized by a particular normal distribution with a variance that differs between regimes. A Markov-Switching model with constant mean and a 2-variance regime (high and low, respectively) was run on the data, confirming that returns fit a 2-regime process.
Results must be taken with caution, since multivariate normality tests rejected the null hypothesis of normality for both subsamples in both markets.

8. References

Affleck-Graves, J. and B. McDonald: "Nonnormalities and Tests of Asset Pricing Theories". The Journal of Finance 44. 1989.

Cambell, J., A.W. Lo and A.C. MacKinlay: "The Econometrics of Financial Markets". Princeton University Press. 1997.

Fama, E.: "Efficient Capital Markets: A Review of Theory and Empirical Work". The Journal of Finance, 25. May 1970.

Fama, E.: "Efficient Capital Markets II". The Journal of Finance 46. December 1991

Hansen, Lars. "Large Sample Properties of Generalized Method of Moments Estimators". Econometrica, 50. 1982.

Kazuyuki K.i, Takuma S. and Tatjana P.: "Measures of multivariate skewness and kurtosis in high-dimensional framework", SUT Journal of Mathematics, 2014.

Linter, J.: "The Valuation of Risky Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets". Review of Economics and Statistics, 47. 1965.

Richardson, M. and T. Smith: "A Test for Multivariate Normality in Stock Returns" The Journal of Business, 66 (2). 1993.

Shanken, J.: "Multivariante Tests of the Zero Beta CAPM". Journal of Financial Economics, 14. 1985.

Sharpe, W.: "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk". The Journal of Finance, 19. 1964.

Swoboda, C.: “El Mercado de Valores de la Argentina y los Cambios a Partir de Diciembre de 2001”. Departamento de Economía - Facultad de Ciencias Económicas (FCE) de la Universidad Nacional de Córdoba (UNC), 2005.

Swoboda, C.: “Rendimiento y Riesgo en el Mercado de Valores de Buenos Aires: Nuevas Pruebas Empíricas para el Período 1996 - 2005”. Departamento de Economía – FCE – UNC, 2006.