# MATRIX GEGENBAUER POLYNOMIALS: THE $2 \times 2$ FUNDAMENTAL CASES

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ABSTRACT. In this paper, we exhibit explicitly a sequence of  $2 \times 2$  matrix valued orthogonal polynomials with respect to a weight  $W_{p,n}$ , for any pair of real numbers p and n such that 0 . The entries $of these polynomiales are expressed in terms of the Gegenbauer polynomials <math>C_k^{\lambda}$ . Also the corresponding three-term recursion relations are given and we make some studies of the algebra of differential operators associated with the weight  $W_{p,n}$ .

## 1. INTRODUCTION

The theory of matrix valued orthogonal polynomials, without any consideration of differential equations, goes back to [18] and [19]. In [3], the study of the matrix valued orthogonal polynomials that are eigenfunctions of certain second order symmetric differential operators was started. The first explicit examples of such polynomials were given in [8], [9], [7], [10] and [4]. See also [5], [6], [1], [2], and the references given there.

On the two dimensional sphere  $S^2 = SO(3)/SO(2)$ , the harmonic analysis with respect to the action of the orthogonal group is contained in the classical theory of the spherical harmonics. In spherical coordinates, the zonal spherical functions on  $S^2$  are the Legendre polynomials. More generally, in the case of the *n*-dimensional sphere  $S^n$  the zonal spherical functions are given in terms of Gegenbauer (or ultraspherical) polynomials of parameter (n-1)/2.

This fruitful connection between orthogonal polynomials and representation theory of compact Lie groups is also established in the matrix case: the matrix valued spherical functions of any K-type are closely related to matrix valued orthogonal polynomials. In this way, several examples of matrix orthogonal polynomials which are eigenfunctions of a symmetric differential operator have been obtained by focusing on a group representation approach. See for example [9], [11], [22], [23], [21] and more recently [16] and [24].

The examples of matrix orthogonal polynomials introduced in this paper are motivated by the spherical functions of fundamental K-types associated with the n-dimensional spheres  $S^n \simeq G/K$ , where (G, K) = (SO(n + 1), SO(n)). These matrix valued spherical functions were studied in detail in [27] and [29]. The "group parameters" of the fundamental K-types are  $p, n \in \mathbb{N}$  such that  $0 and they give rise to <math>2 \times 2$  matrix valued orthogonal polynomials.

In this paper we go beyond these group parameters and we extend these parameters continuously. We would like to remark that the group representation theory is a natural source of examples of matrix valued orthogonal polynomials. We keep this in mind in spite of the fact that the results obtained in this paper are self-contained, the proofs are of analytic nature and they do not depend on any previous results on spherical functions.

Given a weight matrix W, it is very natural to study the algebra  $\mathcal{D}(W)$ , of all differential operators that have a sequence of matrix valued orthogonal polynomials with respect to W as eigenfunctions, see (3). In the classical cases of Hermite, Laguerre and Jacobi weights, the structure of this algebra is well understood: it is a polynomial algebra in a second order differential operator, see [20]. In particular, it is a commutative

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algebra. In the matrix case, the first attempt to go beyond the issue of the existence of one nontrivial element in  $\mathcal{D}(W)$  and to study the full algebra is undertaken in [2], with the assistance of symbolic computation, for a few weights W. The first deep study of the algebra  $\mathcal{D}(W)$  can be founded in [26], where the author worked out one of the examples introduced in [2]. We refer the reader to [13] for basic definitions and main results concerning the algebra  $\mathcal{D}(W)$ . The present paper leads to understand completely a second and more promising example of  $\mathcal{D}(W)$  in a forthcoming paper, [28]. There are very few examples of non-commutative algebras that arise in a natural setup at the intersection of harmonic analysis and algebras. The study of such examples for the algebra  $\mathcal{D}(W)$  considered here is one step in that direction. ++As a consequence of this work, together with F.A. Grünbaum, in [12] we extend to a matrix setup a result that traces its origin and its importance to the work of Claude Shannon in lying the mathematical foundations of information theory, and to a remarkable series of papers by D. Slepian, H. Landau and H. Pollak.

To the best of our knowledge, this is the first example showing in a non-commutative setup that a bispectral property implies that the corresponding global operator of "time and band limiting" admits a commuting local operator. This is a noncommutative analog of the famous prolate spheroidal wave operator.

Now we discuss briefly the content of the paper. In Section 2 we recall the general notions of matrix valued orthogonal polynomials and some results from [13] about the algebra  $\mathcal{D}(W)$ .

In Section 3, we introduce our sequence  $\{P_w\}_{w\in\mathbb{N}_0}$  of  $2\times 2$  matrix valued polynomials on [-1,1] whose entries are given in terms of the classical Gegenbauer polynomials, for real parameters p and n such that  $0 , see (4). We prove that these polynomials satisfy <math>P_w D = \Lambda_w P_w$ , where D is a (right-hand side) hypergeometric differential operator and the eigenvalue is a diagonal matrix. This differential operator Dis symmetric with respect to the matrix weight W introduced in (12). We use these facts to prove that the polynomials  $\{P_w\}_{w\in\mathbb{N}_0}$  are orthogonal with respect to the weight matrix  $W = W_{p,n}$  (Theorem 3.6).

We also connect our weight matrix  $W_{p,n}$  with the weight considered in [15], where the authors give examples of matrix valued Gegenbauer polynomials, extending for an arbitrary parameter  $\nu$  the results in [16] for  $\nu = 1$ . See Remark 3.7.

In Section 4 we prove a three-term recursion relation satisfied by  $\{P_w\}_{w\in\mathbb{N}_0}$ . Section 5 is focused on the study of the algebra  $\mathcal{D}(W)$ . In our case  $\mathcal{D}(W)$  is a noncommutative algebra. We provide a basis  $\{D_1, D_2, D_3, D_4, I\}$  of the subspace of the differential operators in  $\mathcal{D}(W)$  of order at most two. The differential operators  $D_1$  and  $D_2$  are symmetric operators, while  $D_3$  and  $D_4$  are not. We conjecture that  $D_1, D_2, D_3, D_4$ generates the algebra  $\mathcal{D}(W)$ .

#### 2. Background on matrix valued orthogonal polynomials

Let W = W(x) be a weight matrix of size N on the real line, that is a complex  $N \times N$  matrix valued integrable function on the interval (a, b) such that W(x) is positive definite almost everywhere and with finite moments of all orders. Let  $\operatorname{Mat}_N(\mathbb{C})$  be the algebra of all  $N \times N$  complex matrices and let  $\operatorname{Mat}_N(\mathbb{C})[x]$ be the algebra over  $\mathbb{C}$  of all polynomials in the indeterminate x with coefficients in  $\operatorname{Mat}_N(\mathbb{C})$ . We consider the following Hermitian sesquilinear form in the linear space  $\operatorname{Mat}_N(\mathbb{C})[x]$ 

$$\langle P, Q \rangle = \langle P, Q \rangle_W = \int_a^b P(x) W(x) Q(x)^* dx$$

The following properties are satisfied, for all  $P, Q, R \in \operatorname{Mat}_N(\mathbb{C})[x], a, b \in \mathbb{C}, T \in \operatorname{Mat}_N(\mathbb{C})$ 

- (1)  $\langle aP + bQ, R \rangle = a \langle P, R \rangle + b \langle Q, R \rangle,$
- (2)  $\langle TP, R \rangle = T \langle P, R \rangle$ ,
- (3)  $\langle P, Q \rangle^* = \langle Q, P \rangle,$
- (4)  $\langle P, P \rangle \ge 0$ . Moreover, if  $\langle P, P \rangle = 0$ , then P = 0.

Let us denote  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . Given a weight matrix W one can construct sequences of matrix valued orthogonal polynomials, that is sequences  $\{P_n\}_{n \in \mathbb{N}_0}$ , where  $P_n$  is a polynomial of degree n with nonsingular

leading coefficient and  $\langle P_n, P_m \rangle = 0$  for  $n \neq m$ . We observe that there exists a unique sequence of monic orthogonal polynomials  $\{Q_n\}_{n \in \mathbb{N}_0}$  in  $\operatorname{Mat}_N(\mathbb{C})[x]$ . By following a standard argument, given for instance in [18] or [19], one shows that the monic orthogonal polynomials  $\{Q_n\}_{n \in \mathbb{N}_0}$  satisfy a three-term recursion relation

$$xQ_n(x) = A_nQ_{n-1}(x) + B_nQ_n(x) + Q_{n+1}(x), \qquad n \in \mathbb{N}_0,$$

where  $Q_{-1} = 0$  and  $A_n, B_n$  are matrices depending on n and not on x.

Two weights W and  $\widetilde{W}$  are said to be *similar* if there exists a nonsingular matrix M, which does not depend on x, such that

$$\widetilde{W}(x) = MW(x)M^*$$
, for all  $x \in (a, b)$ .

Notice that if  $\{P_n\}_{n\geq 0}$  is a sequence of orthogonal polynomials with respect to W, and  $M \in \operatorname{GL}_N(\mathbb{C})$ , then  $\{P_n M^{-1}\}_{n\geq 0}$  is orthogonal with respect to  $\widetilde{W} = MWM^*$ . A weight matrix W reduces to a smaller size if there exists a nonsingular matrix M such that

$$MW(x)M^* = \begin{pmatrix} W_1(x) & 0\\ 0 & W_2(x) \end{pmatrix}, \quad \text{for all } x \in (a,b),$$

where  $W_1$  and  $W_2$  are weights of smaller size.

For a given weight matrix and a sequence of orthogonal polynomials, it may be of interest the study of the differential operators having these polynomials as eigenfunctions. Let D be a right-hand side ordinary differential operator with matrix polynomial coefficients  $F_i(x)$  of degree less than or equal to i of the form

(1) 
$$D = \sum_{i=0}^{s} \partial^{i} F_{i}(x), \qquad \partial = \frac{d}{dx},$$

with the action of D on a polynomial function P(x) given by

$$(PD)(x) = \sum_{i=0}^{s} \partial^{i}(P)(x)F_{i}(x).$$

We say that the differential operator D is symmetric if  $\langle PD, Q \rangle = \langle P, QD \rangle$ , for all  $P, Q \in \operatorname{Mat}_N(\mathbb{C})[x]$ . It is a matter of careful integration by parts to see that the condition of symmetry for a differential operator of order two is equivalent to a set of three differential equations involving the weight W and the coefficients of the differential operator D.

**Proposition 2.1** ([10] or [4]). Let W(x) be a smooth weight matrix supported on (a, b). Let  $D = \partial^2 F_2(x) + \partial F_1(x) + F_0$ . Then D is symmetric with respect to W if and only if

$$\begin{cases} F_2 W = W F_2^* \\ 2(F_2 W)' - F_1 W = W F_1^* \\ (F_2 W)'' - (F_1 W)' + F_0 W = W F_0^* \end{cases}$$

with the boundary conditions

$$\lim_{x \to a, b} F_2(x) W(x) = 0, \quad \lim_{x \to a, b} \left( F_1(x) W(x) - W F_1^*(x) \right) = 0.$$

We consider the following subalgebra of the algebra of all right-hand side differential operators with coefficients in  $Mat_N(\mathbb{C})[x]$ ,

$$\mathcal{D} = \{ D = \sum_{i=0}^{s} \partial^{i} F_{i} : s \in \mathbb{N}_{0}, F_{i} \in \operatorname{Mat}_{N}(\mathbb{C})[x], \deg F_{i} \leq i \}.$$

**Proposition 2.2** ([13], Propositions 2.6 and 2.7). Let W = W(x) be a weight matrix of size  $N \times N$  and let  $\{Q_n\}_{n\geq 0}$  be the sequence of monic orthogonal polynomials in  $Mat_N(\mathbb{C})[x]$ . If D is a right-hand side ordinary differential operator of order s, as in (1), such that

$$Q_n D = \Lambda_n Q_n, \quad \text{for all } n \in \mathbb{N}_0,$$

with  $\Lambda_n \in \operatorname{Mat}_N(\mathbb{C})$ , then  $F_i = F_i(x) = \sum_{j=0}^i x^j F_j^i$ ,  $F_j^i \in \operatorname{Mat}_N(\mathbb{C})$ , is a polynomial and deg $(F_i) \leq i$ . Moreover D is determined by the sequence  $\{\Lambda_n\}_{n\geq 0}$  and

(2) 
$$\Lambda_n = \sum_{i=0}^s [n]_i F_i^i, \quad \text{for all } n \ge 0,$$

where  $[n]_i = n(n-1)\cdots(n-i+1), [n]_0 = 1.$ 

Given a matrix weight W, the algebra

(3) 
$$\mathcal{D}(W) = \{ D \in \mathcal{D} : P_n D = \Lambda_n(D) P_n, \Lambda_n(D) \in \operatorname{Mat}_N(\mathbb{C}), \text{ for all } n \in \mathbb{N}_0 \}$$

is introduced in [13], where  $\{P_n\}_{n \in \mathbb{N}_0}$  is any sequence of matrix valued orthogonal polynomials with respect to W.

We observe that the definition of  $\mathcal{D}(W)$  depends only on the weight matrix W and not on the particular sequence of orthogonal polynomials, since two sequences  $\{P_w\}_{w\in\mathbb{N}_0}$  and  $\{Q_w\}_{w\in\mathbb{N}_0}$  of matrix orthogonal polynomials with respect to the weight W are related by  $P_w = M_w Q_w$ , for  $w \in \mathbb{N}_0$ , with  $\{M_w\}_{w\in\mathbb{N}_0}$  invertible matrices (see [13, Corollary 2.5]).

**Proposition 2.3** ([13], Proposition 2.8). For each  $n \in \mathbb{N}_0$ , the mapping  $D \mapsto \Lambda_n(D)$  is a representation of  $\mathcal{D}(W)$  in  $\operatorname{Mat}_N(\mathbb{C})$ . Moreover, the sequence of representations  $\{\Lambda_n\}_{n\in\mathbb{N}_0}$  separates the elements of  $\mathcal{D}(W)$ .

We remark that the result in Proposition 2.3 says that the map

$$D \mapsto (\Lambda_0(D), \Lambda_1(D), \Lambda_2(D), \dots)$$

is an injective morphism of  $\mathcal{D}(W)$  into  $\operatorname{Mat}_N(\mathbb{C})^{\mathbb{N}_0}$ , the direct product of infinite copies, indexed by  $\mathbb{N}_0$ , of the algebra  $\operatorname{Mat}_N(\mathbb{C})$ . In particular, if  $D_1, D_2 \in \mathcal{D}(W)$  then

$$D_1 = D_2$$
 if and only if  $\Lambda_n(D_1) = \Lambda_n(D_2)$  for all  $n \in \mathbb{N}_0$ .

For any  $D \in \mathcal{D}(W)$  there exists a unique differential operator  $D^* \in \mathcal{D}(W)$ , the adjoint of D in  $\mathcal{D}(W)$ , such that

$$\langle PD, Q \rangle = \langle P, QD^* \rangle,$$

for all  $P, Q \in \operatorname{Mat}_N(\mathbb{C})[x]$ . See Theorem 4.3 and Corollary 4.5 in [13]. The map  $D \mapsto D^*$  is a \*-operation in the algebra  $\mathcal{D}(W)$ . Moreover, it is shown that  $\mathcal{S}(W)$ , the set of all symmetric operators in  $\mathcal{D}(W)$ , is a real form of the space  $\mathcal{D}(W)$ , i.e.

$$\mathcal{D}(W) = \mathcal{S}(W) \oplus i\mathcal{S}(W),$$

as real vector spaces. In particular, the algebra  $\mathcal{D}(W)$ , together with the involution, is completely determined by  $\mathcal{S}(W)$ .

**Corollary 2.4.** A differential operator  $D \in \mathcal{D}(W)$  is a symmetric operator if and only if

$$\Lambda_n(D)\langle Q_n, Q_n\rangle = \langle Q_n, Q_n\rangle\Lambda_n(D)$$

for all  $n \in \mathbb{N}_0$ .

Also it is worth to recall the following important result from [13].

**Proposition 2.5** (Proposition 2.10). If  $D \in \mathcal{D}$  is symmetric then  $D \in \mathcal{D}(W)$ .

#### 3. MATRIX VALUED ORTHOGONAL POLYNOMIALS ASSOCIATED WITH THE *n*-DIMENSIONAL SPHERES

Motivated by the results obtained in [27] we introduce the following family of polynomials, for  $w \in \mathbb{N}_0$ ,

(4) 
$$P_w(x) = P_w^{n,p}(x) = \begin{pmatrix} \frac{1}{n+1} C_w^{\frac{n+1}{2}}(x) + \frac{1}{p+w} C_{w-2}^{\frac{n+3}{2}}(x) & \frac{1}{p+w} C_{w-1}^{\frac{n+3}{2}}(x) \\ \frac{1}{n-p+w} C_{w-1}^{\frac{n+3}{2}}(x) & \frac{1}{n+1} C_w^{\frac{n+1}{2}}(x) + \frac{1}{n-p+w} C_{w-2}^{\frac{n+3}{2}}(x) \end{pmatrix},$$

with parameters  $p, n \in \mathbb{R}$  such that  $0 . Here <math>C_n^{\lambda}(x)$  denotes the *n*-th Gegenbauer polynomial

$$C_w^{\lambda}(x) = \frac{(2\lambda)_w}{w!} \, _2F_1\left(\frac{-w, \, w+2\lambda}{\lambda+1/2}; \frac{1-x}{2}\right), \qquad x \in [-1,1],$$

where  $(a)_w = a(a+1)\dots(a+w-1)$  denotes the Pochhammer symbol. As usual, we assume  $C_w^{\lambda}(x) = 0$  if w < 0. We recall that  $C_w^{\lambda}$  is a polynomial of degree w, with leading coefficient  $\frac{2^w(\lambda)_w}{w!}$ .

Let us observe that  $\deg(P_w) = w$  and the leading coefficient of  $P_w$  is a nonsingular scalar matrix

(5) 
$$\frac{2^{w}(\frac{n+1}{2})_{w}}{(n+1)w!} \operatorname{Id} = \frac{1}{w!} 2^{w-1}(\frac{n+3}{2})_{w-1} \operatorname{Id}.$$

We start by proving that the polynomials  $P_w$  given in (4) are eigenfunctions of the following differential operator D.

**Theorem 3.1.** For each  $w \in \mathbb{N}_0$ , the matrix polynomial  $P_w$  is an eigenfunction of the differential operator

$$D = \partial^2 \left( 1 - x^2 \right) - \partial \left( (n+2)x + 2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) - \begin{pmatrix} p & 0 \\ 0 & n-p \end{pmatrix},$$

with eigenvalue

$$\Lambda_w(D) = \begin{pmatrix} -w(w+n+1) - p & 0\\ 0 & -w(w+n+1) - n + p \end{pmatrix}.$$

*Proof.* We need to verify that

$$P_w D = \Lambda_w P_w.$$

We will need to use the following properties of the Gegenbauer polynomials (for the first three see [14] page 40, and for the last one see [25], page 83, equation (4.7.27))

(6) 
$$(1-x^2)\frac{d^2}{dx^2}C_m^{\lambda}(x) - (2\lambda+1)x\frac{d}{dx}C_m^{\lambda}(x) + m(m+2\lambda)C_m^{\lambda}(x) = 0,$$

(7) 
$$\frac{d}{dx}C_m^{\lambda}(x) = 2\lambda C_{m-1}^{\lambda+1}(x),$$

(8) 
$$2(m+\lambda)x C_m^{\lambda}(x) = (m+1)C_{m+1}^{\lambda}(x) + (m+2\lambda-1)C_{m-1}^{\lambda}(x),$$

(9) 
$$\frac{(m+2\lambda-1)}{2(\lambda-1)}C_{m+1}^{\lambda-1}(x) = C_{m+1}^{\lambda}(x) - x C_m^{\lambda}(x).$$

Also, combining (8) and (9), we have

(10) 
$$(m+\lambda)C_{m+1}^{\lambda-1}(x) = (\lambda-1)\Big(C_{m+1}^{\lambda}(x) - C_{m-1}^{\lambda}(x)\Big).$$

The entry (1, 1) of the matrix  $P_w D - \Lambda_w P_w$  is

$$(1 - x^{2})(P_{w})_{11}'' - (n+2)x(P_{w})_{11}' - 2(P_{w})_{12}' + w(w+n+1)(P_{w})_{11}$$
  
=  $(1 - x^{2})\left(\frac{1}{n+1}C_{w}^{\frac{n+1}{2}} + \frac{1}{p+w}C_{w-2}^{\frac{n+3}{2}}\right)'' - (n+2)x\left(\frac{1}{n+1}C_{w}^{\frac{n+1}{2}} + \frac{1}{p+w}C_{w-2}^{\frac{n+3}{2}}\right)'$   
 $-\frac{2}{p+w}\left(C_{w-1}^{\frac{n+3}{2}}\right)' + w(w+n+1)\left(\frac{1}{n+1}C_{w}^{\frac{n+1}{2}} + \frac{1}{p+w}C_{w-2}^{\frac{n+3}{2}}\right).$ 

Applying (6) for  $\lambda = \frac{1}{2}(n+1)$ ,  $\lambda = \frac{1}{2}(n+3)$  and (7) for  $\lambda = \frac{1}{2}(n+3)$ , with m = w, we have that the entry (1,1) of  $P_w D - \Lambda_w P_w$ , multiplied by (p+w)/2 is

$$-(n+3)C_{w-2}^{\frac{n+5}{2}} + (n+3) x C_{w-3}^{\frac{n+5}{2}} + (w+n+1)C_{w-2}^{\frac{n+3}{2}} = 0,$$

this last identity follows from equation (9) with  $\lambda = \frac{n+5}{2}$  and m = w - 3. Repeating the previous verification, by changing p by n - p, it follows that the entry (2, 2) of  $P_w D - \Lambda_w P_w$  is also zero.

The entry (1,2) of  $P_w D - \Lambda_w P_w$  is

$$(1-x^2)(P_w)_{12}'' - (n+2)x(P_w)_{12}' - 2(P_w)_{11}' + (w(w+n+1) - n + 2p)(P_w)_{12}$$

if we multiply it by (p+w) we get

(11)

$$(1-x^2)\left(C_{w-1}^{\frac{n+3}{2}}\right)'' - (n+2)x\left(C_{w-1}^{\frac{n+3}{2}}\right)' + (w(w+n+1) - n + 2p)C_{w-1}^{\frac{n+3}{2}} - 2\frac{(p+w)}{n+1}\left(C_w^{\frac{n+1}{2}}\right)' - 2\left(C_{w-2}^{\frac{n+3}{2}}\right)'.$$

Applying (6) for  $\lambda = (n+3)/2$ , m = w - 1, (7) for  $\lambda = (n+1)/2$ , m = w and  $\lambda = (n+3)/2$ , m = w - 1, one obtain that (11) is

$$2x\left(C_{w-1}^{\frac{n+3}{2}}\right)' - 2(w-1)C_{w-1}^{\frac{n+3}{2}} - 2(n+3)C_{w-3}^{\frac{n+5}{2}}.$$

Now, applying (7) and (9), this expression becomes

$$2(n+3)\left(C_{w-1}^{\frac{n+5}{2}}-C_{w-3}^{\frac{n+5}{2}}\right)-2(2w+n+1)C_{w-1}^{\frac{n+3}{2}},$$

which is equal to zero by (10) with  $\lambda = \frac{n+5}{2}$  and m = w - 2. This concludes that the entry (1,2) of  $P_w D - \Lambda_w P_w$  is zero. To complete the proof of the theorem we need to verify that the entry (2,1) is also zero. This is obtained making exactly the same computations, by changing p by n - p.

We introduce the weight matrix

(12) 
$$W(x) = W_{p,n} = (1 - x^2)^{\frac{n}{2} - 1} \begin{pmatrix} p x^2 + n - p & -nx \\ -nx & (n - p)x^2 + p \end{pmatrix}, \quad x \in [-1, 1].$$

**Proposition 3.2.** For  $n \neq 2p$ , the weight W(x) does not reduce to a smaller size.

*Proof.* Assume that there exists a nonsingular matrix  $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$  such that

$$MW(x)M^* = \begin{pmatrix} w_1(x) & 0\\ 0 & w_2(x) \end{pmatrix}.$$

The entry (1,2) of  $MW(x)M^*$  is

 $x^{2} (p m_{11}\overline{m}_{21} + (n-p)m_{12}\overline{m}_{22}) - (m_{11}\overline{m}_{22} + m_{12}\overline{m}_{21})nx + (n-p)m_{11}\overline{m}_{21} + p m_{12}\overline{m}_{22},$ 

from here we see that

$$m_{11}\overline{m}_{22} + m_{12}\overline{m}_{21} = 0,$$

(13) 
$$p m_{11}\overline{m}_{21} + (n-p)m_{12}\overline{m}_{22} = 0,$$

(14) 
$$(n-p)m_{11}\overline{m}_{21} + p\,m_{12}\overline{m}_{22} = 0.$$

By combining equations (13) and (14) we have that  $(n-2p)m_{11}\overline{m}_{21} = 0$ . The assumption  $n \neq 2p$ , together with (9), implies  $\det(M) = 0$ , which is a contradiction.

Remark 3.3. For n = 2p, the weight matrix W reduces to two scalar weights. The corresponding scalar polynomials are Jacobi polynomials  $P_w^{\alpha,\beta}$  with  $(\alpha,\beta) = (n/2+1, n/2-1)$  and  $(\alpha,\beta) = (n/2-1, n/2+1)$ , respectively. In fact, by taking  $M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$  we have that

$$MW(x)M^* = 2p \left(1 - x^2\right)^{\frac{n}{2} - 1} \begin{pmatrix} (1 - x)^2 & 0\\ 0 & (1 + x)^2 \end{pmatrix}.$$

*Remark* 3.4. We have that the weight matrices  $W_{p,n}$  and  $W_{n-p,n}$  are similar. In fact, by taking  $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  we get

$$MW_{p,n}M^* = W_{n-p,n}.$$

From Proposition 2.1 and following straightforward computations, one can prove the following result.

**Proposition 3.5.** The differential operator

$$D = \partial^2 \left( 1 - x^2 \right) - \partial \left( (n+2)x + 2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) - \begin{pmatrix} p & 0 \\ 0 & n-p \end{pmatrix}$$

is symmetric with respect to the weight function W(x).

In the scalar case, if D is a symmetric differential operator with respect to W and  $\{P_w\}_{w\in\mathbb{N}_0}$  is a family of eigenfunctions of D with different eigenvalues, then the sequence  $\{P_w\}_{w\in\mathbb{N}_0}$  is automatically orthogonal with respect to W. In the matrix case this is not always true since

(15) 
$$\Lambda_w \langle P_w, P_{w'} \rangle = \langle P_w D, P_{w'} \rangle = \langle P_w, P_{w'} D \rangle = \langle P_w, P_{w'} \rangle \Lambda_w$$

does not imply that  $\langle P_w, P_{w'} \rangle = 0$ , for  $w \neq w'$ . Therefore, we prove the orthogonality in the next theorem.

**Theorem 3.6.** When  $n \neq 2p$  the matrix polynomials  $\{P_w\}_{w \in \mathbb{N}_0}$  are orthogonal polynomials with respect to the matrix valued inner product

$$\langle P, Q \rangle = \int_{-1}^{1} P(x) W(x) Q(x)^* \, dx.$$

*Proof.* We know that  $P_w$  is a polynomial of degree w and its leading coefficient is a nonsingular diagonal matrix (see (5)). We only have to verify that for  $w \neq w'$ ,  $\langle P_w, P_{w'} \rangle_W = 0$ . Since  $P_w$  is an eigenfunction of the differential operator D, which is symmetric with respect to W, we have that (15) holds with

$$\Lambda_w = \begin{pmatrix} \lambda_{w,1} & 0\\ 0 & \lambda_{w,2} \end{pmatrix} = \begin{pmatrix} -w(w+n+1)-p & 0\\ 0 & -w(w+n+1)-n+p \end{pmatrix}$$

see Theorem 3.1. Therefore, for i, j = 1, 2 we have  $\lambda_{w,i} \langle P_{w,i}, P_{w',j} \rangle = \lambda_{w',j} \langle P_{w,i}, P_{w',j} \rangle$ , where  $P_{w,i}$  is the *i*-th row of the polynomial  $P_w$ , and

$$\langle P_{w,i}, P_{w',j} \rangle = \int_{-1}^{1} P_{w,i}(x) W(x) P_{w',j}^{*}(x) \, dx \in \mathbb{C}.$$

It is not difficult to verify that  $\lambda_{w,i} \neq \lambda_{w',j}$ , for  $w \neq w'$  or  $i \neq j$ . Then we have

(16) 
$$\langle P_{w,i}, P_{w',j} \rangle = 0, \quad \text{for } w \neq w' \text{ or } i \neq j$$

Therefore  $\langle P_w, P_{w'} \rangle = 0$ , for  $w \neq w'$ , which concludes the proof of the theorem.

Remark 3.7. Recently, in [15] the authors study some families on matrix valued polynomials, depending on one real parameter  $\nu > 0$ , of arbitrary size  $(2\ell + 1) \times (2\ell + 1)$  with  $\ell \in \frac{1}{2}\mathbb{N}$ . These weights are not irreducible. For  $\ell = 1, \frac{3}{2}, 2$  appears some irreducible  $2 \times 2$  blocks  $W_{+}^{(\nu)}$  and  $W_{-}^{(\nu)}$ . See Remark 2.8 (ii) there.

For  $\ell = 1, \frac{3}{2}, 2$  appears some irreducible  $2 \times 2$  blocks  $W_{+}^{(\nu)}$  and  $W_{-}^{(\nu)}$ . See Remark 2.8 (ii) there. The case  $\ell = 3/2$  does not match with the examples considered in this paper. The cases  $\ell = 1$  and  $\ell = 2$  are particular cases of our weight matrices  $W_{p,n}$  by choosing our parameters  $(p,n) = (\nu, 2\nu + 1)$  and  $(p,n) = (\nu, 2\nu + 3)$ , for  $\ell = 1$  and  $\ell = 2$  respectively. In fact, with  $L = \begin{pmatrix} 0 & \sqrt{2} \\ -1 & 0 \end{pmatrix}$  and  $D = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$  we get

$$W_{+}^{(\nu)} = \frac{(\nu+2)}{(2\nu+1)} L W_{\nu,2\nu+1} L^* \qquad \text{for } \ell = 1,$$
  
$$W_{-}^{(\nu)} = \frac{(\nu+4)(\nu+2)}{(2\nu+1)(2\nu+3)} D W_{\nu,2\nu+3} D^* \qquad \text{for } \ell = 2.$$

The case  $\nu = 1$  was previously studied in [16] and [17].

#### 4. THREE-TERM RECURSION RELATION

The main result of this section is a three-term recursion relation satisfied by the sequence of orthogonal polynomials studied in this paper. We give a proof by using some properties of the Gegenbauer polynomials.

**Theorem 4.1.** The orthogonal polynomials  $\{P_w\}_{w\in\mathbb{N}_0}$  satisfy the three-term recursion relation

$$x P_w(x) = A_w P_{w-1}(x) + B_w P_w(x) + C_w P_{w+1}(x)$$

where

$$A_w = \begin{pmatrix} \frac{(n+w)(p+w-1)(n-p+w+1)}{(p+w)(n-p+w)(2w+n+1)} & 0\\ 0 & \frac{(n+w)(p+w+1)(n-p+w-1)}{(p+w)(n-p+w)(2w+n+1)} \end{pmatrix},$$
$$B_w = \begin{pmatrix} 0 & \frac{-p}{(p+w)(p+w+1)}\\ \frac{-(n-p)}{(n-p+w)(n-p+w+1)} & 0 \end{pmatrix}, \quad C_w = \frac{w+1}{2w+n+1}I$$

*Proof.* To verify the (1, 1)-entry of the equation in the statement of the theorem we need to prove that

(17) 
$$x \left( \frac{1}{n+1} C_w^{\frac{n-1}{2}-1}(x) + \frac{1}{p+w} C_{w-2}^{\frac{n+3}{2}}(x) \right) = \frac{(n+w)(p+w-1)(n-p+w+1)}{(2w+n+1)(p+w)(n-p+w)} \left( \frac{1}{n+1} C_{w-1}^{\frac{n-1}{2}-1}(x) + \frac{1}{p+w} C_{w-3}^{\frac{n+3}{2}}(x) \right) \\ - \frac{p}{(p+w)(p+w+1)(n-p+w)} C_{w-1}^{\frac{n+3}{2}}(x) + \frac{w+1}{2w+n+1} \left( \frac{1}{n+1} C_{w+1}^{\frac{n-1}{2}-1}(x) + \frac{1}{p+w-1} C_{w-1}^{\frac{n+3}{2}}(x) \right).$$

By replacing the identities given by (8) for  $\lambda = \frac{n+1}{2}$ , m = w and  $\lambda = \frac{n+3}{2}$ , m = w - 2, one obtain that (17) is equivalent to

(18) 
$$\frac{\frac{(w+n)}{(n+1)(2w+n+1)} \left(-1 + \frac{(p+w-1)(n-p+w-1)}{(p+w)(n-p+w)}\right) C_{w-1}^{\frac{n+1}{2}}(x)}{+ \left(-\frac{p}{(p+w)(p+w+1)(n-p+w)} + \frac{w+1}{(2w+n+1)(p+w+1)} - \frac{w-1}{(p+w)(2w+n-1)}\right) C_{w-1}^{\frac{n+3}{2}}(x)} + \frac{(n+w)}{p+w} \left(\left(\frac{n-p+w-1}{(2w+n+1)(n-p+w)} - \frac{1}{2w+n-1}\right) C_{w-3}^{\frac{n+3}{2}}(x) = 0.$$

Thus, by using the relation (10) for  $\lambda = \frac{n+3}{2}$  and m = w - 2, the identity in (18) follows after some straightforward computations.

Now we verify that the equation for the (1, 2)-entry in the statement of the theorem holds. We need to verify that the following identity holds

(19) 
$$\frac{\frac{1}{p+w}xC_{w-1}^{\frac{n+3}{2}}(x) = \frac{(n+w)(n-p+w+1)}{(p+w)(2w+n+1)(n-p+w)}C_{w-2}^{\frac{n+3}{2}}(x)}{-\frac{p}{(p+w)(p+w+1)}\left(\frac{1}{n+1}C_{w}^{\frac{n+1}{2}}(x) + \frac{1}{n-p+w}C_{w-2}^{\frac{n+3}{2}}(x)\right) + \frac{w+1}{(2w+n+1)(p+w+1)}C_{w}^{\frac{n+3}{2}}(x)}$$

From (10) for  $\lambda = \frac{n+3}{2}$  and m = w - 1 we have that the right-hand side of (19) is

$$\frac{n+w+1}{(p+w)(2w+n+1)}C_{w-2}^{\frac{n+3}{2}}(x) + \frac{w}{(p+w)(2w+n+1)}C_{w}^{\frac{n+3}{2}}(x).$$

Therefore, (19) is proved, since it is equivalent to (8) with  $\lambda = \frac{n+3}{2}$  and m = w - 1.

For the entries (2,2) and (2,1) we proceed in a similar way, by observing that we need to do the same computations as in the cases (1,1) and (1,2) respectively, by changing p by n-p. This concludes the proof of the theorem.

The sequence of monic orthogonal polynomials is given by

(20) 
$$Q_w = \frac{w!(n+1)}{2^w \left(\frac{n+1}{2}\right)_w} P_w, \qquad w \in \mathbb{N}_0$$

The first polynomials of the sequence  $\{Q_w\}_{w\in\mathbb{N}_0}$  are

$$Q_{0} = \mathrm{Id}, \qquad Q_{1} = \begin{pmatrix} x & \frac{1}{p+1} \\ \frac{1}{n-p+1} & x \end{pmatrix}, \qquad Q_{2} = \begin{pmatrix} x^{2} - \frac{p}{(n+3)(p+2)} & \frac{2}{p+2}x \\ \frac{2}{n-p+2}x & x^{2} - \frac{n-p}{(n+3)(n-p+2)} \end{pmatrix}$$
$$Q_{3} = \begin{pmatrix} x^{3} - \frac{3(p+1)}{(n+5)(p+3)}x & \frac{3}{p+3}x^{2} - \frac{3}{(n+5)(p+3)} \\ \frac{3}{n-p+3}x^{2} - \frac{3}{(n+5)(n-p+3)} & x^{3} - \frac{3(n-p+1)}{(n+5)(n-p+3)}x \end{pmatrix}.$$

*Remark* 4.2. Observe that from (16) and (20) we have that  $\langle Q_w, Q_w \rangle$  is always a diagonal matrix. Moreover one can verify that

$$\langle Q_w, Q_w \rangle = \|Q_w\|^2 = \frac{\pi 2^{[w/2]} \Gamma(n/2 + 1 + [w/2])}{w!(n+2w+1) \Gamma((n+3)/2)} \prod_{k=1}^{[(w-1)/2]} (n+2k+1) \begin{pmatrix} \frac{p(n-p+w+1)}{p+w} & 0\\ 0 & \frac{(n-p)(p+w+1)}{n-p+w} \end{pmatrix}.$$

# 5. The Algebra $\mathcal{D}(W)$

In this section we discuss some properties of the structure of the algebra  $\mathcal{D}(W)$ , defined in (3), for our weight matrix W(x) introduced in (12). We are not interested in the cases when p = n - p, since the weight reduces to classical scalar weights, see Remark 3.3. We observe that in our example, the polynomials  $\{P_w\}_{w\in\mathbb{N}_0}$ , given in (4), and the monic orthogonal polynomials  $\{Q_w\}_{w\in\mathbb{N}_0}$  have the same sequence of eigenvalues, since they are related by a scalar multiple, see (20).

First of all we observe that the space of differential operators of order zero in  $\mathcal{D}(W)$  consists of scalar multiplies of the identity operator. In fact, a differential operator of order zero, having the sequence of monic orthogonal polynomials  $\{Q_w\}_w$  as eigenfunctions, is a constant matrix L such that

$$Q_w L = \Lambda_w Q_w,$$
 for all  $w \in \mathbb{N}_0.$ 

From (2) we have that  $\Lambda_w = L$  for every w. When w = 1, we obtain that the entries of L satisfy  $L_{11} = L_{22}$ and  $(p+1)L_{12} = (n-p+1)L_{21}$ . Thus, looking at the case w = 2 we get  $(n-2p)L_{12} = 0$ . Therefore we obtain that any operator of order zero L in  $\mathcal{D}(W)$  is a multiple of the identity matrix. Now we study differential operators of order at most two in the algebra  $\mathcal{D}(W)$ . Let  $\{Q_w\}_{w\in\mathbb{N}_0}$  the sequence of monic orthogonal polynomials with respect to W and D a differential operator of order at most two in  $\mathcal{D}(W)$ . From Proposition 2.2 we have

$$D = \partial^2 (A_2 x^2 + A_1 x + A_0) + \partial (B_1 x + B_0) + C \in \mathcal{D}(W)$$

if and only if

$$Q_w D = (w(w-1)A_2 + wB_1 + C)Q_w, \quad \text{for all } w \in \mathbb{N}_0.$$

Here  $A_2, A_1, A_0, B_1, B_0, C$  are  $2 \times 2$  complex matrices. Let us denote  $Q_{w,j}$  the coefficients of the polynomial  $Q_w$ , i.e.,  $Q_w = \sum_{j=0}^w Q_{w,j} x^j$ , with  $Q_{w,w} = I$ . Therefore  $D \in \mathcal{D}(W)$  if and only if

$$j(j-1)Q_{w,j}A_2 + j(j+1)Q_{w,j+1}A_1 + (j+1)(j+2)Q_{w,j+2}A_0 + jQ_{w,j}B_1 + (j+1)Q_{w,j+1}B_0 + Q_{w,j}C - (w(w-1)A_2 + wB_1 + C)Q_{w,j} = 0$$

for all  $w \in \mathbb{N}_0$  and  $j = 0, \dots, w$ . For j = w - 1 and j = 0 we respectively obtain

(21) 
$$(w-1)(w-2)Q_{w,w-1}A_2 + w(w-1)A_1 + (w-1)Q_{w,w-1}B_1 + wB_0 + Q_{w,w-1}C_1 - (w(w-1)A_2 + wB_1 + C)Q_{w,w-1} = 0$$

and

(22) 
$$2Q_{w,2}A_0 + Q_{w,1}B_0 + Q_{w,0}C - (w(w-1)A_2 + wB_1 + C)Q_{w,0} = 0.$$

Now from (21) considering w = 1 and w = 2, and from (22) considering w = 2, we respectively obtain

$$B_0 = (B_1 + C)Q_{1,0} - Q_{1,0}C, \qquad 2A_1 = (2A_2 + 2B_1 + C)Q_{2,1} - Q_{2,1}B_1 - 2B_0 - Q_{2,1}C,$$
  
$$2A_0 = (2A_2 + 2B_1 + C)Q_{2,0} - Q_{2,1}B_0 - Q_{2,0}C.$$

From the expression of  $Q_1$  and  $Q_2$ , given at the end of Section 4, we know that

$$Q_{1,0} = \begin{pmatrix} 0 & \frac{1}{p+1} \\ \frac{1}{n-p+1} \end{pmatrix}, \qquad Q_{2,1} = \begin{pmatrix} 0 & \frac{2}{p+2} \\ \frac{2}{n-p+2} & 0 \end{pmatrix}, \qquad Q_{2,0} = \frac{-p}{(n+3)} \begin{pmatrix} \frac{1}{(p+2)} & 0 \\ 0 & \frac{1}{(n-p+2)} \end{pmatrix}$$

By using (20) and (4) it is easy to see that

$$Q_{w,w-1} = \begin{pmatrix} 0 & \frac{w}{p+w} \\ \frac{w}{n-p+w} & \end{pmatrix}, \quad \text{for all } w \in \mathbb{N}.$$

To determine the matrices  $A_2 = (a_{ij})$ ,  $B_1 = (b_{ij})$  and  $C = (c_{ij})$ , we first combine the entries in the diagonal of the matrix (21) to obtain

$$2(n+2)a_{21} = \frac{\left((n+p+2)b_{21} - 2c_{21}\right)}{p+1} + \frac{(p+2)(p+w)(2c_{12} - (n-p)b_{12})}{(n-p+1)(n-p+2)(n-p+w)},$$
  
$$2(n+2)a_{12} = \frac{\left((2n-p+2)b_{12} - 2c_{12}\right)}{n-p+1} + \frac{(n-p+2)(n-p+w)(2c_{21} - pb_{21})}{(p+1)(p+2)(p+w)}.$$

Since these identities are valid for any integer  $w \ge 3$  we conclude that, if  $n \ne 2p$  then  $2c_{12} = (n-p)b_{12}$ and  $2c_{21} = p b_{21}$ . Therefore  $b_{21} = 2(p+1)a_{21}$  and  $b_{12} = 2(n-p+1)a_{12}$ .

By combining the nondiagonal entries of (21) we have

$$(n-2p+1)\big((n+2)a_{11}-b_{11}\big) = (n-2p-1)\big((n+2)a_{22}-b_{22}\big)$$

and

$$c_{11} - c_{22} = (p+1)(p+2)a_{22} - p(p+1)a_{11} + p b_{11} - (p+1)b_{22}.$$

Equation (22) with w = 3 says that

$$2Q_{3,2}A_0 + Q_{3,1}B_0 + Q_{3,0}C - (6A_2 + 3B_1 + C)Q_{3,0} = 0.$$

Now, by using the expression of  $Q_3 = x^3 + Q_{3,2}x^2 + Q_{3,1}x + Q_{3,0}$  given at the end of Section 4, it is not difficult to see that  $b_{11} = (n+2)a_{11}$ . Thus  $b_{22} = (n+2)a_{22}$ , and  $c_{11} - c_{22} = p(n-p+1)a_{11} - (p+1)(n-p)a_{22}$ .

Therefore, the matrices  $A_2, A_1, A_0, B_1, B_0, C$  are given in terms of the entries of  $A_2$  and  $c_{11}$ , as we state in the following theorem.

**Theorem 5.1.** The differential operators of order at most two in  $\mathcal{D}(W)$  are of the form

$$D = \partial^2 F_2(x) + \partial F_1(x) + F_0$$

where

$$F_{2}(x) = x^{2} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + x \begin{pmatrix} a_{12} - a_{21} & a_{11} - a_{22} \\ a_{22} - a_{11} & a_{21} - a_{12} \end{pmatrix} + \begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix},$$

$$(23) \quad F_{1}(x) = x \begin{pmatrix} (n+2)a_{11} & 2(n-p+1)a_{12} \\ 2(p+1)a_{21} & (n+2)a_{22} \end{pmatrix} + \begin{pmatrix} -pa_{21} + (n-p+2)a_{12} & (n-p+2)a_{11} - (n-p)a_{22} \\ -pa_{11} + (p+2)a_{22} & (p+2)a_{21} - (n-p)a_{12} \end{pmatrix},$$

$$F_{0} = \begin{pmatrix} p (n-p+1)a_{11} + c & (n-p)(n-p+1)a_{12} \\ p (p+1)a_{21} & (p+1)(n-p)a_{22} + c \end{pmatrix}.$$

with  $a_{11}$ ,  $a_{12}$ ,  $a_{21}$ ,  $a_{22}$ , c arbitrary complex numbers.

*Proof.* We have already proved that any differential operator of order at most two in  $\mathcal{D}(W)$  is of this form for some constant  $a_{11}, a_{12}, a_{21}, a_{22}, c \in \mathbb{C}$ . Let  $\mathcal{D}_2$  be the complex vector space of the differential operators in  $\mathcal{D}(W)$  of order at most two. Then we have that dim  $\mathcal{D}_2 \leq 5$ .

From Proposition 2.1 it is not difficult to see that a differential operator D of order two, with coefficients given by (23), is a symmetric operator if and only if

$$a_{11}, a_{22}, c \in \mathbb{R}$$
 and  $p a_{21} = (n-p) \overline{a}_{12}$ .

From Proposition 2.5 any symmetric operator  $D \in \mathcal{D}$  belongs to the algebra  $\mathcal{D}(W)$ . Thus there exists (at least) five  $\mathbb{R}$ -linearly independent symmetric operators in  $\mathcal{D}_2$ . Therefore dim  $\mathcal{D}_2 = 5$  and this concludes the proof of the theorem.

**Corollary 5.2.** There are no operators of order one in the algebra  $\mathcal{D}(W)$ .

The elements of the sequence  $\{Q_w\}_w$  are eigenfunctions of the operators  $D \in \mathcal{D}(W)$  and they satisfy  $Q_w D = \Lambda_w(D)Q_w$ , for  $w \in \mathbb{N}_0$ . We explicitly state the eigenvalues  $\Lambda_w$  using formula (2): for a differential operator  $D = \partial^2 F_2 + \partial F_1 + F_0$  we have

$$\Lambda_w(D) = w(w-1)F_2^2 + wF_1^1 + F_0^0$$

with  $F_i^i$  (i=1,2,3) the leading coefficient of the polynomial coefficient  $F_i$  of the differential operator D. Therefore we get

**Corollary 5.3.** Let  $D \in \mathcal{D}(W)$ , defined as in Theorem 5.1. The monic orthogonal polynomials  $\{Q_w\}_w$  satisfy

$$Q_w D = \Lambda_w(D)Q_w, \quad \text{for } w \in \mathbb{N}_0,$$

where the eigenvalue  $\Lambda_w(D)$  is given by

$$\Lambda_w(D) = \begin{pmatrix} (w+p)(w+n-p+1)a_{11}+c & (w+n-p)(w+n-p+1)a_{12} \\ (w+p)(w+p+1)a_{21} & (w+n-p)(w+p+1)a_{22}+c \end{pmatrix}.$$

Now we introduce a useful basis for the differential operators of order at most two in the algebra  $\mathcal{D}(W)$ : the identity I and

$$D_{1} = \partial^{2} \begin{pmatrix} x^{2} & x \\ -x & -1 \end{pmatrix} + \partial \begin{pmatrix} (n+2)x & n-p+2 \\ -p & 0 \end{pmatrix} + \begin{pmatrix} p(n-p+1) & 0 \\ 0 & 0 \end{pmatrix},$$

$$D_{2} = \partial^{2} \begin{pmatrix} -1 & -x \\ x & x^{2} \end{pmatrix} + \partial \begin{pmatrix} 0 & p-n \\ p+2 & (n+2)x \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & (p+1)(n-p) \end{pmatrix},$$
  

$$D_{3} = \partial^{2} \begin{pmatrix} -x & -1 \\ x^{2} & x \end{pmatrix} + \partial \begin{pmatrix} -p & 0 \\ 2(p+1)x & p+2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ p(p+1) & 0 \end{pmatrix},$$
  

$$D_{4} = \partial^{2} \begin{pmatrix} x & x^{2} \\ -1 & -x \end{pmatrix} + \partial \begin{pmatrix} n-p+2 & 2(n-p+1)x \\ 0 & p-n \end{pmatrix} + \begin{pmatrix} 0 & (n-p)(n-p+1) \\ 0 & 0 \end{pmatrix}.$$

The corresponding eigenvalues are

$$\Lambda_w(D_1) = \begin{pmatrix} (w+p)(w+n-p+1) & 0 \\ 0 & 0 \end{pmatrix}, \qquad \Lambda_w(D_2) = \begin{pmatrix} 0 & 0 \\ 0 & (w+p+1)(w+n-p) \end{pmatrix}, \Lambda_w(D_3) = \begin{pmatrix} 0 & 0 \\ (w+p)(w+p+1) & 0 \end{pmatrix}, \qquad \Lambda_w(D_4) = \begin{pmatrix} 0 & (w+n-p)(w+n-p+1) \\ 0 & 0 \end{pmatrix}.$$

*Remark* 5.4. The differential operator D appearing in Theorem 3.1 is  $D = -D_1 - D_2 + p(n-p)I$ .

We observe here that, for example,

$$\Lambda_w(D_1)\Lambda_w(D_3) \neq \Lambda_w(D_3)\Lambda_w(D_1), \text{ for all } w \in \mathbb{N}_0$$

By using Proposition 2.3 we obtain that  $D_1D_3 \neq D_3D_1$ , which in turn implies the following result.

**Corollary 5.5.** The algebra  $\mathcal{D}(W)$  is not commutative.

By following the same argument, through the sequence of eigenvalues, we obtain the following relations among the differential operators  $D_1, D_2, D_3, D_4$ .

$$\begin{split} D_1 D_2 &= 0, \quad D_2 D_1 = 0, \quad D_1 D_3 = 0, \quad D_4 D_1 = 0, \quad D_2 D_4 = 0, \quad D_3 D_2 = 0, \quad D_3^2 = 0, \quad D_4^2 = 0, \\ D_3 D_1 &= D_2 D_3 - (n-2p) D_3, \quad D_1 D_4 = D_4 D_2 - (n-2p) D_4, \quad D_3 D_4 = D_2^2 - (n-2p) D_2, \\ D_4 D_3 &= D_1^2 + (n-2p) D_1. \end{split}$$

### Conjecture 5.6.

- (1) There are no operators of odd order in  $\mathcal{D}(W)$ .
- (2) The second order differential operators in  $\mathcal{D}(W)$  generate the algebra  $\mathcal{D}(W)$ .

For a differential operator of order two  $D = \partial^2 F_2 + \partial F_1 + F_0 \in \mathcal{D}(W)$ , the explicit expression of the adjoint operator  $D^*$  is

$$D^* = \partial^2 G_2 + \partial G_1 + G_0,$$

where the polynomials  $G_i$ , i = 0, 1, 2, are defined by

$$G_{0} = \langle Q_{0}, Q_{0} \rangle \Lambda_{0}(D)^{*} \langle Q_{0}, Q_{0} \rangle^{-1}, \qquad G_{1} = \langle Q_{1}, Q_{1} \rangle \Lambda_{1}(D)^{*} \langle Q_{1}, Q_{1} \rangle^{-1} Q_{1}(x) - Q_{1}(x) G_{0},$$
  

$$G_{2} = \langle Q_{2}, Q_{2} \rangle \Lambda_{2}(D)^{*} \langle Q_{2}, Q_{2} \rangle^{-1} Q_{2}(x) - \partial(Q_{2}) G_{1}(x) - Q_{2}(x) G_{0},$$

see Theorem 4.3 in [13].

Also from Corollary 4.5 in [13], we obtain the expression for the corresponding eigenvalues for the adjoint operator  $D^*$ , in terms of the eigenvalues of the differential operator D and the norm of the polynomials  $Q_w$ ,

$$\Lambda_w(D^*) = \langle Q_w, Q_w \rangle \Lambda_w(D)^* \langle Q_w, Q_w \rangle^{-1}, \quad \text{for all } w.$$

By using the expressions of  $\langle Q_i, Q_i \rangle$ , given at the end of Section 4, and making straightforward computations, we can verify that

$$D_1^* = D_1, \quad D_2^* = D_2, \quad \text{and} \quad D_3^* = \frac{p}{n-p}D_4$$

Therefore

$$E_3 = (n-p)D_3 + pD_4$$
 and  $E_4 = i((n-p)D_3 - pD_4)$ 

are also symmetric operators, because for any  $D \in \mathcal{D}(W)$  the operators  $D + D^*$  and  $i(D - D^*)$  are symmetric operators. Explicitly,

$$E_{3} = (n-p)D_{3} + pD_{4} = \partial^{2} \begin{pmatrix} -x(n-2p) & x^{2}p - n + p \\ x^{2}(n-p) - p & x(n-2p) \end{pmatrix} + \partial \begin{pmatrix} 2p & 2p(n-p+1)x \\ 2(p+1)(n-p)x & 2(n-p) \end{pmatrix} \\ + \begin{pmatrix} 0 & p(n-p)(n-p+1) \\ p(p+1)(n-p) & 0 \end{pmatrix},$$
  
$$-iE_{4} = (n-p)D_{3} - pD_{4} = \partial^{2} \begin{pmatrix} -nx & -x^{2}p - n + p \\ x^{2}(n-p) + p & nx \end{pmatrix} + \partial \begin{pmatrix} -2p(n-p+1) & -2p(n-p+1)x \\ 2(p+1)(n-p)x & 2(n-p)(p+1) \end{pmatrix} \\ + \begin{pmatrix} 0 & -p(n-p)(n-p+1) \\ p(p+1)(n-p) & 0 \end{pmatrix}.$$

The corresponding eigenvalues are

$$\Lambda_w(E_3) = \begin{pmatrix} 0 & p(n-p+w)(n-p+w+1) \\ (n-p)(p+w)(p+w+1) & 0 \end{pmatrix},$$
  
$$\Lambda_w(-iE_4) = \begin{pmatrix} 0 & -p(n-p+w)(n-p+w+1) \\ (n-p)(p+w)(p+w+1) & 0 \end{pmatrix}.$$

Remark 5.7. In [16] the authors study matrix valued orthogonal polynomials related to spherical functions on the group (SU(2) × SU(2), SU(2)). The weight matrix is  $W_{+}^{(\nu)}$ , with  $\nu = 1$  in the notation of Remark 3.7. Let us denote  $\tilde{D}_1$ ,  $\tilde{D}_2$  and  $\tilde{D}_3$  the differential operators  $D_1, D_2$  and  $D_3$  appearing in Theorem 8.1 in [16]. Then we have the following relations with our operators  $D_1$ ,  $D_2$ ,  $D_3$  and  $D_4$  for the case n = 3 and p = 1

$$\widetilde{D}_1 = L(D_1 + D_2 - 3)L^{-1}, \quad \widetilde{D}_2 = LD_2L^{-1}, \quad \widetilde{D}_3 = -\sqrt{2}L(2D_3 + D_4)L^{-1}.$$

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