# MATRIX GEGENBAUER POLYNOMIALS: THE $2 \times 2$ FUNDAMENTAL CASES 

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#### Abstract

In this paper, we exhibit explicitly a sequence of $2 \times 2$ matrix valued orthogonal polynomials with respect to a weight $W_{p, n}$, for any pair of real numbers $p$ and $n$ such that $0<p<n$. The entries of these polynomiales are expressed in terms of the Gegenbauer polynomials $C_{k}^{\lambda}$. Also the corresponding three-term recursion relations are given and we make some studies of the algebra of differential operators associated with the weight $W_{p, n}$.


## 1. Introduction

The theory of matrix valued orthogonal polynomials, without any consideration of differential equations, goes back to [18] and [19]. In [3], the study of the matrix valued orthogonal polynomials that are eigenfunctions of certain second order symmetric differential operators was started. The first explicit examples of such polynomials were given in [8, [9], 7], 10] and [4]. See also [5], [6], 1], 2], and the references given there.

On the two dimensional sphere $S^{2}=\mathrm{SO}(3) / \mathrm{SO}(2)$, the harmonic analysis with respect to the action of the orthogonal group is contained in the classical theory of the spherical harmonics. In spherical coordinates, the zonal spherical functions on $S^{2}$ are the Legendre polynomials. More generally, in the case of the $n$ dimensional sphere $S^{n}$ the zonal spherical functions are given in terms of Gegenbauer (or ultraspherical) polynomials of parameter $(n-1) / 2$.

This fruitful connection between orthogonal polynomials and representation theory of compact Lie groups is also established in the matrix case: the matrix valued spherical functions of any $K$-type are closely related to matrix valued orthogonal polynomials. In this way, several examples of matrix orthogonal polynomials which are eigenfunctions of a symmetric differential operator have been obtained by focusing on a group representation approach. See for example [9], [11, [22], [23], 21] and more recently [16] and [24].

The examples of matrix orthogonal polynomials introduced in this paper are motivated by the spherical functions of fundamental $K$-types associated with the $n$-dimensional spheres $S^{n} \simeq G / K$, where $(G, K)=$ $(\mathrm{SO}(n+1), \mathrm{SO}(n))$. These matrix valued spherical functions were studied in detail in [27] and [29]. The "group parameters" of the fundamental $K$-types are $p, n \in \mathbb{N}$ such that $0<p<[n / 2]$ and they give rise to $2 \times 2$ matrix valued orthogonal polynomials.

In this paper we go beyond these group parameters and we extend these parameters continuously. We would like to remark that the group representation theory is a natural source of examples of matrix valued orthogonal polynomials. We keep this in mind in spite of the fact that the results obtained in this paper are self-contained, the proofs are of analytic nature and they do not depend on any previous results on spherical functions.

Given a weight matrix $W$, it is very natural to study the algebra $\mathcal{D}(W)$, of all differential operators that have a sequence of matrix valued orthogonal polynomials with respect to $W$ as eigenfunctions, see (3). In the classical cases of Hermite, Laguerre and Jacobi weights, the structure of this algebra is well understood: it is a polynomial algebra in a second order differential operator, see 20. In particular, it is a commutative

[^0]algebra. In the matrix case, the first attempt to go beyond the issue of the existence of one nontrivial element in $\mathcal{D}(W)$ and to study the full algebra is undertaken in [2], with the assistance of symbolic computation, for a few weights $W$. The first deep study of the algebra $\mathcal{D}(W)$ can be founded in [26], where the author worked out one of the examples introduced in [2]. We refer the reader to [13] for basic definitions and main results concerning the algebra $\mathcal{D}(W)$. The present paper leads to understand completely a second and more promising example of $\mathcal{D}(W)$ in a forthcoming paper, [28]. There are very few examples of non-commutative algebras that arise in a natural setup at the intersection of harmonic analysis and algebras. The study of such examples for the algebra $\mathcal{D}(W)$ considered here is one step in that direction. ++ As a consequence of this work, together with F.A. Grünbaum, in [12] we extend to a matrix setup a result that traces its origin and its importance to the work of Claude Shannon in lying the mathematical foundations of information theory, and to a remarkable series of papers by D. Slepian, H. Landau and H. Pollak.

To the best of our knowledge, this is the first example showing in a non-commutative setup that a bispectral property implies that the corresponding global operator of "time and band limiting" admits a commuting local operator. This is a noncommutative analog of the famous prolate spheroidal wave operator.

Now we discuss briefly the content of the paper. In Section 2 we recall the general notions of matrix valued orthogonal polynomials and some results from [13] about the algebra $\mathcal{D}(W)$.

In Section 3, we introduce our sequence $\left\{P_{w}\right\}_{w \in \mathbb{N}_{0}}$ of $2 \times 2$ matrix valued polynomials on $[-1,1]$ whose entries are given in terms of the classical Gegenbauer polynomials, for real parameters $p$ and $n$ such that $0<p<n$, see (4). We prove that these polynomials satisfy $P_{w} D=\Lambda_{w} P_{w}$, where $D$ is a (right-hand side) hypergeometric differential operator and the eigenvalue is a diagonal matrix. This differential operator $D$ is symmetric with respect to the matrix weight $W$ introduced in (12). We use these facts to prove that the polynomials $\left\{P_{w}\right\}_{w \in \mathbb{N}_{0}}$ are orthogonal with respect to the weight matrix $W=W_{p, n}$ (Theorem 3.6).

We also connect our weight matrix $W_{p, n}$ with the weight considered in [15], where the authors give examples of matrix valued Gegenbauer polynomials, extending for an arbitrary parameter $\nu$ the results in [16] for $\nu=1$. See Remark 3.7 .

In Section 4 we prove a three-term recursion relation satisfied by $\left\{P_{w}\right\}_{w \in \mathbb{N}_{0}}$. Section 5 is focused on the study of the algebra $\mathcal{D}(W)$. In our case $\mathcal{D}(W)$ is a noncommutative algebra. We provide a basis $\left\{D_{1}, D_{2}, D_{3}, D_{4}, I\right\}$ of the subspace of the differential operators in $\mathcal{D}(W)$ of order at most two. The differential operators $D_{1}$ and $D_{2}$ are symmetric operators, while $D_{3}$ and $D_{4}$ are not. We conjecture that $D_{1}, D_{2}, D_{3}, D_{4}$ generates the algebra $\mathcal{D}(W)$.

## 2. BACKGROUND ON MATRIX VALUED ORTHOGONAL POLYNOMIALS

Let $W=W(x)$ be a weight matrix of size $N$ on the real line, that is a complex $N \times N$ matrix valued integrable function on the interval $(a, b)$ such that $W(x)$ is positive definite almost everywhere and with finite moments of all orders. Let $\operatorname{Mat}_{N}(\mathbb{C})$ be the algebra of all $N \times N$ complex matrices and let $\operatorname{Mat}_{N}(\mathbb{C})[x]$ be the algebra over $\mathbb{C}$ of all polynomials in the indeterminate $x$ with coefficients in $\operatorname{Mat}_{N}(\mathbb{C})$. We consider the following Hermitian sesquilinear form in the linear space $\operatorname{Mat}_{N}(\mathbb{C})[x]$

$$
\langle P, Q\rangle=\langle P, Q\rangle_{W}=\int_{a}^{b} P(x) W(x) Q(x)^{*} d x
$$

The following properties are satisfied, for all $P, Q, R \in \operatorname{Mat}_{N}(\mathbb{C})[x], a, b \in \mathbb{C}, T \in \operatorname{Mat}_{N}(\mathbb{C})$
(1) $\langle a P+b Q, R\rangle=a\langle P, R\rangle+b\langle Q, R\rangle$,
(2) $\langle T P, R\rangle=T\langle P, R\rangle$,
(3) $\langle P, Q\rangle^{*}=\langle Q, P\rangle$,
(4) $\langle P, P\rangle \geq 0$. Moreover, if $\langle P, P\rangle=0$, then $P=0$.

Let us denote $\mathbb{N}_{0}=\mathbb{N} \cup\{0\}$. Given a weight matrix $W$ one can construct sequences of matrix valued orthogonal polynomials, that is sequences $\left\{P_{n}\right\}_{n \in \mathbb{N}_{0}}$, where $P_{n}$ is a polynomial of degree $n$ with nonsingular
leading coefficient and $\left\langle P_{n}, P_{m}\right\rangle=0$ for $n \neq m$. We observe that there exists a unique sequence of monic orthogonal polynomials $\left\{Q_{n}\right\}_{n \in \mathbb{N}_{0}}$ in $\operatorname{Mat}_{N}(\mathbb{C})[x]$. By following a standard argument, given for instance in [18] or [19], one shows that the monic orthogonal polynomials $\left\{Q_{n}\right\}_{n \in \mathbb{N}_{0}}$ satisfy a three-term recursion relation

$$
x Q_{n}(x)=A_{n} Q_{n-1}(x)+B_{n} Q_{n}(x)+Q_{n+1}(x), \quad n \in \mathbb{N}_{0}
$$

where $Q_{-1}=0$ and $A_{n}, B_{n}$ are matrices depending on $n$ and not on $x$.
Two weights $W$ and $\widetilde{W}$ are said to be similar if there exists a nonsingular matrix $M$, which does not depend on $x$, such that

$$
\widetilde{W}(x)=M W(x) M^{*}, \quad \text { for all } x \in(a, b)
$$

Notice that if $\left\{P_{n}\right\}_{n \geq 0}$ is a sequence of orthogonal polynomials with respect to $W$, and $M \in \mathrm{GL}_{N}(\mathbb{C})$, then $\left\{P_{n} M^{-1}\right\}_{n \geq 0}$ is orthogonal with respect to $\widetilde{W}=M W M^{*}$. A weight matrix $W$ reduces to a smaller size if there exists a nonsingular matrix $M$ such that

$$
M W(x) M^{*}=\left(\begin{array}{cc}
W_{1}(x) & 0 \\
0 & W_{2}(x)
\end{array}\right), \quad \text { for all } x \in(a, b)
$$

where $W_{1}$ and $W_{2}$ are weights of smaller size.
For a given weight matrix and a sequence of orthogonal polynomials, it may be of interest the study of the differential operators having these polynomials as eigenfunctions. Let $D$ be a right-hand side ordinary differential operator with matrix polynomial coefficients $F_{i}(x)$ of degree less than or equal to $i$ of the form

$$
\begin{equation*}
D=\sum_{i=0}^{s} \partial^{i} F_{i}(x), \quad \partial=\frac{d}{d x} \tag{1}
\end{equation*}
$$

with the action of $D$ on a polynomial function $P(x)$ given by

$$
(P D)(x)=\sum_{i=0}^{s} \partial^{i}(P)(x) F_{i}(x)
$$

We say that the differential operator $D$ is symmetric if $\langle P D, Q\rangle=\langle P, Q D\rangle$, for all $P, Q \in \operatorname{Mat}_{N}(\mathbb{C})[x]$. It is a matter of careful integration by parts to see that the condition of symmetry for a differential operator of order two is equivalent to a set of three differential equations involving the weight $W$ and the coefficients of the differential operator $D$.

Proposition 2.1 ([10] or [4]). Let $W(x)$ be a smooth weight matrix supported on $(a, b)$. Let $D=\partial^{2} F_{2}(x)+$ $\partial F_{1}(x)+F_{0}$. Then $D$ is symmetric with respect to $W$ if and only if

$$
\left\{\begin{aligned}
F_{2} W & =W F_{2}^{*} \\
2\left(F_{2} W\right)^{\prime}-F_{1} W & =W F_{1}^{*} \\
\left(F_{2} W\right)^{\prime \prime}-\left(F_{1} W\right)^{\prime}+F_{0} W & =W F_{0}^{*}
\end{aligned}\right.
$$

with the boundary conditions

$$
\lim _{x \rightarrow a, b} F_{2}(x) W(x)=0, \quad \lim _{x \rightarrow a, b}\left(F_{1}(x) W(x)-W F_{1}^{*}(x)\right)=0
$$

We consider the following subalgebra of the algebra of all right-hand side differential operators with coefficients in $\operatorname{Mat}_{N}(\mathbb{C})[x]$,

$$
\mathcal{D}=\left\{D=\sum_{i=0}^{s} \partial^{i} F_{i}: s \in \mathbb{N}_{0}, F_{i} \in \operatorname{Mat}_{N}(\mathbb{C})[x], \operatorname{deg} F_{i} \leq i\right\}
$$

Proposition 2.2 ([13], Propositions 2.6 and 2.7). Let $W=W(x)$ be a weight matrix of size $N \times N$ and let $\left\{Q_{n}\right\}_{n \geq 0}$ be the sequence of monic orthogonal polynomials in $\operatorname{Mat}_{N}(\mathbb{C})[x]$. If $D$ is a right-hand side ordinary differential operator of order $s$, as in (11), such that

$$
Q_{n} D=\Lambda_{n} Q_{n}, \quad \text { for all } n \in \mathbb{N}_{0}
$$

with $\Lambda_{n} \in \operatorname{Mat}_{N}(\mathbb{C})$, then $F_{i}=F_{i}(x)=\sum_{j=0}^{i} x^{j} F_{j}^{i}, F_{j}^{i} \in \operatorname{Mat}_{N}(\mathbb{C})$, is a polynomial and $\operatorname{deg}\left(F_{i}\right) \leq i$. Moreover $D$ is determined by the sequence $\left\{\Lambda_{n}\right\}_{n \geq 0}$ and

$$
\begin{equation*}
\Lambda_{n}=\sum_{i=0}^{s}[n]_{i} F_{i}^{i}, \quad \text { for all } n \geq 0 \tag{2}
\end{equation*}
$$

where $[n]_{i}=n(n-1) \cdots(n-i+1),[n]_{0}=1$.
Given a matrix weight $W$, the algebra

$$
\begin{equation*}
\mathcal{D}(W)=\left\{D \in \mathcal{D}: P_{n} D=\Lambda_{n}(D) P_{n}, \Lambda_{n}(D) \in \operatorname{Mat}_{N}(\mathbb{C}), \text { for all } n \in \mathbb{N}_{0}\right\} \tag{3}
\end{equation*}
$$

is introduced in [13], where $\left\{P_{n}\right\}_{n \in \mathbb{N}_{0}}$ is any sequence of matrix valued orthogonal polynomials with respect to $W$.

We observe that the definition of $\mathcal{D}(W)$ depends only on the weight matrix $W$ and not on the particular sequence of orthogonal polynomials, since two sequences $\left\{P_{w}\right\}_{w \in \mathbb{N}_{0}}$ and $\left\{Q_{w}\right\}_{w \in \mathbb{N}_{0}}$ of matrix orthogonal polynomials with respect to the weight $W$ are related by $P_{w}=M_{w} Q_{w}$, for $w \in \mathbb{N}_{0}$, with $\left\{M_{w}\right\}_{w \in \mathbb{N}_{0}}$ invertible matrices (see [13, Corollary 2.5]).
Proposition 2.3 ([13], Proposition 2.8). For each $n \in \mathbb{N}_{0}$, the mapping $D \mapsto \Lambda_{n}(D)$ is a representation of $\mathcal{D}(W)$ in $\operatorname{Mat}_{N}(\mathbb{C})$. Moreover, the sequence of representations $\left\{\Lambda_{n}\right\}_{n \in \mathbb{N}_{0}}$ separates the elements of $\mathcal{D}(W)$.

We remark that the result in Proposition 2.3 says that the map

$$
D \mapsto\left(\Lambda_{0}(D), \Lambda_{1}(D), \Lambda_{2}(D), \ldots \ldots\right)
$$

is an injective morphism of $\mathcal{D}(W)$ into $\operatorname{Mat}_{N}(\mathbb{C})^{\mathbb{N}_{0}}$, the direct product of infinite copies, indexed by $\mathbb{N}_{0}$, of the algebra $\operatorname{Mat}_{N}(\mathbb{C})$. In particular, if $D_{1}, D_{2} \in \mathcal{D}(W)$ then

$$
D_{1}=D_{2} \quad \text { if and only if } \quad \Lambda_{n}\left(D_{1}\right)=\Lambda_{n}\left(D_{2}\right) \text { for all } n \in \mathbb{N}_{0}
$$

For any $D \in \mathcal{D}(W)$ there exists a unique differential operator $D^{*} \in \mathcal{D}(W)$, the adjoint of $D$ in $\mathcal{D}(W)$, such that

$$
\langle P D, Q\rangle=\left\langle P, Q D^{*}\right\rangle
$$

for all $P, Q \in \operatorname{Mat}_{N}(\mathbb{C})[x]$. See Theorem 4.3 and Corollary 4.5 in [13]. The map $D \mapsto D^{*}$ is a *-operation in the algebra $\mathcal{D}(W)$. Moreover, it is shown that $\mathcal{S}(W)$, the set of all symmetric operators in $\mathcal{D}(W)$, is a real form of the space $\mathcal{D}(W)$, i.e.

$$
\mathcal{D}(W)=\mathcal{S}(W) \oplus i \mathcal{S}(W)
$$

as real vector spaces. In particular, the algebra $\mathcal{D}(W)$, together with the involution, is completely determined by $\mathcal{S}(W)$.

Corollary 2.4. A differential operator $D \in \mathcal{D}(W)$ is a symmetric operator if and only if

$$
\Lambda_{n}(D)\left\langle Q_{n}, Q_{n}\right\rangle=\left\langle Q_{n}, Q_{n}\right\rangle \Lambda_{n}(D)^{*}
$$

for all $n \in \mathbb{N}_{0}$.
Also it is worth to recall the following important result from [13].
Proposition 2.5 (Proposition 2.10). If $D \in \mathcal{D}$ is symmetric then $D \in \mathcal{D}(W)$.

## 3. Matrix valued orthogonal polynomials associated with the n-dimensional spheres

Motivated by the results obtained in [27] we introduce the following family of polynomials, for $w \in \mathbb{N}_{0}$,

$$
P_{w}(x)=P_{w}^{n, p}(x)=\left(\begin{array}{cc}
\frac{1}{n+1} C_{w}^{\frac{n+1}{2}}(x)+\frac{1}{p+w} C_{w-2}^{\frac{n+3}{2}}(x) & \frac{1}{p+w} C_{w-1}^{\frac{n+3}{2}}(x)  \tag{4}\\
\frac{1}{n-p+w} C_{w-1}^{\frac{n+3}{2}}(x) & \frac{1}{n+1} C_{w}^{\frac{n+1}{2}}(x)+\frac{1}{n-p+w} C_{w-2}^{\frac{n+3}{2}}(x)
\end{array}\right)
$$

with parameters $p, n \in \mathbb{R}$ such that $0<p<n$. Here $C_{n}^{\lambda}(x)$ denotes the $n$-th Gegenbauer polynomial

$$
C_{w}^{\lambda}(x)=\frac{(2 \lambda)_{w}}{w!}{ }_{2} F_{1}\left(\begin{array}{c}
-w, w+2 \lambda \\
\lambda+1 / 2
\end{array} ; \frac{1-x}{2}\right), \quad x \in[-1,1]
$$

where $(a)_{w}=a(a+1) \ldots(a+w-1)$ denotes the Pochhammer symbol. As usual, we assume $C_{w}^{\lambda}(x)=0$ if $w<0$. We recall that $C_{w}^{\lambda}$ is a polynomial of degree $w$, with leading coefficient $\frac{2^{w}(\lambda)_{w}}{w!}$.

Let us observe that $\operatorname{deg}\left(P_{w}\right)=w$ and the leading coefficient of $P_{w}$ is a nonsingular scalar matrix

$$
\begin{equation*}
\frac{2^{w}\left(\frac{n+1}{2}\right)_{w}}{(n+1) w!} \operatorname{Id}=\frac{1}{w!} 2^{w-1}\left(\frac{n+3}{2}\right)_{w-1} \operatorname{Id} \tag{5}
\end{equation*}
$$

We start by proving that the polynomials $P_{w}$ given in (4) are eigenfunctions of the following differential operator $D$.

Theorem 3.1. For each $w \in \mathbb{N}_{0}$, the matrix polynomial $P_{w}$ is an eigenfunction of the differential operator

$$
D=\partial^{2}\left(1-x^{2}\right)-\partial\left((n+2) x+2\left(\begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}\right)\right)-\left(\begin{array}{cc}
p & 0 \\
0 & n-p
\end{array}\right),
$$

with eigenvalue

$$
\Lambda_{w}(D)=\left(\begin{array}{cc}
-w(w+n+1)-p & 0 \\
0 & -w(w+n+1)-n+p
\end{array}\right)
$$

Proof. We need to verify that

$$
P_{w} D=\Lambda_{w} P_{w}
$$

We will need to use the following properties of the Gegenbauer polynomials (for the first three see 14 page 40, and for the last one see [25], page 83, equation (4.7.27))

$$
\begin{align*}
& \left(1-x^{2}\right) \frac{d^{2}}{d x^{2}} C_{m}^{\lambda}(x)-(2 \lambda+1) x \frac{d}{d x} C_{m}^{\lambda}(x)+m(m+2 \lambda) C_{m}^{\lambda}(x)=0  \tag{6}\\
& \frac{d}{d x} C_{m}^{\lambda}(x)=2 \lambda C_{m-1}^{\lambda+1}(x)  \tag{7}\\
& 2(m+\lambda) x C_{m}^{\lambda}(x)=(m+1) C_{m+1}^{\lambda}(x)+(m+2 \lambda-1) C_{m-1}^{\lambda}(x)  \tag{8}\\
& \frac{(m+2 \lambda-1)}{2(\lambda-1)} C_{m+1}^{\lambda-1}(x)=C_{m+1}^{\lambda}(x)-x C_{m}^{\lambda}(x) \tag{9}
\end{align*}
$$

Also, combining (8) and (9), we have

$$
\begin{equation*}
(m+\lambda) C_{m+1}^{\lambda-1}(x)=(\lambda-1)\left(C_{m+1}^{\lambda}(x)-C_{m-1}^{\lambda}(x)\right) \tag{10}
\end{equation*}
$$

The entry $(1,1)$ of the matrix $P_{w} D-\Lambda_{w} P_{w}$ is

$$
\begin{aligned}
(1- & \left.x^{2}\right)\left(P_{w}\right)_{11}^{\prime \prime}-(n+2) x\left(P_{w}\right)_{11}^{\prime}-2\left(P_{w}\right)_{12}^{\prime}+w(w+n+1)\left(P_{w}\right)_{11} \\
= & \left(1-x^{2}\right)\left(\frac{1}{n+1} C_{w}^{\frac{n+1}{2}}+\frac{1}{p+w} C_{w-2}^{\frac{n+3}{2}}\right)^{\prime \prime}-(n+2) x\left(\frac{1}{n+1} C_{w}^{\frac{n+1}{2}}+\frac{1}{p+w} C_{w-2}^{\frac{n+3}{2}}\right)^{\prime} \\
& -\frac{2}{p+w}\left(C_{w-1}^{\frac{n+3}{2}}\right)^{\prime}+w(w+n+1)\left(\frac{1}{n+1} C_{w}^{\frac{n+1}{2}}+\frac{1}{p+w} C_{w-2}^{\frac{n+3}{2}}\right) .
\end{aligned}
$$

Applying (6) for $\lambda=\frac{1}{2}(n+1), \lambda=\frac{1}{2}(n+3)$ and (7) for $\lambda=\frac{1}{2}(n+3)$, with $m=w$, we have that the entry $(1,1)$ of $P_{w} D-\Lambda_{w} P_{w}$, multiplied by $(p+w) / 2$ is

$$
-(n+3) C_{w-2}^{\frac{n+5}{2}}+(n+3) x C_{w-3}^{\frac{n+5}{2}}+(w+n+1) C_{w-2}^{\frac{n+3}{2}}=0
$$

this last identity follows from equation (9) with $\lambda=\frac{n+5}{2}$ and $m=w-3$. Repeating the previous verification, by changing $p$ by $n-p$, it follows that the entry $(2,2)$ of $P_{w} D-\Lambda_{w} P_{w}$ is also zero.

The entry $(1,2)$ of $P_{w} D-\Lambda_{w} P_{w}$ is

$$
\left(1-x^{2}\right)\left(P_{w}\right)_{12}^{\prime \prime}-(n+2) x\left(P_{w}\right)_{12}^{\prime}-2\left(P_{w}\right)_{11}^{\prime}+(w(w+n+1)-n+2 p)\left(P_{w}\right)_{12}
$$

if we multiply it by $(p+w)$ we get

$$
\begin{equation*}
\left(1-x^{2}\right)\left(C_{w-1}^{\frac{n+3}{2}}\right)^{\prime \prime}-(n+2) x\left(C_{w-1}^{\frac{n+3}{2}}\right)^{\prime}+(w(w+n+1)-n+2 p) C_{w-1}^{\frac{n+3}{2}}-2 \frac{(p+w)}{n+1}\left(C_{w}^{\frac{n+1}{2}}\right)^{\prime}-2\left(C_{w-2}^{\frac{n+3}{2}}\right)^{\prime} \tag{11}
\end{equation*}
$$

Applying (6) for $\lambda=(n+3) / 2, m=w-1$, (7) for $\lambda=(n+1) / 2, m=w$ and $\lambda=(n+3) / 2, m=w-1$, one obtain that (11) is

$$
2 x\left(C_{w-1}^{\frac{n+3}{2}}\right)^{\prime}-2(w-1) C_{w-1}^{\frac{n+3}{2}}-2(n+3) C_{w-3}^{\frac{n+5}{2}}
$$

Now, applying (7) and (9), this expression becomes

$$
2(n+3)\left(C_{w-1}^{\frac{n+5}{2}}-C_{w-3}^{\frac{n+5}{2}}\right)-2(2 w+n+1) C_{w-1}^{\frac{n+3}{2}}
$$

which is equal to zero by (10) with $\lambda=\frac{n+5}{2}$ and $m=w-2$. This concludes that the entry $(1,2)$ of $P_{w} D-\Lambda_{w} P_{w}$ is zero. To complete the proof of the theorem we need to verify that the entry $(2,1)$ is also zero. This is obtained making exactly the same computations, by changing $p$ by $n-p$.

We introduce the weight matrix

$$
W(x)=W_{p, n}=\left(1-x^{2}\right)^{\frac{n}{2}-1}\left(\begin{array}{cc}
p x^{2}+n-p & -n x  \tag{12}\\
-n x & (n-p) x^{2}+p
\end{array}\right), \quad x \in[-1,1]
$$

Proposition 3.2. For $n \neq 2 p$, the weight $W(x)$ does not reduce to a smaller size.
Proof. Assume that there exists a nonsingular matrix $M=\left(\begin{array}{ll}m_{11} & m_{12} \\ m_{21} & m_{22}\end{array}\right)$ such that

$$
M W(x) M^{*}=\left(\begin{array}{cc}
w_{1}(x) & 0 \\
0 & w_{2}(x)
\end{array}\right)
$$

The entry $(1,2)$ of $M W(x) M^{*}$ is

$$
x^{2}\left(p m_{11} \bar{m}_{21}+(n-p) m_{12} \bar{m}_{22}\right)-\left(m_{11} \bar{m}_{22}+m_{12} \bar{m}_{21}\right) n x+(n-p) m_{11} \bar{m}_{21}+p m_{12} \bar{m}_{22}
$$

from here we see that

$$
\begin{align*}
m_{11} \bar{m}_{22}+m_{12} \bar{m}_{21} & =0 \\
p m_{11} \bar{m}_{21}+(n-p) m_{12} \bar{m}_{22} & =0  \tag{13}\\
(n-p) m_{11} \bar{m}_{21}+p m_{12} \bar{m}_{22} & =0 \tag{14}
\end{align*}
$$

By combining equations (13) and (14) we have that $(n-2 p) m_{11} \bar{m}_{21}=0$. The assumption $n \neq 2 p$, together with (9), implies $\operatorname{det}(M)=0$, which is a contradiction.
Remark 3.3. For $n=2 p$, the weight matrix $W$ reduces to two scalar weights. The corresponding scalar polynomials are Jacobi polynomials $P_{w}^{\alpha, \beta}$ with $(\alpha, \beta)=(n / 2+1, n / 2-1)$ and $(\alpha, \beta)=(n / 2-1, n / 2+1)$, respectively. In fact, by taking $M=\left(\begin{array}{cc}1 & 1 \\ -1 & 1\end{array}\right)$ we have that

$$
M W(x) M^{*}=2 p\left(1-x^{2}\right)^{\frac{n}{2}-1}\left(\begin{array}{cc}
(1-x)^{2} & 0 \\
0 & (1+x)^{2}
\end{array}\right)
$$

Remark 3.4. We have that the weight matrices $W_{p, n}$ and $W_{n-p, n}$ are similar. In fact, by taking $M=\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$ we get

$$
M W_{p, n} M^{*}=W_{n-p, n}
$$

From Proposition 2.1 and following straightforward computations, one can prove the following result.
Proposition 3.5. The differential operator

$$
D=\partial^{2}\left(1-x^{2}\right)-\partial\left((n+2) x+2\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)\right)-\left(\begin{array}{cc}
p & 0 \\
0 & n-p
\end{array}\right)
$$

is symmetric with respect to the weight function $W(x)$.
In the scalar case, if $D$ is a symmetric differential operator with respect to $W$ and $\left\{P_{w}\right\}_{w \in \mathbb{N}_{0}}$ is a family of eigenfunctions of $D$ with different eigenvalues, then the sequence $\left\{P_{w}\right\}_{w \in \mathbb{N}_{0}}$ is automatically orthogonal with respect to $W$. In the matrix case this is not always true since

$$
\begin{equation*}
\Lambda_{w}\left\langle P_{w}, P_{w^{\prime}}\right\rangle=\left\langle P_{w} D, P_{w^{\prime}}\right\rangle=\left\langle P_{w}, P_{w^{\prime}} D\right\rangle=\left\langle P_{w}, P_{w^{\prime}}\right\rangle \Lambda_{w^{\prime}} \tag{15}
\end{equation*}
$$

does not imply that $\left\langle P_{w}, P_{w^{\prime}}\right\rangle=0$, for $w \neq w^{\prime}$. Therefore, we prove the orthogonality in the next theorem.
Theorem 3.6. When $n \neq 2 p$ the matrix polynomials $\left\{P_{w}\right\}_{w \in \mathbb{N}_{0}}$ are orthogonal polynomials with respect to the matrix valued inner product

$$
\langle P, Q\rangle=\int_{-1}^{1} P(x) W(x) Q(x)^{*} d x
$$

Proof. We know that $P_{w}$ is a polynomial of degree $w$ and its leading coefficient is a nonsingular diagonal matrix (see (5)). We only have to verify that for $w \neq w^{\prime},\left\langle P_{w}, P_{w^{\prime}}\right\rangle_{W}=0$. Since $P_{w}$ is an eigenfunction of the differential operator $D$, which is symmetric with respect to $W$, we have that (15) holds with

$$
\Lambda_{w}=\left(\begin{array}{cc}
\lambda_{w, 1} & 0 \\
0 & \lambda_{w, 2}
\end{array}\right)=\left(\begin{array}{cc}
-w(w+n+1)-p & 0 \\
0 & -w(w+n+1)-n+p
\end{array}\right),
$$

see Theorem 3.1. Therefore, for $i, j=1,2$ we have $\lambda_{w, i}\left\langle P_{w, i}, P_{w^{\prime}, j}\right\rangle=\lambda_{w^{\prime}, j}\left\langle P_{w, i}, P_{w^{\prime}, j}\right\rangle$, where $P_{w, i}$ is the $i$-th row of the polynomial $P_{w}$, and

$$
\left\langle P_{w, i}, P_{w^{\prime}, j}\right\rangle=\int_{-1}^{1} P_{w, i}(x) W(x) P_{w^{\prime}, j}^{*}(x) d x \in \mathbb{C}
$$

It is not difficult to verify that $\lambda_{w, i} \neq \lambda_{w^{\prime}, j}$, for $w \neq w^{\prime}$ or $i \neq j$. Then we have

$$
\begin{equation*}
\left\langle P_{w, i}, P_{w^{\prime}, j}\right\rangle=0, \quad \text { for } w \neq w^{\prime} \text { or } i \neq j \tag{16}
\end{equation*}
$$

Therefore $\left\langle P_{w}, P_{w^{\prime}}\right\rangle=0$, for $w \neq w^{\prime}$, which concludes the proof of the theorem.

Remark 3.7. Recently, in [15] the authors study some families on matrix valued polynomials, depending on one real parameter $\nu>0$, of arbitrary size $(2 \ell+1) \times(2 \ell+1)$ with $\ell \in \frac{1}{2} \mathbb{N}$. These weights are not irreducible. For $\ell=1, \frac{3}{2}, 2$ appears some irreducible $2 \times 2$ blocks $W_{+}^{(\nu)}$ and $W_{-}^{(\nu)}$. See Remark 2.8 (ii) there.

The case $\ell=3 / 2$ does not match with the examples considered in this paper. The cases $\ell=1$ and $\ell=2$ are particular cases of our weight matrices $W_{p, n}$ by choosing our parameters $(p, n)=(\nu, 2 \nu+1)$ and $(p, n)=(\nu, 2 \nu+3)$, for $\ell=1$ and $\ell=2$ respectively. In fact, with $L=\left(\begin{array}{cc}0 & \sqrt{2} \\ -1 & 0\end{array}\right)$ and $D=\left(\begin{array}{cc}1 & 0 \\ 0 & -2\end{array}\right)$ we get

$$
\begin{array}{ll}
W_{+}^{(\nu)}=\frac{(\nu+2)}{(2 \nu+1)} L W_{\nu, 2 \nu+1} L^{*} & \text { for } \ell=1 \\
W_{-}^{(\nu)}=\frac{(\nu+4)(\nu+2)}{(2 \nu+1)(2 \nu+3)} D W_{\nu, 2 \nu+3} D^{*} & \text { for } \ell=2
\end{array}
$$

The case $\nu=1$ was previously studied in [16] and [17].

## 4. Three-term recursion relation

The main result of this section is a three-term recursion relation satisfied by the sequence of orthogonal polynomials studied in this paper. We give a proof by using some properties of the Gegenbauer polynomials.

Theorem 4.1. The orthogonal polynomials $\left\{P_{w}\right\}_{w \in \mathbb{N}_{0}}$ satisfy the three-term recursion relation

$$
x P_{w}(x)=A_{w} P_{w-1}(x)+B_{w} P_{w}(x)+C_{w} P_{w+1}(x)
$$

where

$$
\begin{aligned}
& A_{w}=\left(\begin{array}{cc}
\frac{(n+w)(p+w-1)(n-p+w+1)}{(p+w)(n-p+w)(2 w+n+1)} & 0 \\
0 & \frac{(n+w)(p+w+1)(n-p+w-1)}{(p+w)(n-p+w)(2 w+n+1)}
\end{array}\right) \\
& B_{w}=\left(\begin{array}{cc}
0 & \frac{-p}{(p+w)(p+w+1)} \\
\frac{-(n-p)}{(n-p+w)(n-p+w+1)} & 0
\end{array}\right), \quad C_{w}=\frac{w+1}{2 w+n+1} I .
\end{aligned}
$$

Proof. To verify the $(1,1)$-entry of the equation in the statement of the theorem we need to prove that

$$
\begin{gather*}
x\left(\frac{1}{n+1} C_{w}^{\frac{n-1}{2}-1}(x)+\frac{1}{p+w} C_{w-2}^{\frac{n+3}{2}}(x)\right)=\frac{(n+w)(p+w-1)(n-p+w+1)}{(2 w+n+1)(p+w)(n-p+w)}\left(\frac{1}{n+1} C_{w-1}^{\frac{n-1}{2}-1}(x)+\frac{1}{p+w} C_{w-3}^{\frac{n+3}{2}}(x)\right) \\
-\frac{p}{(p+w)(p+w+1)(n-p+w)} C_{w-1}^{\frac{n+3}{2}}(x)+\frac{w+1}{2 w+n+1}\left(\frac{1}{n+1} C_{w+1}^{\frac{n-1}{2}-1}(x)+\frac{1}{p+w-1} C_{w-1}^{\frac{n+3}{2}}(x)\right) . \tag{17}
\end{gather*}
$$

By replacing the identities given by (8) for $\lambda=\frac{n+1}{2}, m=w$ and $\lambda=\frac{n+3}{2}, m=w-2$, one obtain that (17) is equivalent to

$$
\begin{align*}
& \frac{(w+n)}{(n+1)(2 w+n+1)}\left(-1+\frac{(p+w-1)(n-p+w-1)}{(p+w)(n-p+w)}\right) C_{w-1}^{\frac{n+1}{2}}(x) \\
& \quad+\left(-\frac{p}{(p+w)(p+w+1)(n-p+w)}+\frac{w+1}{(2 w+n+1)(p+w+1)}-\frac{w-1}{(p+w)(2 w+n-1)}\right) C_{w-1}^{\frac{n+3}{2}}(x)  \tag{18}\\
& \quad+\frac{(n+w)}{p+w}\left(\left(\frac{n-p+w-1}{(2 w+n+1)(n-p+w)}-\frac{1}{2 w+n-1}\right) C_{w-3}^{\frac{n+3}{2}}(x)=0 .\right.
\end{align*}
$$

Thus, by using the relation (10) for $\lambda=\frac{n+3}{2}$ and $m=w-2$, the identity in (18) follows after some straightforward computations.

Now we verify that the equation for the (1,2)-entry in the statement of the theorem holds. We need to verify that the following identity holds

$$
\begin{align*}
& \frac{1}{p+w}
\end{align*} x C_{w-1}^{\frac{n+3}{2}}(x)=\frac{(n+w)(n-p+w+1)}{(p+w)(2 w+n+1)(n-p+w)} C_{w-2}^{\frac{n+3}{2}}(x) .
$$

From (10) for $\lambda=\frac{n+3}{2}$ and $m=w-1$ we have that the right-hand side of (19) is

$$
\frac{n+w+1}{(p+w)(2 w+n+1)} C_{w-2}^{\frac{n+3}{2}}(x)+\frac{w}{(p+w)(2 w+n+1)} C_{w}^{\frac{n+3}{2}}(x) .
$$

Therefore, (19) is proved, since it is equivalent to (8) with $\lambda=\frac{n+3}{2}$ and $m=w-1$.
For the entries $(2,2)$ and $(2,1)$ we proceed in a similar way, by observing that we need to do the same computations as in the cases $(1,1)$ and $(1,2)$ respectively, by changing $p$ by $n-p$. This concludes the proof of the theorem.

The sequence of monic orthogonal polynomials is given by

$$
\begin{equation*}
Q_{w}=\frac{w!(n+1)}{2^{w}\left(\frac{n+1}{2}\right)_{w}} P_{w}, \quad w \in \mathbb{N}_{0} \tag{20}
\end{equation*}
$$

The first polynomials of the sequence $\left\{Q_{w}\right\}_{w \in \mathbb{N}_{0}}$ are

$$
\begin{aligned}
& Q_{0}=\mathrm{Id}, \quad Q_{1}=\left(\begin{array}{cc}
x & \frac{1}{p+1} \\
\frac{1}{n-p+1} & x
\end{array}\right), \quad Q_{2}=\left(\begin{array}{cc}
x^{2}-\frac{p}{(n+3)(p+2)} & \frac{2}{p+2} x \\
\frac{2}{n-p+2} x & x^{2}-\frac{n-p}{(n+3)(n-p+2)}
\end{array}\right), \\
& Q_{3}=\left(\begin{array}{cc}
x^{3}-\frac{3(p+1)}{(n+5)(p+3)} x & \frac{3}{p+3} x^{2}-\frac{3}{(n+5)(p+3)} \\
\frac{3}{n-p+3} x^{2}-\frac{3}{(n+5)(n-p+3)} & x^{3}-\frac{3(n-p+1)}{(n+5)(n-p+3)} x
\end{array}\right) .
\end{aligned}
$$

Remark 4.2. Observe that from (16) and (20) we have that $\left\langle Q_{w}, Q_{w}\right\rangle$ is always a diagonal matrix. Moreover one can verify that

$$
\left\langle Q_{w}, Q_{w}\right\rangle=\left\|Q_{w}\right\|^{2}=\frac{\pi 2^{[w / 2]} \Gamma(n / 2+1+[w / 2])}{w!(n+2 w+1) \Gamma((n+3) / 2)} \prod_{k=1}^{[(w-1) / 2]}(n+2 k+1)\left(\begin{array}{cc}
\frac{p(n-p+w+1)}{p+w} & 0 \\
0 & \frac{(n-p)(p+w+1)}{n-p+w}
\end{array}\right) .
$$

5. The algebra $\mathcal{D}(W)$

In this section we discuss some properties of the structure of the algebra $\mathcal{D}(W)$, defined in (3), for our weight matrix $W(x)$ introduced in (12). We are not interested in the cases when $p=n-p$, since the weight reduces to classical scalar weights, see Remark 3.3. We observe that in our example, the polynomials $\left\{P_{w}\right\}_{w \in \mathbb{N}_{0}}$, given in (4), and the monic orthogonal polynomials $\left\{Q_{w}\right\}_{w \in \mathbb{N}_{0}}$ have the same sequence of eigenvalues, since they are related by a scalar multiple, see (20).

First of all we observe that the space of differential operators of order zero in $\mathcal{D}(W)$ consists of scalar multiplies of the identity operator. In fact, a differential operator of order zero, having the sequence of monic orthogonal polynomials $\left\{Q_{w}\right\}_{w}$ as eigenfunctions, is a constant matrix $L$ such that

$$
Q_{w} L=\Lambda_{w} Q_{w}, \quad \text { for all } w \in \mathbb{N}_{0}
$$

From (2) we have that $\Lambda_{w}=L$ for every $w$. When $w=1$, we obtain that the entries of $L$ satisfy $L_{11}=L_{22}$ and $(p+1) L_{12}=(n-p+1) L_{21}$. Thus, looking at the case $w=2$ we get $(n-2 p) L_{12}=0$. Therefore we obtain that any operator of order zero $L$ in $\mathcal{D}(W)$ is a multiple of the identity matrix.

Now we study differential operators of order at most two in the algebra $\mathcal{D}(W)$. Let $\left\{Q_{w}\right\}_{w \in \mathbb{N}_{0}}$ the sequence of monic orthogonal polynomials with respect to $W$ and $D$ a differential operator of order at most two in $\mathcal{D}(W)$. From Proposition 2.2 we have

$$
D=\partial^{2}\left(A_{2} x^{2}+A_{1} x+A_{0}\right)+\partial\left(B_{1} x+B_{0}\right)+C \in \mathcal{D}(W)
$$

if and only if

$$
Q_{w} D=\left(w(w-1) A_{2}+w B_{1}+C\right) Q_{w}, \quad \text { for all } w \in \mathbb{N}_{0}
$$

Here $A_{2}, A_{1}, A_{0}, B_{1}, B_{0}, C$ are $2 \times 2$ complex matrices. Let us denote $Q_{w, j}$ the coefficients of the polynomial $Q_{w}$, i.e., $Q_{w}=\sum_{j=0}^{w} Q_{w, j} x^{j}$, with $Q_{w, w}=I$. Therefore $D \in \mathcal{D}(W)$ if and only if

$$
\begin{aligned}
& j(j-1) Q_{w, j} A_{2}+j(j+1) Q_{w, j+1} A_{1}+(j+1)(j+2) Q_{w, j+2} A_{0}+j Q_{w, j} B_{1} \\
& +(j+1) Q_{w, j+1} B_{0}+Q_{w, j} C-\left(w(w-1) A_{2}+w B_{1}+C\right) Q_{w, j}=0
\end{aligned}
$$

for all $w \in \mathbb{N}_{0}$ and $j=0, \ldots, w$. For $j=w-1$ and $j=0$ we respectively obtain

$$
\begin{align*}
& (w-1)(w-2) Q_{w, w-1} A_{2}+w(w-1) A_{1}+(w-1) Q_{w, w-1} B_{1}+w B_{0}+Q_{w, w-1} C \\
& \quad-\left(w(w-1) A_{2}+w B_{1}+C\right) Q_{w, w-1}=0 \tag{21}
\end{align*}
$$

and

$$
\begin{equation*}
2 Q_{w, 2} A_{0}+Q_{w, 1} B_{0}+Q_{w, 0} C-\left(w(w-1) A_{2}+w B_{1}+C\right) Q_{w, 0}=0 \tag{22}
\end{equation*}
$$

Now from (21) considering $w=1$ and $w=2$, and from (22) considering $w=2$, we respectively obtain

$$
\begin{aligned}
B_{0} & =\left(B_{1}+C\right) Q_{1,0}-Q_{1,0} C, \quad 2 A_{1}=\left(2 A_{2}+2 B_{1}+C\right) Q_{2,1}-Q_{2,1} B_{1}-2 B_{0}-Q_{2,1} C, \\
2 A_{0} & =\left(2 A_{2}+2 B_{1}+C\right) Q_{2,0}-Q_{2,1} B_{0}-Q_{2,0} C .
\end{aligned}
$$

From the expression of $Q_{1}$ and $Q_{2}$, given at the end of Section [4 we know that

$$
Q_{1,0}=\left(\begin{array}{cc}
0 & \frac{1}{p+1} \\
\frac{1}{n-p+1} &
\end{array}\right), \quad Q_{2,1}=\left(\begin{array}{cc}
0 & \frac{2}{p+2} \\
\frac{2}{n-p+2} & 0
\end{array}\right), \quad Q_{2,0}=\frac{-p}{(n+3)}\left(\begin{array}{cc}
\frac{1}{(p+2)} & 0 \\
0 & \frac{1}{(n-p+2)}
\end{array}\right)
$$

By using (20) and (4) it is easy to see that

$$
Q_{w, w-1}=\left(\begin{array}{cc}
\frac{0}{w-p+w} & \frac{w}{p+w}
\end{array}\right), \quad \text { for all } w \in \mathbb{N} .
$$

To determine the matrices $A_{2}=\left(a_{i j}\right), B_{1}=\left(b_{i j}\right)$ and $C=\left(c_{i j}\right)$, we first combine the entries in the diagonal of the matrix (21) to obtain

$$
\begin{aligned}
& 2(n+2) a_{21}=\frac{\left((n+p+2) b_{21}-2 c_{21}\right)}{p+1}+\frac{(p+2)(p+w)\left(2 c_{12}-(n-p) b_{12}\right)}{(n-p+1)(n-p+2)(n-p+w)} \\
& 2(n+2) a_{12}=\frac{\left((2 n-p+2) b_{12}-2 c_{12}\right)}{n-p+1}+\frac{(n-p+2)(n-p+w)\left(2 c_{21}-p b_{21}\right)}{(p+1)(p+2)(p+w)}
\end{aligned}
$$

Since these identities are valid for any integer $w \geq 3$ we conclude that, if $n \neq 2 p$ then $2 c_{12}=(n-p) b_{12}$ and $2 c_{21}=p b_{21}$. Therefore $b_{21}=2(p+1) a_{21}$ and $b_{12}=2(n-p+1) a_{12}$.

By combining the nondiagonal entries of (21) we have

$$
(n-2 p+1)\left((n+2) a_{11}-b_{11}\right)=(n-2 p-1)\left((n+2) a_{22}-b_{22}\right)
$$

and

$$
c_{11}-c_{22}=(p+1)(p+2) a_{22}-p(p+1) a_{11}+p b_{11}-(p+1) b_{22} .
$$

Equation (22) with $w=3$ says that

$$
2 Q_{3,2} A_{0}+Q_{3,1} B_{0}+Q_{3,0} C-\left(6 A_{2}+3 B_{1}+C\right) Q_{3,0}=0
$$

Now, by using the expression of $Q_{3}=x^{3}+Q_{3,2} x^{2}+Q_{3,1} x+Q_{3,0}$ given at the end of Section 4 it is not difficult to see that $b_{11}=(n+2) a_{11}$. Thus $b_{22}=(n+2) a_{22}$, and $c_{11}-c_{22}=p(n-p+1) a_{11}-(p+1)(n-p) a_{22}$.

Therefore, the matrices $A_{2}, A_{1}, A_{0}, B_{1}, B_{0}, C$ are given in terms of the entries of $A_{2}$ and $c_{11}$, as we state in the following theorem.

Theorem 5.1. The differential operators of order at most two in $\mathcal{D}(W)$ are of the form

$$
D=\partial^{2} F_{2}(x)+\partial F_{1}(x)+F_{0}
$$

where

$$
\begin{align*}
F_{2}(x) & =x^{2}\left(\begin{array}{ll}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right)+x\left(\begin{array}{cc}
a_{12}-a_{21} & a_{11}-a_{22} \\
a_{22}-a_{11} & a_{21}-a_{12}
\end{array}\right)+\left(\begin{array}{cc}
a_{22} & a_{21} \\
a_{12} & a_{11}
\end{array}\right) \\
F_{1}(x) & =x\left(\begin{array}{cc}
(n+2) a_{11} & 2(n-p+1) a_{12} \\
2(p+1) a_{21} & (n+2) a_{22}
\end{array}\right)+\left(\begin{array}{cc}
-p a_{21}+(n-p+2) a_{12} & (n-p+2) a_{11}-(n-p) a_{22} \\
-p a_{11}+(p+2) a_{22} & (p+2) a_{21}-(n-p) a_{12}
\end{array}\right)  \tag{23}\\
F_{0} & =\left(\begin{array}{cc}
p(n-p+1) a_{11}+c & (n-p)(n-p+1) a_{12} \\
p(p+1) a_{21} & (p+1)(n-p) a_{22}+c
\end{array}\right) .
\end{align*}
$$

with $a_{11}, a_{12}, a_{21}, a_{22}, c$ arbitrary complex numbers.
Proof. We have already proved that any differential operator of order at most two in $\mathcal{D}(W)$ is of this form for some constant $a_{11}, a_{12}, a_{21}, a_{22}, c \in \mathbb{C}$. Let $\mathcal{D}_{2}$ be the complex vector space of the differential operators in $\mathcal{D}(W)$ of order at most two. Then we have that $\operatorname{dim} \mathcal{D}_{2} \leq 5$.

From Proposition 2.1] it is not difficult to see that a differential operator $D$ of order two, with coefficients given by (23), is a symmetric operator if and only if

$$
a_{11}, a_{22}, c \in \mathbb{R} \quad \text { and } \quad p a_{21}=(n-p) \bar{a}_{12}
$$

From Proposition 2.5 any symmetric operator $D \in \mathcal{D}$ belongs to the algebra $\mathcal{D}(W)$. Thus there exists (at least) five $\mathbb{R}$-linearly independent symmetric operators in $\mathcal{D}_{2}$. Therefore $\operatorname{dim} \mathcal{D}_{2}=5$ and this concludes the proof of the theorem.

Corollary 5.2. There are no operators of order one in the algebra $\mathcal{D}(W)$.
The elements of the sequence $\left\{Q_{w}\right\}_{w}$ are eigenfunctions of the operators $D \in \mathcal{D}(W)$ and they satisfy $Q_{w} D=\Lambda_{w}(D) Q_{w}$, for $w \in \mathbb{N}_{0}$. We explicitly state the eigenvalues $\Lambda_{w}$ using formula (2): for a differential operator $D=\partial^{2} F_{2}+\partial F_{1}+F_{0}$ we have

$$
\Lambda_{w}(D)=w(w-1) F_{2}^{2}+w F_{1}^{1}+F_{0}^{0}
$$

with $F_{i}^{i}(\mathrm{i}=1,2,3)$ the leading coefficient of the polynomial coefficient $F_{i}$ of the differential operator $D$. Therefore we get

Corollary 5.3. Let $D \in \mathcal{D}(W)$, defined as in Theorem 5.1. The monic orthogonal polynomials $\left\{Q_{w}\right\}_{w}$ satisfy

$$
Q_{w} D=\Lambda_{w}(D) Q_{w}, \quad \text { for } w \in \mathbb{N}_{0}
$$

where the eigenvalue $\Lambda_{w}(D)$ is given by

$$
\Lambda_{w}(D)=\left(\begin{array}{cc}
(w+p)(w+n-p+1) a_{11}+c & (w+n-p)(w+n-p+1) a_{12} \\
(w+p)(w+p+1) a_{21} & (w+n-p)(w+p+1) a_{22}+c
\end{array}\right)
$$

Now we introduce a useful basis for the differential operators of order at most two in the algebra $\mathcal{D}(W)$ : the identity $I$ and

$$
D_{1}=\partial^{2}\left(\begin{array}{cc}
x^{2} & x \\
-x & -1
\end{array}\right)+\partial\left(\begin{array}{cc}
(n+2) x & n-p+2 \\
-p & 0
\end{array}\right)+\left(\begin{array}{cc}
p(n-p+1) & 0 \\
0 & 0
\end{array}\right)
$$

$$
\begin{aligned}
D_{2} & =\partial^{2}\left(\begin{array}{cc}
-1 & -x \\
x & x^{2}
\end{array}\right)+\partial\left(\begin{array}{cc}
0 & p-n \\
p+2 & (n+2) x
\end{array}\right)+\left(\begin{array}{cc}
0 & 0 \\
0 & (p+1)(n-p)
\end{array}\right), \\
D_{3} & =\partial^{2}\left(\begin{array}{cc}
-x & -1 \\
x^{2} & x
\end{array}\right)+\partial\left(\begin{array}{cc}
-p & 0 \\
2(p+1) x & p+2
\end{array}\right)+\left(\begin{array}{cc}
0 & 0 \\
p(p+1) & 0
\end{array}\right), \\
D_{4} & =\partial^{2}\left(\begin{array}{cc}
x & x^{2} \\
-1 & -x
\end{array}\right)+\partial\left(\begin{array}{cc}
n-p+2 & 2(n-p+1) x \\
0 & p-n
\end{array}\right)+\left(\begin{array}{cc}
0 & (n-p)(n-p+1) \\
0 & 0
\end{array}\right) .
\end{aligned}
$$

The corresponding eigenvalues are

$$
\left.\begin{array}{ll}
\Lambda_{w}\left(D_{1}\right)=\left(\begin{array}{cc}
(w+p)(w+n-p+1) & 0 \\
0 & 0
\end{array}\right), & \Lambda_{w}\left(D_{2}\right)=\left(\begin{array}{c}
0 \\
0 \\
(w+p+1)(w+n-p)
\end{array}\right), \\
\Lambda_{w}\left(D_{3}\right)=\left(\begin{array}{cc}
0 \\
(w+p)(w+p+1) & 0
\end{array}\right), & \Lambda_{w}\left(D_{4}\right)=\left(\begin{array}{c}
0 \\
0
\end{array}(w+n-p)(w+n-p+1)\right. \\
0
\end{array}\right) .
$$

Remark 5.4. The differential operator $D$ appearing in Theorem 3.1 is $D=-D_{1}-D_{2}+p(n-p) I$.
We observe here that, for example,

$$
\Lambda_{w}\left(D_{1}\right) \Lambda_{w}\left(D_{3}\right) \neq \Lambda_{w}\left(D_{3}\right) \Lambda_{w}\left(D_{1}\right), \quad \text { for all } w \in \mathbb{N}_{0}
$$

By using Proposition [2.3 we obtain that $D_{1} D_{3} \neq D_{3} D_{1}$, which in turn implies the following result.
Corollary 5.5. The algebra $\mathcal{D}(W)$ is not commutative.
By following the same argument, through the sequence of eigenvalues, we obtain the following relations among the differential operators $D_{1}, D_{2}, D_{3}, D_{4}$.

$$
\begin{aligned}
& D_{1} D_{2}=0, \quad D_{2} D_{1}=0, \quad D_{1} D_{3}=0, \quad D_{4} D_{1}=0, \quad D_{2} D_{4}=0, \quad D_{3} D_{2}=0, \quad D_{3}^{2}=0, \quad D_{4}^{2}=0, \\
& D_{3} D_{1}=D_{2} D_{3}-(n-2 p) D_{3}, \quad D_{1} D_{4}=D_{4} D_{2}-(n-2 p) D_{4}, \quad D_{3} D_{4}=D_{2}^{2}-(n-2 p) D_{2}, \\
& D_{4} D_{3}=D_{1}^{2}+(n-2 p) D_{1} .
\end{aligned}
$$

## Conjecture 5.6.

(1) There are no operators of odd order in $\mathcal{D}(W)$.
(2) The second order differential operators in $\mathcal{D}(W)$ generate the algebra $\mathcal{D}(W)$.

For a differential operator of order two $D=\partial^{2} F_{2}+\partial F_{1}+F_{0} \in \mathcal{D}(W)$, the explicit expression of the adjoint operator $D^{*}$ is

$$
D^{*}=\partial^{2} G_{2}+\partial G_{1}+G_{0},
$$

where the polynomials $G_{i}, i=0,1,2$, are defined by

$$
\begin{aligned}
& G_{0}=\left\langle Q_{0}, Q_{0}\right\rangle \Lambda_{0}(D)^{*}\left\langle Q_{0}, Q_{0}\right\rangle^{-1}, \quad G_{1}=\left\langle Q_{1}, Q_{1}\right\rangle \Lambda_{1}(D)^{*}\left\langle Q_{1}, Q_{1}\right\rangle^{-1} Q_{1}(x)-Q_{1}(x) G_{0}, \\
& G_{2}=\left\langle Q_{2}, Q_{2}\right\rangle \Lambda_{2}(D)^{*}\left\langle Q_{2}, Q_{2}\right\rangle^{-1} Q_{2}(x)-\partial\left(Q_{2}\right) G_{1}(x)-Q_{2}(x) G_{0},
\end{aligned}
$$

see Theorem 4.3 in [13].
Also from Corollary 4.5 in [13, we obtain the expression for the corresponding eigenvalues for the adjoint operator $D^{*}$, in terms of the eigenvalues of the differential operator $D$ and the norm of the polynomials $Q_{w}$,

$$
\Lambda_{w}\left(D^{*}\right)=\left\langle Q_{w}, Q_{w}\right\rangle \Lambda_{w}(D)^{*}\left\langle Q_{w}, Q_{w}\right\rangle^{-1}, \quad \text { for all } w .
$$

By using the expressions of $\left\langle Q_{i}, Q_{i}\right\rangle$, given at the end of Section 4 and making straightforward computations, we can verify that

$$
D_{1}^{*}=D_{1}, \quad D_{2}^{*}=D_{2}, \quad \text { and } \quad D_{3}^{*}=\frac{p}{n-p} D_{4} .
$$

Therefore

$$
E_{3}=(n-p) D_{3}+p D_{4} \quad \text { and } \quad E_{4}=i\left((n-p) D_{3}-p D_{4}\right)
$$

are also symmetric operators, because for any $D \in \mathcal{D}(W)$ the operators $D+D^{*}$ and $i\left(D-D^{*}\right)$ are symmetric operators. Explicitly,

$$
\begin{aligned}
E_{3}=(n-p) D_{3}+p D_{4}= & \partial^{2}\left(\begin{array}{cc}
-x(n-2 p) & x^{2} p-n+p \\
x^{2}(n-p)-p & x(n-2 p)
\end{array}\right)+\partial\left(\begin{array}{cc}
2 p & 2 p(n-p+1) x \\
2(p+1)(n-p) x & 2(n-p)
\end{array}\right) \\
& +\left(\begin{array}{cc}
0 & p(n-p)(n-p+1) \\
p(p+1)(n-p) & 0
\end{array}\right) \\
-i E_{4}=(n-p) D_{3}-p D_{4}= & \partial^{2}\left(\begin{array}{cc}
-n x & -x^{2} p-n+p \\
x^{2}(n-p)+p & n x
\end{array}\right)+\partial\left(\begin{array}{cc}
-2 p(n-p+1) & -2 p(n-p+1) x \\
2(p+1)(n-p) x & 2(n-p)(p+1)
\end{array}\right) \\
& +\left(\begin{array}{cc}
0 & -p(n-p)(n-p+1) \\
p(p+1)(n-p) & 0
\end{array}\right) .
\end{aligned}
$$

The corresponding eigenvalues are

$$
\begin{aligned}
\Lambda_{w}\left(E_{3}\right) & =\left(\begin{array}{cc}
0 & p(n-p+w)(n-p+w+1) \\
(n-p)(p+w)(p+w+1) & 0
\end{array}\right) \\
\Lambda_{w}\left(-i E_{4}\right) & =\left(\begin{array}{cc}
0 & -p(n-p+w)(n-p+w+1) \\
(n-p)(p+w)(p+w+1) & 0
\end{array}\right) .
\end{aligned}
$$

Remark 5.7. In [16] the authors study matrix valued orthogonal polynomials related to spherical functions on the group $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{SU}(2))$. The weight matrix is $W_{+}^{(\nu)}$, with $\nu=1$ in the notation of Remark 3.7. Let us denote $\widetilde{D}_{1}, \widetilde{D}_{2}$ and $\widetilde{D}_{3}$ the differential operators $D_{1}, D_{2}$ and $D_{3}$ appearing in Theorem 8.1 in [16]. Then we have the following relations with our operators $D_{1}, D_{2}, D_{3}$ and $D_{4}$ for the case $n=3$ and $p=1$

$$
\widetilde{D}_{1}=L\left(D_{1}+D_{2}-3\right) L^{-1}, \quad \widetilde{D}_{2}=L D_{2} L^{-1}, \quad \widetilde{D}_{3}=-\sqrt{2} L\left(2 D_{3}+D_{4}\right) L^{-1}
$$

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