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Asymptotic structure of the null surface formulation and the classical graviton

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The dynamical equations for the null surface formulation of general relativity for purely radiative spacetimes are derived. Those asymptotically flat spacetimes describe the nonlinear evolution of gravitational radiation and represent a classical graviton. The evolution equations constitute a set of three partial differential equations in a six-dimensional space and the source term is the free initial data of incoming gravitational radiation. The Huygens part of the wave propagation, backreaction terms, and source terms are identified in the resulting equations. An analysis of the range of validity of these equations based on the development of caustics is also given.

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I. INTRODUCTION

The null surface formulation of general relativity, or NSF for short, presents Einstein's theory of gravity as an interaction between null surfaces and a scalar field. The null surfaces yield the conformal structure and the scalar field yields the conformal factor of a Ricci flat metric [1].

Although the resulting equations are technically involved there is a well-defined perturbation scheme to obtain the conformal structure at each order of the perturbation. This is important when comparing perturbative approaches with numerical evolution of the Einstein's equations. For example, a perturbative solution written in the harmonic gauge fixes once and for all the conformal structure of the background metric. However, the null geodesics of the background metric either yield spacelike or timelike curves in terms of the full solution. Quantities like gravitational radiation or Bondi mass or momentum are therefore difficult to interpret when perturbing around a fixed background as the "null" curves from the background metric end either at i^+ or i^0.

Another feature of NSF worth mentioning is that the formulation explicitly includes the free data on the initial characteristic surface and the NSF equations propagate this free data into the spacetime. However, in the original approach, it is very difficult to distinguish if all the radiation data are given explicitly or it is hidden in its many different terms of the final equation [1].

To improve the original formulation one needs to define first the asymptotic structure of NSF and then derive an equivalent set of field equations with a clear interpretation of the different parts of the resulting equations together with the range of validity of the intrinsic coordinates used in the derivation. In this work we present an approach to the NSF that follows these ideas.

Beyond the above mentioned motivations that one might have for rederiving the Einstein's equations for asymptotically flat NSF it is worth mentioning another reason that also motivated the present work.

This issue, very much related to the gravitational radiation, is the quantization of a vacuum, globally hyperbolic, asymptotically flat spacetime containing no horizons. Such a spacetime will be called a classical graviton. Many years ago it was shown that both at future or past null infinity one can perform a free field quantization of the two degrees of freedom associated to the radiation data [2,3]. Later, the phase space for classical gravitons was derived together with a complex structure, thus giving a Hilbert space for quantum gravitons [4]. Although recently there have been attempts to obtain a quantum scattering matrix linking the fields at future and past null infinity [5,6], what is missing is a dynamical evolution equation that could link the Hilbert spaces associated with $\mathcal{I}^-$ and $\mathcal{I}^+$ to construct an S-matrix theory for the quantum graviton. Linking the radiative data with the fields inside the spacetime via the NSF equations fulfills this need.

In this work, we derive the field equations for asymptotically flat NSF. These equations introduce the free Bondi data, representing incoming gravitational radiation, as a source term for the main variable of NSF. To avoid issues with gravitational tails at future null infinity we assume the free data is given on past null infinity, $\mathcal{I}^-$ and the null cone cuts are formed from the intersection of the past null cones from points on the spacetime with $\mathcal{I}^-$. (Note that this is the time reversed version of all the references of NSF).

In Sec. II, we first show that it is always possible to obtain a region on past null infinity where the intersection of the past null cone from a point with $\mathcal{I}^-$ is a closed 2-surface with the topology of a sphere. We then present the metricity conditions and the Einstein equations assuming regularity conditions on the null cone cuts. In Sec. III, we present a kinematical analysis of the main variable of NSF using the available asymptotic structure of spacetimes with null boundaries and show its relationship to the free radiation data at null infinity. We then derive the NSF equations together with an analysis of the range of validity of those equations and a comparison with previous results. In particular, we analyze each term of the equation and identify the one that is responsible for the appearance of caustics. This is done by studying the range of validity of the intrinsic coordinates used for the setting of the field equations. Finally, in the Conclusion, we summarize our results and discuss the advantages of using the derived set of field equations in an asymptotic quantization procedure.

II. ASYMPTOTIC STRUCTURE AND NULL CONE CUTS

The null surface formulation describes the conformal structure of general relativity in terms of a function defined on a six-dimensional space. In this section, we will review the basic ideas and equations of NSF adding some new results that are used in this work.

We first introduce the notion of asymptotic flatness at null infinity. This concept appears as a need to define isolated systems in general relativity and it was laid out by the pioneering work of H. Bondi [7], R. Sachs [8], R. Penrose [9], and R. Geroch [10].

If one moves far apart from an isolated body, we know that the spacetime should be flat. Getting "apart from" is not a trivial problem in general relativity; we have to take into account what we want to observe and measure. In our case, we are interested in gravitational radiation and all the information that we can get of it, hence, we want to move apart along a null geodesic. For this, we want to define a parameter along that geodesic to measure the distance from the source to our position, for example affine distance s, and get far enough away is equivalent to $s \to \pm\infty$. At this point is important to note that the points $s = \pm\infty$ do not belong to our coordinate system. To "add" the points at infinity we introduce a new coordinate $\Omega = s^{-1}$ which solves the problem since we can place infinity at $\Omega = 0$. Thus, if the spacetime does not contains singularities, every null geodesic reaches infinity at $\Omega = 0$ and the set of all those endpoints form a null surface, called $\mathcal{I}$, which is a boundary surface for the spacetime manifold M itself.

More precisely, let (M, g_{ab}) be a spacetime, an asymptotically flat spacetime is defined as $(\hat{M}, \hat{g}_{ab})$, a manifold $\hat{M}$ with its boundary $\mathcal{I}$ together with a smooth Lorentzian metric $\hat{g}_{ab}$ and a smooth function Ω on $\hat{M}$ such that:

(1) $\hat{M} = M \cup \mathcal{I}$.

(2) At M, $\hat{g}_{ab} = \Omega^2 g_{ab}$ and $\Omega > 0$.

(3) At $\mathcal{I}$, $\Omega = 0$, $\nabla_a\Omega \neq 0$ and $\hat{g}^{ab}\nabla_a\Omega\nabla_b\Omega = 0$.

In this framework, $\hat{g}_{ab}$ is called the unphysical metric and it is related to the physical metric g_{ab} via a conformal transformation. The third condition asserts that $\mathcal{I}$ is a null surface.

As was mentioned above, the future (past) end points of null geodesics lie at future (past) null infnity, $\mathcal{I}^+$ ($\mathcal{I}^-$). Thus, $\mathcal{I} = \mathcal{I}^+ \cup \mathcal{I}^-$.

Let M be an asymptotically flat spacetime and $\mathcal{I}^-$ its past null boundary with Bondi coordinates $(v, \zeta, \bar{\zeta})$ with $(\zeta, \bar{\zeta}) \in S^2$ and $v \in \mathbb{R}$. Consider a fixed point $x^a \in M$ and denote by N_x^- its past null cone. The intersection between N_x^- and $\mathcal{I}^-$ defines a null cone cut. Locally this intersection can be described by

$$v = Z(x^a, \zeta, \bar{\zeta}) \tag{1}$$

where $(v, \zeta, \bar{\zeta})$ are the Bondi coordinates on $\mathcal{I}^-$. The function Z has a second interpretation. For fixed values of $(\zeta, \bar{\zeta})$ the level surface

$$Z(x^a, \zeta, \bar{\zeta}) = v = \text{const}, \tag{2}$$

i.e., the collection of points x^a in the spacetime that satisfy Eq. (2), represents the future null cone N^+ from a point $(v, \zeta, \bar{\zeta})$ at $\mathcal{I}^-$.

It is clear from its definition that Z can only be given locally as a function of $(\zeta, \bar{\zeta})$. We recall that a generic light cone will have singularities due to the focusing effect of the gravitational field. The appearance of those singularities are described by the vanishing of the geodesic deviations vectors associated with neighboring geodesics and are called caustics of the null cones. Thus, its intersection with $\mathcal{I}^-$ will not be a regular function on the sphere [11].

There is however one class of spacetimes where Z can be regarded as a smooth function. The framework we envisage is a Ricci flat spacetime without singularities constructed from the nonlinear dynamical evolution of incoming gravitational radiation. We assume the smooth radiation data at $\mathcal{I}^-$ have compact support, so that it vanishes in a neighborhood of i^-. These data correspond to a spacetime that is flat before the incoming radiation is turned on with a well-defined past infinity i^-.

The past null cone from a point x^a in the spacetime will be flat if for the unphysical metric the point x^a is close to i^-. As the point x^a gets further and further into the future of i^- the question is whether the null cone will develop caustics or remain a smooth hypersurface up to its intersection with $\mathcal{I}^-$.

To analyze the development of caustics in the past null cone from x^a we introduce the optical parameters, shear σ and divergence ρ of a small pencil of null rays around a given null geodesic $\ell^a = (\frac{\partial}{\partial s})^a$, with s its affine length. Since ρ blows up at a caustic point, to find those points one looks for solutions of the evolution equation for those parameters [12]

$$\frac{\partial\rho}{\partial s} = \rho^2 + \sigma\bar{\sigma} \tag{3}$$

$$\frac{\partial\sigma}{\partial s} = 2\rho\sigma + \psi_0, \tag{4}$$

with $\psi_0 = C_{abcd}\ell^a m^b \ell^c m^d$, and analyzes the regularity of the solutions.

In Minkowski space ψ_0 vanishes and the solution to the optical equations is given by

$$\rho_0 = -\frac{1}{s - s_0}, \qquad \sigma_0 = 0. \tag{5}$$

with $s = s_0$ representing the apex of the cone. Note that ρ_0 vanishes when $s \to \infty$, i.e., when the world line reaches $\mathcal{I}^-$. Thus, the intersection of N_x^- with $\mathcal{I}^-$ is a smooth two surface if one is sufficiently close to i^-. However, as one moves the apex further into the future, there will be a region of the spacetime with nonvanishing Weyl scalar ψ_0. In this case the shear is nonvanishing, and it follows from Eq. (3) that the derivative of $\rho(s)$ has an extra positive term. If there is a value of s where $\rho = 0$ then, within a finite distance, the divergence blows up at a caustic point. Considering that s ranges from s_0 to ∞ it is a nontrivial question to see if for a sufficiently small ψ_0 one obtains regular solutions of the optical equations on the entire range of s.

This problem is best analyzed if one introduces the unphysical metric $\hat{g}_{ab}$ which is related to the physical g_{ab} by

$$\hat{g}_{ab} = \Omega^2 g_{ab}$$

with $\Omega > 0$ on the spacetime. Since the past null cone from x^a is constructed from the conformal structure, and any point of the spacetime is at a finite (unphysical) null affine distance from $\mathcal{I}^-$ it is more convenient to use the unphysical metric to analyze the regularity of N_x^-. The unphysical affine parameter $\hat{s}$ is defined by the unphysical geodesic vector, $\hat{\ell}^a = (\frac{\partial}{\partial\hat{s}})^a$, and one can show that,

$$\frac{d\hat{s}}{ds} = \Omega^2. \tag{6}$$

The optical equations now read

$$\frac{\partial\hat{\rho}}{\partial\hat{s}} = \hat{\rho}^2 + \hat{\sigma}\,\bar{\hat{\sigma}} + \hat{\phi}_{00}, \tag{7}$$

and

$$\frac{\partial\hat{\sigma}}{\partial\hat{s}} = 2\hat{\rho}\,\hat{\sigma} + \hat{\psi}_0, \tag{8}$$

with $\hat{\phi}_{00} = \hat{R}_{ab}\hat{l}^a\hat{l}^b$ and $\hat{\psi}_0 = \hat{C}_{abcd}\hat{l}^a\hat{m}^b\hat{l}^c\hat{m}^d$. By assumption they are both regular functions of s. We now apply the standard theorem on ODEs, if a regular solution (for any value of $(\zeta, \bar{\zeta})$) exists for a finite distance $\hat{s}$ when $\hat{\psi}_0 = 0$, then there will be a regular solution at this finite distance for a sufficiently small value $\hat{\psi}_0 \neq 0$. Since a vanishing Weyl tensor yields a flat null cone with apex x^a with smooth cut at $\mathcal{I}^-$, the cut corresponding to an apex x'^a in the future of x^a but with nonvanishing Weyl tensor will also be smooth. Thus, the family of NC cuts are smooth functions when the corresponding apexes are sufficiently close to $\mathcal{I}^-$. In the remaining of this work we assume this situation. After the field equations are obtained, we will address again the issue of caustics analyzing the structure of the equations.

Note that the unphysical Ricci term $\hat{\phi}_{00}$ does not play a dynamical role in the above result. It is just a kinematical term constructed from Ω. This can be seen from the relation between the physical and the unphysical Ricci tensor, which reads:

$$R_{ab} = \hat{R}_{ab} + 2\Omega^{-1}(\nabla_a\nabla_b\Omega).$$

Setting $R_{ab} = 0$ and contracting with $\hat{\ell}^a\hat{\ell}^b$ we finally obtain,

$$\hat{\phi}_{00} = -\Omega^{-1}\frac{\partial^2\Omega}{\partial\hat{s}^2}.$$

Although it appears that $\hat{\phi}_{00}$ diverges at null infinity in fact it does not. Instead it is a regularity condition imposed on any well-defined conformal factor. If we give a specific form for Ω as a function of s we can then check the prescribed behavior for its behavior. For example, if we use the physical affine length as a radial coordinate, a suitable function could be

$$\Omega^2(s) = \frac{1}{1 + s^2}$$

since when $s \to \infty$ then $\Omega \to 0$ and it is positive everywhere. With this choice, the relationship between s and $\hat{s}$ can be obtained from (6) giving

$$s = \tan(\hat{s})$$

Note that $\hat{s} = \frac{\pi}{2}$ at null infinity. Note also that

$$\Omega(\hat{s}) = \cos(\hat{s}). \tag{9}$$

Therefore, $\hat{\phi}_{00} = 1$ with this choice of conformal factor.

It is illustrative to go back and solve again the optical parameters for the flat cone. Setting $\hat{\psi}_0 = 0$ immediately yields $\hat{\sigma} = 0$ and the equation for the divergence becomes

$$\frac{\partial\hat{\rho}}{\partial\hat{s}} = \hat{\rho}^2 + 1$$

whose solution, satisfying the appropriate initial and boundary conditions is given by

$$\hat{\rho}(\hat{s}) = -\frac{1}{\tan(\hat{s})} \tag{10}$$

When $\hat{s} \to 0$ then $\hat{\rho} \to -\infty$, satisfying the initial divergence of a null cone. Also when $\hat{s} \to \frac{\pi}{2}$ then $\hat{\rho} \to 0$ i.e., $\hat{\rho}$ vanishes at $\mathcal{I}^-$ for this particular null cone. It is clear that a small $\hat{\psi}_0$ in (8) will induce a small $\hat{\sigma}$ and $\hat{\rho}$ will remain finite at past null infinity.

For future reference, we note that if we consider a point x^a in a neighborhood of $\mathcal{I}^-$ with nonvanishing but small gravitational data, such that N_x^- is a smooth null surface in the spacetime, then by means of the reciprocity theorem, the future null cone N^+ from a fixed point at $\mathcal{I}^-$ that is connected to x^a by a null geodesic is also smooth. As the point x^a gets further away from i^- the and the unphysical affine length increases, a finite number of caustics could develop. Since the NSF breaks down at the appearance of caustics it is a valid question to ask if within the formalism it is possible to spot the validity of the field equations. We will return to this issue later.

A. Axiomatic description of NSF

In this subsection we derive kinematic and dynamic equations of NSF. The function Z has been introduced before and it is the main variable of the theory. We assume the spacetime to be a classical graviton and the cut Z to be sufficiently close to i^- that it is a smooth, closed 2-surface. We will show how to impose conditions such that $Z =$ const. is a characteristic surface of a Lorentzian metric. These are called metricity conditions. Using global conditions on the sphere we will then present a set of real equations equivalent to the original complex metricity conditions.

A useful derivative that will be used throughout this work is the covariant derivative on the unit sphere. Using stereographic coordinates $(\zeta, \bar{\zeta})$ we define the eth operator $\eth$ as [13]

$$\eth_\zeta f^{(\check{s})} = 2P^{1-\check{s}} \frac{\partial}{\partial \zeta}(P^{\check{s}} f^{(\check{s})}) \tag{11}$$

where $P = \frac{1}{2}(1+\zeta\bar{\zeta})$ and $f^{(\check{s})}$ is a function with spin weight $\check{s}$. Changing $\check{s}$ by $-\check{s}$ in the P factor, and ζ by $\bar{\zeta}$ the $\eth$ turns into $\bar{\eth}$ and the derivative with respect to $\bar{\zeta}$ is thus defined. Taking $\eth$ and $\bar{\eth}$ derivatives of Z we define four $(\zeta, \bar{\zeta})$-dependent functions

$$\theta^i(x^a, \zeta, \bar{\zeta}) = (Z, \eth Z, \bar{\eth} Z, \bar{\eth}\eth Z) = (v, \omega, \bar{\omega}, r)$$

for $i = 0, +, -, 1$. We assume the inverse transformation

$$x^a = x^a(\theta^i, \zeta, \bar{\zeta}). \tag{12}$$

is also defined for each value of the parameters $(\zeta, \bar{\zeta})$.

The four functions $\theta^i(x^a, \zeta, \bar{\zeta})$ have well-defined geometrical meanings. For fixed values of $(\zeta, \bar{\zeta})$, $Z(x^a, \zeta, \bar{\zeta}) = v =$ const.. yields the future null cone from a point $(v, \zeta, \bar{\zeta})$ at $\mathcal{I}^-$. On that cone, each geodesic is labelled by $\eth Z = \omega =$ const., $\bar{\eth} Z = \bar{\omega} =$ const. On a given null geodesic, each point is labelled by $\bar{\eth}\eth Z = r =$ const.

We also introduce an important variable in NSF given by

$$\Lambda(x^a, \zeta, \bar{\zeta}) \equiv \eth^2 Z(x^a, \zeta, \bar{\zeta}). \tag{13}$$

[As we will see below, the conformal metric can be written completely in terms of Λ and its derivatives.] This function also has a well-defined meaning. As will be shown in Sec. III, it is the difference between the shear of the Bondi congruence and the shear of past the null cone from x^a evaluated at a null cone cut of $\mathcal{I}^-$.

A trivial observation follows from its definition, namely,

$$\bar{\eth}^2 \Lambda(x^a, \zeta, \bar{\zeta}) = \eth^2 \bar{\Lambda}(x^a, \zeta, \bar{\zeta}). \tag{14}$$

Now, if the change of coordinates (12) is performed, the previous expression (13) can now be written as

$$\eth^2 Z = \Lambda(Z, \eth Z, \bar{\eth} Z, \bar{\eth}\eth Z, \zeta, \bar{\zeta}), \tag{15}$$

and can be regarded as a partial differential equation for Z on the sphere. Likewise, Eq. (14) is no longer a trivial equation but an integrability condition for Λ. Since Λ is now a function of $(\theta^i, \zeta, \bar{\zeta})$, $\eth$ is now written as

$$\eth = \eth_\zeta + \eth\theta^i \partial_i$$

(the explicit form is given in Sec. III) with $\eth_\zeta$ given in (11). Note also that the points of spacetime x^a have disappeared. They are recovered as the constants of integration of Eq. (15), i.e., points of the solution space of Eq. (15).

In fact, we can take a completely different point of view starting with an arbitrary function Λ which satisfies the integrability conditions and asking what extra conditions are needed on Λ so that given a solution $Z(x^a, \zeta, \bar{\zeta})$ of Eq. (15) the level surface $Z(x^a, \zeta, \bar{\zeta}) =$ const is a null surface for a metric in the solution space. The answer to this problem yields many interesting results.

(1) A condition on Λ, a partial differential equation on the six-dimensional space coordinatized by $(\theta^i, \zeta, \bar{\zeta})$, which is a generalization of the Wünschmann condition [14].

(2) The components of the conformal metric in the θ^i coordinate system are completely algebraically determined from the knowledge of Λ and its derivatives.

The detailed calculations of the NSF approach are given in many references. Below we outline the relevant equations and results.

Since the level surfaces of Z are null, the derivative with respect to x^a, $\partial_a Z$, must be a null covector field over M. It then follows that

$$g^{ab}(x^d)\partial_a Z(x^d,\zeta,\bar{\zeta})\partial_b Z(x^d,\zeta,\bar{\zeta}) = 0. \tag{16}$$

The idea is to algebraically determine the metric components by fixing x^a and selecting different values of $(\zeta, \bar{\zeta})$. Note however, that a solution of (16) can only be obtained up to an arbitrary rescaling, i.e., the solution yields a conformal metric. Taking $ð$ and $\bar{ð}$ derivatives of the above equation, one writes down nine linearly independent equations from which the components of the conformal metric are obtained. All the nontrivial components can be written in terms of $\Lambda(Z, ðZ, \bar{ð}Z, \bar{ð}ðZ, \zeta, \bar{\zeta})$ and its derivatives [15]. In fact, one can see that the explicit form of these components is $g^{ab} = g^{01}h^{ab}[\Lambda]$ where h^{ab} is the Λ dependent part of g^{ab} (see Appendix A for details).

It is clear that once all the components have been obtained extra $ð$ and $\bar{ð}$ derivatives will impose conditions on Λ since all the components are algebraically related to Λ.

These conditions, called metricity conditions, were originally given as follows [15–17]. Introducing the functions

$$g^{00} = g^{ab}\partial_a Z\partial_b Z, g^{01} = g^{ab}\partial_a Z\partial_b ð\bar{ð}Z,$$

the complex metricity conditions then read

$$ðg^{01} = g^{ab}ð(\partial_a Z\partial_b ð\bar{ð}Z)$$
$$ð^3 g^{00} = 0$$

Note that since $ðg^{01}$ has s.w. $= 1$, an expansion in spherical harmonics only contains terms with $\ell \geq 1$ while $ð^3 g^{00}$ being a s.w.3 object has terms $\ell \geq 3$. They read

$$ðg^{01} = g^{01}(h^{+1} + \partial_1\bar{ð}\Lambda), \tag{17}$$

$$ð^3 g^{00} = g^{01}(3h^{+i}\partial_i\Lambda + \partial_1 ð\Lambda) = 0. \tag{18}$$

It is worth mentioning that (18), the generalization of the Wünschmann condition to four dimensions [14], is solely a condition on Λ. We will always assume this condition is satisfied; otherwise, the metric is not Lorentzian.[1]

The metric g^{ab} can be written as $g^{ab} = g^{01}h^{ab}[\Lambda]$ where h^{ab} depends only on Λ and its derivatives. The conformal factor g^{01} has a gauge freedom since the ζ derivative given by (17) is invariant under multiplication by an arbitrary positive function of x^a.

[1]The Wünschmann condition, also obtained by E. Cartan, was originally derived to find an equivalence class of solutions to a third-order ODE. E. Cartan then showed that, if the condition was satisfied, it was possible to introduce a conformal metric on the solution space and the equivalence class was then given as the class of conformal isometries.

The first metricity condition, Eq. (17), is a complex pde for a single real variable g^{01}. A solution exists only if the integrability conditions are identically satisfied. This is indeed the case. Taking $\bar{ð}$ of (17) we obtain

$$2\bar{ð}ð(g^{01}) = g^{01}(\partial_1\bar{ð}^2\Lambda - h^{ij}\partial_i\Lambda\partial_j\bar{\Lambda}). \tag{19}$$

Thus, $\bar{ð}ðg^{01} = ð\bar{ð}g^{01}$ if $\bar{ð}^2\Lambda = ð^2\bar{\Lambda}$, but this is true from the starting assumption, Eq. (14). Note also that we could use the real Eq. (19) instead of the complex Eq. (17) if we are interested in regular solutions for g^{01}. This follows from the fact that the only regular solution of an equation such as $\bar{ð}f^{(1)} = 0$ for a $\check{s} = 1$ function $f^{(1)}$ is $f^{(1)} = 0$. Thus, we could either use (17) or (19) as our first metricity condition.

It is possible to obtain a similar equivalent equation for the complex second metricity condition (18). From

$$\bar{ð}^3(ð^3 g^{00}) = 0,$$

we obtain a real metricity condition,

$$\begin{aligned}0 = h^{ab}[&(\partial_a\bar{ð}ð\bar{ð}^2\Lambda + 6\partial_a\bar{ð}^2\Lambda)\partial_b Z + 3\partial_a\bar{ð}^3\Lambda\partial_b ðZ \\ &+ 3\partial_a ð^3\bar{\Lambda}\partial_b\bar{ð}Z + 9\partial_a\bar{ð}^2\Lambda\partial_b\bar{ð}ðZ - 4\partial_a\Lambda\partial_b\bar{\Lambda} \\ &+ 8\partial_a ð\bar{\Lambda}\partial_b\bar{ð}\Lambda + 2\partial_a ð\bar{ð}\Lambda\partial_b\bar{\Lambda} + 2\partial_a\bar{ð}ð\bar{\Lambda}\partial_b\Lambda]\end{aligned} \tag{20}$$

(The derivation is given in Appendix A). Note that these equations are completely equivalent to the original metricity conditions and they have some nice features: they are manifestly real whereas the original equations are complex. In Sec. IV, we will use the spin weight zero equations to derive equations for g^{01} and Λ that are equivalent to the Ricci flat Einstein's equations including the free gravitational data that represent the gravitational radiation at null infinity.

Finally we impose the Einstein's field equations on g^{01} and Λ. It can be shown that the trace free Ricci flat equations can be written as

$$\partial_r^2\nu - \nu R_{11}(\Lambda) = 0 \tag{21}$$

where $g^{01} = \nu^2$ and $R_{11}(\Lambda)$ is the $a = b = 1$ component of $R_{ab}(h)$, which explicitly reads:

$$\begin{aligned}R_{11}(\Lambda) &= R_{ab}(\Lambda)\theta_1^a\theta_1^b \\ &= \frac{1}{4q}\partial_r^2\Lambda\partial_r^2\bar{\Lambda} + \frac{3}{8q^2}(\partial_r q)^2 - \frac{1}{4q}\partial_r^2 q,\end{aligned} \tag{22}$$

with

$$q = 1 - \partial_r\Lambda\partial_r\bar{\Lambda}$$

where $\theta_1^a = (\frac{\partial}{\partial r})^a$ is the null vector which defines the parameter r as the affine parameter with respect to h^{ab}.

The Einstein's equations represents an PDE for ν and Λ. Note also that trace-free equations have an arbitrary cosmological constant λ which can be positive, negative or zero depending on which cosmology we are dealing with. In the next section, we will assume asymptotically flat spacetime where $\lambda = 0$.

Summarizing, the NSF is a theory of hypersurfaces and a scalar field on a six-dimensional space. Two metricity conditions yield a Lorentzian metric and a real PDE the Einstein's equations.

III. PEELING AND THE NSF

As stated in Sec. II, we assume that the spacetime is asymptotically flat along past null directed directions with past null boundary $\mathcal{I}^-$, and coordinates $(v, \zeta, \bar{\zeta})$. In this framework we analyze the peeling behavior of g^{01} and Λ. Asymptotically g^{ab} goes to η^{ab} at null infinity, thus, $h^{ab} \to \eta^{ab}$ and $g^{01} \to 1$.

To obtain the asymptotic behavior of Λ we first give a kinematical property directly related to its geometrical meaning. Using Sach's theorem, one can show that

$$\eth^2 Z = \sigma_B(Z, \zeta, \bar{\zeta}) - \sigma_Z(x^a, \zeta, \bar{\zeta}) \tag{23}$$

where σ_B and σ_Z are the leading parts of the Bondi and null cone shear respectively evaluated at $\mathcal{I}^-$. (The leading part of σ is defined as $\sigma_Z \equiv \lim_{s\to\infty} s^2\sigma$ with s the affine length along a null geodesic of the past null cone and σ the shear of the null cone. A similar definition for the Bondi null congruence yiels $\sigma_B(v, \zeta, \bar{\zeta})$) In this framework, the peeling of Λ means to obtain its behavior when the apex x^a approaches $\mathcal{I}^-$ along one null geodesic that ends at the point $(v, \zeta, \bar{\zeta})$. It is clear from (23) that σ_B does not change as the apex is moved to $\mathcal{I}^-$. Likewise, one could argue that the shear of a null cone vanishes when the apex of the cone moves to null infinity but we would like to present here a direct proof analyzing the geometrical parameters associated with the past null cone from a point x^a.

We assume that the apex of the null cone is near $\mathcal{I}^-$ along some null direction, and that the Ricci tensor vanishes. Thus, the optical equations,

$$\frac{\partial\rho}{\partial s} = \rho^2 + \sigma\bar{\sigma} \tag{24}$$

$$\frac{\partial\sigma}{\partial s} = 2\rho\sigma + \psi_0 \tag{25}$$

can be solved in the linearized approximation. The linearized solution for ρ of Eq. (24) is given by

$$\rho = -\frac{1}{s - s_0} \tag{26}$$

since the first nontrivial correction in ρ is quadratic in a perturbation scheme The Eq. (25) for σ now reads

$$\frac{\partial\sigma}{\partial s} = -2\frac{\sigma}{s - s_0} + \psi_0.$$

Multiplying both sides by $(s - s_0)^2$ yields

$$(s - s_0)^2\frac{\partial\sigma}{\partial s} + 2\sigma(s - s_0) = \psi_0(s - s_0)^2.$$

We thus have

$$\frac{\partial}{\partial s}[(s - s_0)^2\sigma] = \psi_0(s - s_0)^2$$

which can be integrated as

$$(s - s_0)^2\sigma = \int_{s_0}^{s}[\psi_0(s - s_0)^2]ds$$

Finally, using $\sigma_Z \equiv \lim_{s\to\infty} s^2\sigma$, we get

$$\begin{aligned}\sigma_Z &= \lim_{s\to\infty}\frac{s^2}{(s - s_0)^2}\int_{s_0}^{s}\psi_0(s - s_0)^2 ds + \cdots \\ &= \int_{s_0}^{\infty}[\psi_0(s - s_0)^2]ds + \cdots\end{aligned} \tag{27}$$

The above equation gives the dependence of σ_Z on the Weyl tensor. Remembering that the apex is labelled by s_0 and that $\psi_0(s)$ goes as s^{-5} the integral has a finite value at $\mathcal{I}^-$. It also follows from the above equation that if we let the apex go to $\mathcal{I}^-$, i.e., $s_0 \to \infty$, σ_x vanishes.

Note that the above result does not change if we replace the affine length s by the coordinate r since they are related by

$$\frac{dr}{ds} = g^{01}. \tag{28}$$

Since g^{01} is positive and goes to 1 at null infinity, r is a monotonically increasing function of s.

We thus have,

$$\lim_{r\to\infty}\Lambda = \sigma_B(v, \zeta, \bar{\zeta}). \tag{29}$$

It follows from the above equation that Λ is a free field at $\mathcal{I}^-$. This could play an important role at a classical or quantum scattering of gravitons.

It is also important to obtain the peeling behavior of other scalars like $\lim_{r\to\infty}\bar{\eth}\Lambda$ or $\lim_{r\to\infty}\bar{\eth}^2\Lambda$, or equivalently, the behavior of $\bar{\eth}\sigma_Z$ or $\bar{\eth}^2\sigma_Z$ in the asymptotic region since they are needed to obtain the field equations for Λ. Using the explicit form of $\bar{\eth}$ acting on an arbitrary function $F(u, \omega, \bar{\omega}, r, \zeta, \bar{\zeta})$ given by

$$\bar{\eth} = \bar{\eth} + \bar{\omega}\partial_v + r\partial_\omega + \bar{\Lambda}\partial_{\bar{\omega}} + (\eth\bar{\Lambda} - 2\bar{\omega})\partial_r \tag{30}$$

it is clear that we need to solve the integral in expression (27) in order to have an explicit form of σ_Z in terms of s or r. We thus take the peeling form of $\psi_0 = \frac{\psi_0^0}{s^5}$ in the asymptotic region, obtaining

$$\sigma_Z = O(r^{-2}) \tag{31}$$

where r represents the location of the apex of the light cone with shear σ_Z (see the Appendix B for the calculations). Applying the operator (30) over expression (31) for σ_Z it is easy to show that,

$$\bar{\eth}\sigma_Z = O(r^{-1})$$

and

$$\bar{\eth}^2\sigma_Z = O(r^0).$$

Following the above results the desired limits are:

$$\lim_{r\to\infty} \bar{\eth}\,\Lambda = \bar{\eth}\sigma_B \tag{32}$$

and

$$\lim_{r\to\infty} \bar{\eth}^2\Lambda = \bar{\eth}^2\sigma_B - \bar{\eth}^2\sigma_Z \tag{33}$$

To obtain the explicit form $O(r^0)$ we use this peeling behavior together with the field equations (19), (20) and (21).

IV. FREE DATA AND EINSTEIN EQUATIONS

So far, we showed that there exists a nonempty region on a Ricci flat, asymptotically flat spacetime where the null cone cuts are smooth closed surfaces. Furthermore, the asymptotic behavior of Λ and its derivatives have been studied. Now, we will derive a set of real, spin weight zero equations equivalent to the vacuum Einstein's equations.

The original derivation of the field equation for Λ in terms of the free data $\sigma_B(v, \zeta, \bar{\zeta})$ is very involved [1]. The final equation itself has many terms and its complexity makes it virtually impossible to check the validity of all the terms. The aim of this section is to develop a real, spin-weight zero equation for Λ that we think has several advantages over the previous result. We assume Eq. (20), the generalized Wünschmann condition, is satisfied to have a Lorentzian metric and concentrate on the coupled set of equations,

$$0 = 2\bar{\eth}\eth(\nu^2) - \nu^2(\partial_r\bar{\eth}^2\Lambda - h^{ij}\partial_i\Lambda\partial_j\bar{\Lambda}) \tag{34}$$

$$\partial_r^2\nu - \nu R_{11}(\Lambda) = 0 \tag{35}$$

Eq. (34) and (35) are a coupled system for Λ and ν equivalent to the vacuum Einstein's equations. As a guideline to the steps needed to obtain the final set of field equations we first apply the procedure to derive the linearized version.

A. The linearized approximation

In this section, we only keep up to linear terms in the Bondi free data and Λ. This represents the first-order deviation from flat Minkowski space. Since the Ricci flat equation contains quadratic terms in Λ and the asymptotic flatness condition implies that $g^{01} = 1$ at null infinity, then up to second order, $g^{01} = 1$ in the linearized approximation. Thus, the remaining equations are the complex metricity conditions,

$$\partial_r\bar{\eth}\Lambda + \partial_\omega\Lambda = 2\frac{\eth g^{01}}{g^{01}} = 0 \tag{36}$$

$$\partial_r\eth\Lambda - 3\partial_{\bar{\omega}}\Lambda = 0 \tag{37}$$

from which one should obtain three real conditions since the integrability conditions for (36) is identically satisfied.

Instead of taking the real and imaginary parts of the metricity conditions we take $\eth$ and $\bar{\eth}$ derivatives on the above equations until sw 0 quantities are obtained. The procedure is quite involved but straightforward. The reader may skip the derivation and advance to the summary of the equations at the end of the subsection.

We first obtain two real equations,

$$\partial_r\bar{\eth}^2\Lambda = 2\frac{\bar{\eth}\eth g^{01}}{g^{01}} = 0$$

$$\left(4\partial_v - \partial_{\bar{\omega}}\bar{\eth} - \partial_\omega\eth - 4\partial_r + \frac{1}{2}\bar{\eth}\eth\partial_r\right)\bar{\eth}^2\Lambda = 0 \tag{38}$$

Inserting (38) we get

$$[4\partial_v - \partial_{\bar{\omega}}\bar{\eth} - \partial_\omega\eth]\bar{\eth}^2\Lambda = 0 \tag{39}$$

The next step is to find relations between Λ and σ. Since (36) and (37) are two complex equations of a different spin weight, we can equate their spin weight (applying adequately ∂_ω and $\partial_{\bar{\omega}}$) and then write a new set of complex metricity conditions, i. e., two equivalent equations obtained adding and subtracting them. Combining these two new equations, it can be shown that the wave equation for Λ is obtained (see Appendix C). It must be noticed that if we keep only the "sum equation," we are setting an integrability condition between (36) and (37), which is equivalent to computing the commutator $[\partial_\omega, \partial_{\bar{\omega}}]$ of Λ as it follows:

$$3[\partial_\omega, \partial_{\bar{\omega}}]\Lambda = \partial_r[3\partial_{\bar{\omega}}\bar{\eth}\Lambda + \partial_\omega\eth\Lambda] = 0. \tag{40}$$

Integrating in r gives

$$3\partial_{\bar{\omega}}\bar{\eth}\Lambda + \partial_{\omega}\eth\Lambda = 4\partial_{v}\sigma_{B}. \tag{41}$$

Taking $\bar{\eth}^2$ of the above equation, using the reality condition $\bar{\eth}^2\Lambda = \eth^2\bar{\Lambda}$ together with (38) yields

$$(-6\partial_v + 3\partial_{\bar{\omega}}\bar{\eth} + \partial_{\omega}\eth)\bar{\eth}^2\Lambda = 4\partial_v\bar{\eth}^2\sigma_B. \tag{42}$$

The previous equation is complex; hence, it is equivalent to the following real equations:

$$(-3\partial_v + \partial_{\bar{\omega}}\bar{\eth} + \partial_{\omega}\eth)\bar{\eth}^2\Lambda = \partial_v(\bar{\eth}^2\sigma_B + \eth^2\bar{\sigma}_B) \tag{43}$$

$$(\partial_{\bar{\omega}}\bar{\eth} - \partial_{\omega}\eth)\bar{\eth}^2\Lambda = 2\partial_v(\bar{\eth}^2\sigma_B - \eth^2\bar{\sigma}_B). \tag{44}$$

Adding (43) and (39) gives

$$\partial_v(\bar{\eth}^2\Lambda - \bar{\eth}^2\sigma_B - \eth^2\bar{\sigma}_B) = 0, \tag{45}$$

which is an equation relating the free data σ and our variable Λ via a spin-weight zero and real equation. Furthermore, one can also show that (see Appendix A),

$$\partial_{\bar{\omega}}(\bar{\eth}^2\Lambda - \bar{\eth}^2\sigma_B - \eth^2\bar{\sigma}_B) = 0. \tag{46}$$

We summarize below the main results of the linearized NSF formulation:

$$\begin{aligned} g^{01} &= 1 \\ \partial_r\bar{\eth}^2\Lambda &= 0 \\ \partial_v(\bar{\eth}^2\Lambda - \bar{\eth}^2\sigma_B - \eth^2\bar{\sigma}_B) &= 0. \\ \partial_{\bar{\omega}}(\bar{\eth}^2\Lambda - \bar{\eth}^2\sigma_B - \eth^2\bar{\sigma}_B) &= 0. \end{aligned} \tag{47}$$

It follows from the above equations that Λ satisfies

$$\bar{\eth}^2\Lambda = \bar{\eth}^2\sigma_B + \eth^2\bar{\sigma}_B + F(\zeta,\bar{\zeta}), \tag{48}$$

where $F(\zeta,\bar{\zeta})$ is a real function on the sphere involving terms with $\ell \geq 2$ and represents the supertranslation invariance of Eq. (48).

It can also be shown (see Appendix C) that a solution of Eq. (48) automatically satisfies the imaginary equation (44). Thus, (47) and (48) are the complete set of linearized Einstein's equations in the NSF formulation.

One can also directly obtain the field equation for the light cone cut function Z by simply replacing u and Λ by Z and $\eth^2 Z$ in (48), obtaining

$$\bar{\eth}^2\eth^2 Z = \bar{\eth}^2\sigma_B(Z,\zeta,\bar{\zeta}) + \eth^2\bar{\sigma}_B(Z,\zeta,\bar{\zeta}) + F(\zeta,\bar{\zeta}). \tag{49}$$

Although one can fix a gauge and thus set $F = 0$, one can also keep this arbitrary function since the physical observables do not depend on F.

One can also write F in terms of the Bondi shear as follows. We first note that

$$F(\zeta,\bar{\zeta}) = [\bar{\eth}^2(\Lambda - \sigma_B) - \eth^2\bar{\sigma}_B]_{v\to\infty}.$$

Also from Sach's theorem, we have

$$\lim_{v\to-\infty}(\Lambda - \sigma_B) = \lim_{v\to-\infty}\sigma_x = 0,$$

since the light cone shear vanishes for flat space. We thus have

$$F(\zeta,\bar{\zeta}) = -[\eth^2\bar{\sigma}_B]_{v\to-\infty} = -[\eth^2\bar{\sigma}^{\text{in}}{}_B].$$

Since the imaginary part of the Bondi shear vanishes when there is no incoming radiations we finally obtain

$$F(\zeta,\bar{\zeta}) = -[\eth^2\bar{\sigma}^{\text{in}}{}_{ReB}].$$

As we mentioned before, F is irrelevant for the metric reconstruction since any derivatives involved in the calculations of the metric components will annihilate this term.

Equation (49) can be thought of as the real spacetime version of the good cut equation originally obtained for self-dual (or anti-self-dual) complex spacetimes. Although it was derived following a linearized version of NSF, it is clear that Eq. (49) is nonlinear in Z as the Bondi shear is an arbitrary function of Z.

B. Exact Einstein's equations

We follow the previous procedure to obtain the full equations. Writing (19) as

$$\partial_r\bar{\eth}^2\Lambda = 2\frac{\bar{\eth}\eth(g^{01})}{g^{01}} + h^{ij}\partial_i\Lambda\partial_j\bar{\Lambda} \tag{50}$$

and integrating over r, we have

$$\bar{\eth}^2\Lambda = (\bar{\eth}^2\Lambda)_{r\to\infty} - \int_r^\infty \left(2\frac{\bar{\eth}\eth(g^{01})}{g^{01}} + h^{ij}\partial_i\Lambda\partial_j\bar{\Lambda}\right)dr'. \tag{51}$$

This is our main equation for Λ. To obtain the asymptotic value of $(\bar{\eth}^2\Lambda)_{r\to\infty}$ in terms of the free data, we first rewrite Eq. (20) as

$$\left[4\partial_v - \partial_{\bar{\omega}}\bar{\eth} - \partial_{\omega}\eth + 4\partial_r + \frac{1}{2}\bar{\eth}\eth\partial_r\right]\bar{\eth}^2\Lambda + \frac{1}{2}M_{IIRe} = 0, \tag{52}$$

where the higher-order terms are included in M_{IIRe} (see Appendix D for the full form of M_{IIRe}). We are interested in calculating the explicit behavior of all the terms of Eq. (52) when $r\to\infty$. Using the peeling behavior given in the previous section, we can determine the asymptotic value of every one of them, except for $\partial_{\bar{\omega}}\bar{ð}^3\Lambda$ and c.c. For this reason, an auxiliary expression must be obtained. Combining the metricity conditions and following a calculation given in Appendix D, we obtain

$$[\partial_{\bar{\omega}}\bar{ð} + \partial_{\omega}ð - 3\partial_v + 6\partial_r]\bar{ð}^2\Lambda - N_{\mathrm{Re}} = \bar{ð}^2\dot{\sigma}_B + ð^2\dot{\bar{\sigma}}_B, \tag{53}$$

which is an expression that will be referred to as the auxiliary condition, with N_{Re} a real expression containing second- and higher-order terms arising in the algebraic manipulation (see Appendix D for the full form of N_{Re}). Adding Eq. (52) and Eq. (53) gives

$$\left[\partial_v + 10\partial_r + \frac{1}{2}\bar{ð}ð\partial_r\right]\bar{ð}^2\Lambda + \frac{1}{2}M_{II} - N_{\mathrm{Re}} = \bar{ð}^2\dot{\sigma}_B + ð^2\dot{\bar{\sigma}}_B. \tag{54}$$

This equivalent metricity condition has the advantage that it is written in terms of the free data. The term $\dot{\sigma}_B$ is called the Bondi news function since it can be used as asymptotic initial data for the gravitational radiation field.

Although the terms N_{Re} and M_{IIRe} are quite long, it is straightforward to show that

$$\left[\frac{1}{2}M_{II} - N_{\mathrm{Re}}\right]_{r\to\infty} = \dot{\sigma}\,\dot{\bar{\sigma}}\,.$$

This term also has physical meaning; it is the rate of change of the mass aspect and arises from the Bianchi identities at null infinity when written in Bondi coordinates. Since the $\lim_{r\to\infty}$ and the ∂_v operator commute we have

$$\lim_{r\to\infty}\partial_v(\bar{ð}^2\Lambda - \bar{ð}^2\sigma_B - ð^2\bar{\sigma}_B) = \dot{\sigma}\,\dot{\bar{\sigma}}\,|_{\ell\geq 2}. \tag{55}$$

Before integrating the above expression between the upper limit u and lower limit $v\to-\infty$, we note the following peeling property (C16):

$$\lim_{r\to\infty}\partial_{\bar{\omega}}(\bar{ð}^2\Lambda - \bar{ð}^2\sigma_B - ð^2\bar{\sigma}_B) = 0. \tag{56}$$

Thus, integrating (55) gives

$$\lim_{r\to\infty}\bar{ð}^2\Lambda = \bar{ð}^2\sigma_B + ð^2\bar{\sigma}_B + \int_{-\infty}^{v}\dot{\sigma}\,\dot{\bar{\sigma}}\,|_{\ell\geq 2}du' + F(\zeta,\bar{\zeta}), \tag{57}$$

with $F(\zeta,\bar{\zeta})$ a real function on the sphere involving only terms with $\ell\geq 2$. Finally, inserting (57) in Eq. (51), we get

$$\begin{aligned}\bar{ð}^2\Lambda = {}& \bar{ð}^2\sigma_B + ð^2\bar{\sigma}_B + \int_{-\infty}^{v}\dot{\sigma}\,\dot{\bar{\sigma}}\,|_{\ell\geq 2}du' + F(\zeta,\bar{\zeta}) \\ & - \int_r^{\infty}\left(2\frac{\bar{ð}ð(g^{01})}{g^{01}} + h^{ij}\partial_i\Lambda\partial_j\bar{\Lambda}\right)dr',\end{aligned} \tag{58}$$

where $\sigma_B = \sigma_B(v,\zeta,\bar{\zeta})$.

This is our main equation for Λ in terms of the free data, which together with equation (21), and the Wünschmann condition (18) or (20) constitutes a coupled system of three real PDEs for Λ and ν.

It is possible to distinguish three different terms on Eq. (58) following the dependence of the solution on the initial data. If the data are related to the solution along null characteristics, one often refers to them as Huygens data, following the principle of the Dutch physicist of light propagation on wavefronts. The solution of the wave equation in flat space at a point x^a only depends on the data given in the intersection of the past light cone from x^a with the initial surface. Such data are called Huygens data. If the metric is not flat, the solution also depends on the data given inside this intersection. Note that in this case a timelike curve connects the initial data with the solution and will be called non-Huygens. Usually the main part of the gravitational radiation travels along null characteristics. It thus propagates Huygens data from the source of the gravitational radiation. Finally, back reaction terms arise from the nonlinearity of the field equations and are given at and inside the past light cone from x^a. The three different terms on equation (58) can then be given by,

(1) Huygens data: the first two terms on the rhs of Eq. (58) which was previously obtained in linearized NSF.
(2) Non-Huygens data: the u integral term in Eq. (58). It corresponds to the initial radiation data that is timelike connected to the solution. In the time reversed version of this construction, i.e., using $\mathcal{I}^+$ instead of $\mathcal{I}^-$, this corresponds to the falloff of gravitational tails.
(3) Back Reaction term: given by the r integral in Eq. (58). It plays the role of a source for the solution of the field equation (even though we only have incoming radiation as the free data).

It is worth mentioning that the above equation should be equivalent to Eq. (15) of Ref. [1] which reads

$$\bar{ð}^2\Lambda = \bar{ð}^2\sigma_B + ð^2\bar{\sigma}_B + \frac{1}{2}\int_{-\infty}^{v}\mathcal{N}du', \tag{59}$$

where the quantity $\mathcal{N}$ is a long and involved expression in terms of Λ and derivatives ($\mathcal{N}$ is explicitly given in the Appendix E). However, the terms involved in $\mathcal{N}$ of expression (59) are quite complicated and it is not easy to check if they are the same. Nevertheless, Eq. (58) is more tractable and, what is more important, one can clearly keep track of how the gravitational data enter in the source term and what information from the spacetime is included. Furthermore, (58) is a regular equation on the sphere with coordinates $(\zeta, \bar{\zeta})$ whereas Eq. (15) of Ref. [1] is only given for a neighborhood of the sphere.

One should also point out that this equation is only valid whenever $(u, \omega, \bar{\omega}, r)$ are well behaved. Hence, it is important to know the range of validity of (58) and to analyze where the singulary problems are hidden in this equation. This is analyzed in the following section.

V. ANALYSIS OF THE SINGULARITIES AND THE ROLE OF THE CAUSTICS

It is important to note that the rhs of (57) is a smooth function over the sphere. Thus, any singular behavior of this equation must come from the r integral term of Eq. (58). In this section, we will discuss the appearance of singularities and caustic regions in Eq. (58).

We first state a theorem on conjugate points, adapted to our construction, that is extremely important for the results below.

Theorem 1: Consider two null cone congruences defined along a single null geodesic, one future pointing and the other one past pointing. Assume the future congruence with apex at an affine point s_0 has a conjugate point (a point where the geodesic deviation vector of the congruence vanishes) at a future point s_1. Then the past directed null cone congruence with apex at s_1 has a conjugate point at s_0.

This reciprocity theorem applies directly to our construction since the same function $u = Z$ describes the null cone cuts at $\mathcal{I}^-$ or the future null cone from a point at $\mathcal{I}^-$.

Using this theorem, one can show [11] that the $(u, \omega, \bar{\omega})$ coordinates are well behaved in a caustic region but $r \to -\infty$ and $\Lambda \to -\infty$ there, with the following relation,

$$\Lambda = \alpha r, \tag{60}$$

where $\alpha = \alpha(u, \omega, \bar{\omega}, \zeta, \bar{\zeta})$ is well behaved around a conjugate point. It then follows that $\partial_r \Lambda$ is well behaved around a singular point.

One can also show [11] that near a conjugate point, the conformal factor g^{01} is proportional to the divergence ρ_+ of the future null con from the point $(v,\ \zeta,\ \bar{\zeta})$ at $\mathcal{I}^-$. Thus, in the neighborhood of a singularity, one has

$$g^{01} = \beta \rho_+ \tag{61}$$

with $\beta = \beta(u, \omega, \bar{\omega}, \zeta, \bar{\zeta})$. Furthermore, using the relation

$$\frac{dr}{ds} = g^{01},$$

one can compute the behavior of derivatives like

$$\partial_r g^{01} = \frac{1}{g^{01}} \partial_s g^{01} \propto \frac{1}{\rho_+} \partial_s \rho_+ \propto \rho_+.$$

Thus, the r derivatives of g^{01} also diverge at a conjugate point.

Using the above results one can make several remarks. Our first observation is that in the domain of validity of the coordinate system $(u, \omega, \bar{\omega}, r)$ our main equation (58) never sees the singularities; i.e., if we start with a regular solution, then the term

$$\int_r^\infty (2\bar{\eth}\eth g^{01} + g^{ab} \partial_a \Lambda \partial_b \bar{\Lambda}) \frac{dr'}{g^{01}}$$

is always finite for a finite value of the coordinate r. To see the singularity one must take the limit $r \to -\infty$ or use the affine length s and rewrite the integral term as

$$\int_s^\infty (2\bar{\eth}\eth g^{01} + g^{ab} \partial_a \Lambda \partial_b \bar{\Lambda}) ds'.$$

It follows immediately from the above equation that at a singular affine distance $s = s_c$ the integrand diverges, at least as the divergence ρ.

Thus, it appears that one can follow two different approaches to obtain the solution of Eq. (58). Either one restricts oneself to a regular region, i.e., finite values of r or one decides to include singular points and the development of caustics. In that case one should rewrite the metricity conditions as well as the Einstein equation using the affine length s.

VI. CONCLUSIONS

We begin with a brief exposition of our main results.

(i) It was first shown that, when the null cone cuts are smooth 2-surfaces, one can write a set of two real metricity conditions that are equivalent to the original complex ones.

(ii) We then derived the concept of "peeling" for the main variables of the NSF and the asymptotic values of all the different fields in our formalism.

(iii) Using the above results, we obtained the field equation for Λ. The equation explicitly contains the free radiation data that play the role of a source term for the solution.

A brief comparison with the original derivation contained in Ref. [1] shows that the new derivations is less involved and contains an extra quadratic term in the radiation data. Furthermore, the radiation data contribute to the regular part of Eq. (58) and one can isolate and give an easier caustic analysis of the extra integral term in the equation.

One first shows that for any finite values of the coordinates $(v, \omega, \bar{\omega}, r)$ the solution is regular; i.e., its solution allows us to build a region of spacetime close to null infinity. To analyze the breakdown of regularity, one performs a change of variables, writing the main equation for Λ in terms of the affine parameter s. It is important to explore the development of caustics and this issue will be dealt with in the future.

It is also worth mentioning a few words on the use of this equation on an asymptotic quantization of the gravitational field, i.e., the quantum graviton. As it has been recently suggested by S. Hawking, classical black holes might not exist since black hole evaporation appears to be a generic feature of quantum gravity. If so, then this scenario with a regular asymptotically flat spacetime which develops from the solution of Eq. (58) might be useful in a quantization procedure. To implement the procedure, one would start with a free field quantization of the radiation data and propagate this quantum fields into the developing spacetime via Eq. (58).

Although it is very premature to compare our proposal with well-established approaches for quantum gravity such as loop quantum gravity, we could make a few remarks, mostly intended to highlight its differences. The starting approach behind loop quantum gravity is Ashtekar's canonical formulation in terms of new variables [18]. One of the difficult issues in the canonical approach is obtaining a well-defined procedure to define semiclassical states. The asymptotic quantization [2], on the other hand, has a well-defined meaning of "in" and "out" states. What was missing many years ago was an operator that would mediate between the incoming and outgoing states. We believe that, after providing an analogous description from $\mathcal{I}^+$, we can construct an S-matrix theory for the quantum graviton. If this can be done, then it would certainly be a useful contribution to the subject.

Another problem worth pursuing is the evaporation of "black holes." A starting approach would be an upside down quantization on $\mathcal{I}^+$ together with a classical geometry. As was recently pointed out by Hawking and collaborators [19], the gauge group on the event horizon has many similarities with the BMS group. The supertraslation symmetries would provide the needed degrees of freedom for the gravitational radiation that escapes to $\mathcal{I}^+$.

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APPENDIX A: METRIC COMPONENTS AND THE WÜNSCHMANN CONDITION

From the equation

$$g^{ab}\partial_a Z\partial_b Z = 0, \tag{A1}$$

we can extract all the metric components and the conditions for Z to be the null surfaces of g^{ab}. It is clear that $g^{ab}\partial_a Z\partial_b Z = g^{ab}\theta^0_a\theta^0_b = g^{00}$ and, hence,

$$g^{00} = 0.$$

Computing $\eth(A1)$ and $\bar{\eth}(A1)$, we get

$$g^{0+} = g^{0-} = 0.$$

From $\bar{\eth}\eth$ over (A1), we get

$$g^{+-} = -g^{01}.$$

Calculating $\eth^2(A1)$ and $\bar{\eth}^2(A1)$, we find

$$g^{++} = \overline{g^{--}} = g^{01}h^{++}$$

with

$$h^{++} = \overline{h^{--}} = -\partial_r\Lambda.$$

Applying $\bar{\eth}\eth^2(A1)$ and $\eth\bar{\eth}^2(A1)$, we obtain the following two components:

$$g^{+1} = \overline{g^{-1}} = g^{01}h^{+1},$$

where h^{+1} is given by

$$h^{+1} = \overline{h^{-1}} = \frac{1}{2}\frac{A - \frac{1}{2}\bar{A}\partial_r\Lambda}{1 - \frac{1}{4}\partial_r\Lambda\partial_r\bar{\Lambda}}$$

with

$$A = -\partial_r\bar{\eth}\Lambda + \partial_\omega\Lambda - h^{--}\partial_{\bar{\omega}}\Lambda.$$

Finally, applying $\bar{\eth}^2\eth^2$ on (A1),

$$g^{11} = g^{01}h^{11}$$

with

$$h^{11} = -2 - \frac{1}{2}(\partial_r\bar{\eth}^2\Lambda + h^{ij}\partial_i\Lambda\partial_j\bar{\Lambda}) + h^{-i}\partial_i\bar{\eth}\Lambda + h^{+j}\partial_j\eth\bar{\Lambda}.$$

It is important to note that the conformal structure $g^{ij} = g^{01}h^{ij}[\Lambda]$ arises naturally. The metric just obtained, however, will be in general $(\zeta, \bar{\zeta})$ dependent. To make sure it does not, we impose extra conditions on Λ. Here we obtain the real version of the metricity condition (the Wünschmann condition), Eq. (20). We start with

$$\eth^3[g^{ab}(x^d)\partial_a Z\partial_b Z] = 0,$$

which gives

$$g^{ab}[\partial_a\eth\Lambda\partial_b Z + 3\partial_a\Lambda\partial_b\eth\!\!/ Z] = 0.$$

Applying now $\bar{\eth}^3$ to the previous result, we obtain

$$
\begin{aligned}
0 = g^{ab}[&\partial_a\bar{\eth}\eth\bar{\eth}^2\Lambda\partial_b Z + 6\partial_a\bar{\eth}^2\Lambda\partial_b Z + 3\partial_a\bar{\eth}^3\Lambda\partial_b\eth Z \\
&+ 3\partial_a\eth^3\bar{\Lambda}\partial_b\bar{\eth}Z + 9\partial_a\bar{\eth}^2\Lambda\partial_b\eth\bar{\eth}Z + 3\partial_a\bar{\eth}\eth\Lambda\partial_b\bar{\Lambda} \\
&+ 3\partial_a\eth\bar{\eth}\,\bar{\Lambda}\,\partial_b\Lambda + 9\partial_a\bar{\eth}\Lambda\partial_b\eth\bar{\Lambda} \\
&+ 6\partial_a\Lambda\partial_b\bar{\Lambda} + \partial_a\eth\Lambda\partial_b\bar{\eth}\,\bar{\Lambda}],
\end{aligned}
$$

where the commutation relation

$$[\bar{\eth},\eth]F^{(\check{s})} = 2\check{s}F^{(\check{s})} \tag{A2}$$

and the condition $\bar{\eth}^2\Lambda = \eth^2\bar{\Lambda}$ were used.

Since $g^{ab} = g^{01}h^{ab}$, the above equation becomes Eq. (20) of the main text.

APPENDIX B: BEHAVIOR OF σ_Z AT THE ASYMPTOTIC REGION

In this Appendix, we will obtain the asymptotic behavior of $\sigma_Z(u,\omega,\bar{\omega},r,\zeta,\bar{\zeta})$ together with the asymptotic values of $\bar{\eth}\sigma_Z$ and $\bar{\eth}^2\sigma_Z$, and $\bar{\eth}\Lambda$ and $\bar{\eth}^2\Lambda$. To do this, we use the physical affine parameter s. We start with expression (27) of the main text,

$$\sigma_Z = \int_{s_0}^{\infty}[\psi_0(s-s_0)^2]ds. \tag{B1}$$

Inserting the asymptotic scalar $\psi_0 = \frac{\psi_0^0}{s^5}$ in the previous expression gives

$$\sigma_Z = \int_{s_0}^{\infty}\left[\frac{\psi_0^0}{s^5}(s-s_0)^2\right]ds = -\frac{\psi_0^0}{12s_0^2}, \tag{B2}$$

where $\psi_0^0 = \psi_0^0(u,\omega,\bar{\omega},\zeta,\bar{\zeta})$. Using $dr = g^{01}ds_0$ gives

$$\sigma_Z = -\frac{\psi_0^0}{12r^2} = O(r^{-2}).$$

This last expression corresponds to Eq. (31) of the main text. To obtain the asymptotic behavior of $\bar{\eth}\sigma_Z$ and $\bar{\eth}^2\sigma_Z$, we apply the $\bar{\eth}$ operator on σ_Z and then take their limit $r\to\infty$. We take the $O(\Lambda^0)$ terms of the $\bar{\eth}$ operator, namely, $\bar{\eth} = \bar{\eth}_{\bar{\zeta}} + \bar{\omega}(\partial_v - 2\partial_R) + R\partial_\omega$, in order to obtain the lowest decay power of $\bar{\eth}\sigma_Z$ and $\bar{\eth}^2\sigma_Z$. Analyzing $\bar{\eth}\sigma_Z$ term by term, we have

$$
\begin{aligned}
\bar{\eth}_{\bar{\zeta}}\sigma_Z &= -\frac{\bar{\eth}_{\bar{\zeta}}\psi_0^0}{12r^2} \to r^{-2} \\
\bar{\omega}\partial_u\sigma_Z &= -\frac{\bar{\omega}\partial_u\psi_0^0}{12r^2} \to r^{-2} \\
r\partial_\omega\sigma_Z &= -\frac{r\partial_\omega\psi_0^0}{12r^2} \to r^{-1} \\
-2\bar{\omega}\partial_r\sigma_Z &= 2\bar{\omega}\partial_r\frac{\bar{\eth}'\psi_0^0}{12r^2} \to r^{-3}.
\end{aligned}
$$

Since $r\partial_\omega\sigma_Z$ is the leading term in the asymptotic region, we have

$$\lim_{r\to\infty}\bar{\eth}\,\sigma_Z = O(r^{-1}).$$

In the same way, we apply $\bar{\eth}$ to $\bar{\eth}\sigma_Z$. Following a similar analysis, we find

$$\lim_{r\to\infty}\bar{\eth}^2\sigma_Z = O(r^0)$$

as it is asserted in the main text.

APPENDIX C: THE LINEARIZED APPROXIMATION

The commutation relation between the operators ∂_ω and $\partial_{\bar{\omega}}$ applied over Λ is needed in Sec. IV A. There it was claimed that the commutator $[\partial_\omega;\partial_{\bar{\omega}}]\Lambda = 0$ yields the wave equation for Λ. In this appendix, we will demonstrate this assertion. For this, we recall the following commutation relations:

$$
\begin{aligned}
&[\partial_i;\partial_j] = 0 \\
&[\partial_i,\bar{\eth}] = \delta_i^-(\partial_u - 2\partial_r) + \delta_i^1\partial_\omega + \partial_i\bar{\Lambda}\partial_{\bar{\omega}} + \partial_i\eth\bar{\Lambda}\partial_r \\
&\qquad \equiv [\partial_i,\bar{\eth}]^0 + [\partial_i,\bar{\eth}]^1
\end{aligned} \tag{C1}
$$

$$
\begin{aligned}
&[\partial_i,\eth] = \delta_i^+(\partial_u - 2\partial_r) + \delta_i^1\partial_{\bar{\omega}} + \partial_i\Lambda\partial_\omega + \partial_i\bar{\eth}\Lambda\partial_r \\
&\qquad \equiv [\partial_i,\eth]^0 + [\partial_i,\eth]^1,
\end{aligned} \tag{C2}
$$

where $[\partial_i,\bar{\eth}]^0$ and $[\partial_i,\bar{\eth}]^1$ represents the $[O(\Lambda)^0]$ and the $[O(\Lambda)^1]$ terms respectively of the commutator, namely,

$$[\partial_i,\eth]^0 \equiv \delta_i^+(\partial_v - 2\partial_r) + \delta_i^1\partial_{\bar{\omega}}$$

and

$$[\partial_i,\eth]^1 \equiv \partial_i\Lambda\partial_\omega + \partial_i\bar{\eth}\Lambda\partial_r.$$

This split is essential when these commutation relations are used either in the linear approximation or in the context of the exact equations. To write the wave equation for Λ at a linear level, we start with the set of the complex metricity conditions, Eqs. (36) and (37) of the main text, i.e.,

$$\partial_r\bar{\eth}\Lambda + \partial_\omega\Lambda = 0 \tag{C3}$$

$$\partial_r\eth\Lambda - 3\partial_{\bar{\omega}}\Lambda = 0. \tag{C4}$$

Using the relation $[\partial_r,\bar{\eth}]^0$ and its c.c from expressions (C1) and (C2), we rewrite the previous equations as

$$\bar{\eth}\partial_r\Lambda + 2\partial_\omega\Lambda = 0 \tag{C5}$$

$$\eth\partial_r\Lambda - 2\partial_{\bar{\omega}}\Lambda = 0. \tag{C6}$$

Since (C5) and (C6) are complex, we can obtain another set of two equations totally equivalent from their sum and difference. Before this we must equate their spin weight. We first perform

$$\partial_r\bar{\eth}(C6) - \partial_r\eth(C5) = 0 \tag{C7}$$

which gives

$$4\partial_r^2\Lambda - 2\partial_r[\bar{\eth}\partial_{\bar{\omega}} + \eth\partial_\omega]\Lambda = 0. \tag{C8}$$

We use $[\partial_{\bar{\omega}}, \bar{\eth}]^0$ and its c.c of (C1) and (C2) to commute out $\partial_{\bar{\omega}}$ and ∂_ω in the last two terms to obtain

$$4\partial_r(\partial_v - \partial_r)\Lambda - 2\partial_r[\partial_{\bar{\omega}}\bar{\eth}\Lambda + \partial_\omega\eth\Lambda] = 0 \tag{C9}$$

Now, we apply $\partial_{\bar{\omega}}(C3)$ and $\partial_\omega(C4)$ to equate the spins weight of the system (C3)–(C4), and calculate $\partial_{\bar{\omega}}(C3) + \partial_\omega(C4)$ to have

$$\partial_r(\partial_{\bar{\omega}}\bar{\eth}\Lambda + \partial_\omega\eth\Lambda) - 2\partial_{\bar{\omega}}\partial_\omega\Lambda = 0. \tag{C10}$$

Inserting (C10) in (C9) gives

$$4(\partial_r\partial_v - \partial_r^2 - \partial_{\bar{\omega}}\partial_\omega)\Lambda = 0,$$

which, in our coordinates, corresponds to

$$2\eta^{ab}\partial_a\partial_b\Lambda = 0$$

and, finally,

$$\Box\Lambda = 0. \tag{C11}$$

Thus, as was asserted in Sec. IV A, the wave equation for Λ can be obtained from the complex metricity equations. Thus, Eq. (37) in the main text is equivalent to show that Λ satisfies the wave equation and, hence, (37) has a well-established physical meaning. In previous works, we have explicitly used the wave equation for Λ to obtain the linearized equation for Z. In this work, we integrate in u the expression (42) of the main text for $\partial_v\bar{\eth}^2\Lambda$, giving Eq. (47).

We also claimed that the boundary value after integrating (42) is only $(\zeta, \bar{\zeta})$—dependent. To show this we first look at for a spin weight 1 expression for Λ containing the required terms. Basically, we will find a sw $= 1$ version of the second complex metricity condition and the auxiliary equation, expression (41), which relates Λ and σ, calculating $\bar{\eth}^2\eth^3 g^{00}$ and $\bar{\eth}(41)$ respectively. Starting with $\bar{\eth}^2\eth^3 g^{00}$, we obtain

$$2\partial_v\bar{\eth}\Lambda - \eth\partial_\omega\bar{\eth}\Lambda - \partial_{\bar{\omega}}\bar{\eth}^2\Lambda = 0. \tag{C12}$$

Now, $\bar{\eth}(41)$ gives

$$-2\partial_v\bar{\eth}\Lambda + \eth\partial_\omega\bar{\eth}\Lambda + 3\partial_{\bar{\omega}}\bar{\eth}^2\Lambda = 4\partial_v\bar{\eth}\sigma_B. \tag{C13}$$

Adding (C12) and (C13), we get

$$\partial_{\bar{\omega}}\bar{\eth}^2\Lambda = 2\partial_v\bar{\eth}\sigma_B. \tag{C14}$$

Taking the $O(\Lambda^0)$ of the operator $\bar{\eth}$, in the commutation relations one shows that $2\partial_v\bar{\eth}\sigma_B = \partial_{\bar{\omega}}\bar{\eth}^2\sigma_B$. Thus,

$$\partial_{\bar{\omega}}(\bar{\eth}^2\Lambda - \bar{\eth}^2\sigma_B) = 0. \tag{C15}$$

Furthermore, since $\partial_{\bar{\omega}}\eth^2\bar{\sigma}_B = 0$ (at a linear level in Λ and σ), we have that

$$\partial_{\bar{\omega}}(\bar{\eth}^2\Lambda - \bar{\eth}^2\sigma_B - \eth^2\bar{\sigma}_B) = 0 \tag{C16}$$

and its c.c. Finally, the real equation relating σ and Λ is given by

$$\bar{\eth}^2\Lambda = \bar{\eth}^2\sigma_B + \eth^2\bar{\sigma}_B + F(\zeta, \bar{\zeta}), \tag{C17}$$

which corresponds to Eq. (48) of the main text.

In order to show that the solution for the real equation, Eq. (C17) is also the solution to the imaginary equation (44), we write additional equations, namely,

$$\bar{\eth}^2\sigma_B = [\partial_{\bar{\zeta}}^2 + 2\bar{\omega}\partial_v\partial_{\bar{\zeta}} + \bar{\omega}^2\partial_u^2]\sigma_B$$
$$\eth^2\bar{\sigma}_B = [\partial_\zeta^2 + 2\omega\partial_u\partial_\zeta + \omega^2\partial_u^2]\bar{\sigma}_B$$

and, for a function $F = F(u, \omega, \bar{\omega}, \zeta, \bar{\zeta})$,

$$(\partial_{\bar{\omega}}\bar{\eth} - \partial_\omega\eth)F = (\partial_{\bar{\zeta}} + \bar{\omega}\partial_v)\partial_{\bar{\omega}}F - (\partial_\zeta + \omega\partial_v)\partial_\omega F. \tag{C18}$$

Inserting $\bar{\eth}^2\sigma_B$ and $\eth^2\bar{\sigma}_B$ in $\bar{\eth}^2\Lambda$ given by expression (48) of the main text, we obtain

$$\bar{\eth}^2\Lambda = [\partial_{\bar{\zeta}}^2 + 2\bar{\omega}\partial_v\partial_{\bar{\zeta}} + \bar{\omega}^2\partial_u^2]\sigma_B + [\partial_\zeta^2 + 2\omega\partial_v\partial_\zeta + \omega^2\partial_u^2]\bar{\sigma}_B,$$

and substituting $F(u, \omega, \bar{\omega}, \zeta, \bar{\zeta}) = \bar{\eth}^2\Lambda$ in the above equations, we get

$$(\partial_{\bar{\omega}}\bar{\eth} - \partial_\omega\eth)\bar{\eth}^2\Lambda = 2(\partial_{\bar{\zeta}} + \bar{\omega}\partial_v)(\partial_v\partial_{\bar{\zeta}} + \bar{\omega}\partial_u^2)\sigma_B - 2(\partial_\zeta + \omega\partial_v)(\partial_v\partial_\zeta + \omega\partial_u^2)\bar{\sigma}_B, \tag{C19}$$

which corresponds to the left-hand side of (44). For the rhs, we have

$$\partial_u\bar{ð}^2\sigma_B = [\partial_{\bar{\zeta}}^2 + 2\bar{\omega}\partial_v\partial_{\bar{\zeta}} + \bar{\omega}^2\partial_u^2]\partial_v\sigma_B \\ = (\partial_{\bar{\zeta}} + \bar{\omega}\partial_v)(\partial_v\partial_{\bar{\zeta}} + \bar{\omega}\partial_u^2)\sigma_B \tag{C20}$$

$$\partial_u ð^2\bar{\sigma}_B = [\partial_{\zeta}^2 + 2\omega\partial_v\partial_{\zeta} + \omega^2\partial_u^2]\partial_v\bar{\sigma}_B \\ = (\partial_{\zeta} + \omega\partial_v)(\partial_v\partial_{\zeta} + \omega\partial_u^2)\bar{\sigma}_B, \tag{C21}$$

which shows that the solution of the real equation for Λ satisfies the imaginary condition.

APPENDIX D: EXACT EINSTEIN'S EQUATION

In this Appendix we perform the extra calculations belonging to Sec. IV B. We first obtain the explicit form of the term M_{IIRe} in Eq. (52) of the main text. In order to do this, we split the linear and upper-order terms in Λ of the Eq. (20) as follows:

$$\partial_r\bar{ð}ð\bar{ð}^2\Lambda + 6\partial_r\bar{ð}^2\Lambda + 9\partial_v\bar{ð}^2\Lambda - 3\partial_{\bar{\omega}}\bar{ð}^3\Lambda - 3\partial_\omega ð^3\bar{\Lambda} \\ + 3(h^{++}\partial_\omega\bar{ð}^3\Lambda + h^{+1}\partial_r\bar{ð}^3\Lambda + h^{--}\partial_{\bar{\omega}}ð^3\bar{\Lambda} + h^{-1}\partial_r ð^3\bar{\Lambda}) \\ - 2h^{ij}(2\partial_i\Lambda\partial_j\bar{\Lambda} - 4\partial_i ð\bar{\Lambda}\partial_j\bar{ð}\Lambda - \partial_i ð\bar{ð}\Lambda\partial_j\bar{\Lambda} - \partial_i\bar{ð}ð\bar{\Lambda}\partial_j\Lambda) \\ + 9(h^{1+}\partial_\omega\bar{ð}^2\Lambda + h^{1-}\partial_{\bar{\omega}}\bar{ð}^2\Lambda + h^{11}\partial_r\bar{ð}^2\Lambda) = 0. \tag{D1}$$

We use the full commutation relations (C1) and (C2) to reexpress the first term of (D1), $\partial_r\bar{ð}ð\bar{ð}^2\Lambda$, as

$$\partial_r\bar{ð}ð\bar{ð}^2\Lambda = \bar{ð}ð\partial_r\bar{ð}^2\Lambda + \partial_{\bar{\omega}}\bar{ð}^3\Lambda + \partial_\omega ð^3\Lambda - \partial_v\bar{ð}^2\Lambda + 2\partial_r\bar{ð}^2\Lambda \\ + [\partial_r, \bar{ð}]^1 ð\bar{ð}^2\Lambda + \bar{ð}[\partial_r, ð]^1\bar{ð}^2\Lambda + [\bar{ð}, \partial_{\bar{\omega}}]^1\bar{ð}^2\Lambda$$

and replacing it in (D1), we get

$$4\partial_v\bar{ð}^2\Lambda - \partial_{\bar{\omega}}\bar{ð}^3\Lambda - \partial_\omega ð^3\bar{\Lambda} + 4\partial_r\bar{ð}^2\Lambda \\ + \frac{1}{2}\bar{ð}ð\partial_r\bar{ð}^2\Lambda + \frac{1}{2}M_{IIRe} = 0, \tag{D2}$$

which is Eq. (52) in the main text, with

$$M_{IIRe} = 9(h^{1+}\partial_\omega\bar{ð}^2\Lambda + h^{1-}\partial_{\bar{\omega}}\bar{ð}^2\Lambda + h^{11}\partial_r\bar{ð}^2\Lambda) \\ + 3(h^{++}\partial_\omega\bar{ð}^3\Lambda + h^{+1}\partial_r\bar{ð}^3\Lambda \\ + h^{--}\partial_{\bar{\omega}}ð^3\bar{\Lambda} + h^{-1}\partial_r ð^3\bar{\Lambda}) \\ + h^{ij}(-4\partial_i\Lambda\partial_j\bar{\Lambda} + 8\partial_i ð\bar{\Lambda}\partial_j\bar{ð}\Lambda \\ + 2\partial_i ð\bar{ð}\Lambda\partial_j\bar{\Lambda} + 2\partial_i\bar{ð}ð\bar{\Lambda}\partial_j\Lambda) + C(\Lambda), \tag{D3}$$

where

$$C(\Lambda) = [\partial_r, \bar{ð}]^1 ð\bar{ð}^2\Lambda + \bar{ð}[\partial_r, ð]^1\bar{ð}^2\Lambda + [\bar{ð}, \partial_{\bar{\omega}}]^1\bar{ð}^2\Lambda \\ = [\partial_r, \bar{ð}]^1 ð^3\Lambda + [\partial_r, ð]^1\bar{ð}^3\Lambda + [\bar{ð}, \partial_{\bar{\omega}}]^1\bar{ð}^2\Lambda \\ + [ð, \partial_\omega]^1 ð^2\bar{\Lambda} + ([\bar{ð}, \partial_r]^1\Lambda\partial_\omega + [\bar{ð}, \partial_{1r}]^1\bar{ð}\Lambda\partial_r \\ + \partial_r\Lambda[\bar{ð}, \partial_\omega]^1 + \partial_r\bar{ð}\Lambda[\bar{ð}, \partial_r]^1)\bar{ð}^2\Lambda \\ = [\partial_r, \bar{ð}]^1 ð^3\Lambda + [\partial_r, ð]^1\bar{ð}^3\Lambda + [\bar{ð}, \partial_{\bar{\omega}}]^1\bar{ð}^2\Lambda \\ + [ð, \partial_\omega]^1 ð^2\bar{\Lambda} - [(\partial_r\bar{\Lambda}\partial_{\bar{\omega}}\Lambda + \partial_r ð\bar{\Lambda}\partial_r\Lambda)\partial_\omega\bar{ð}^2\Lambda \\ + (\partial_r\bar{\Lambda}\partial_{\bar{\omega}}\bar{ð}\Lambda + \partial_r ð\bar{\Lambda}\partial_r\bar{ð}\Lambda)\partial_r\bar{ð}^2\Lambda + \text{c.c.}], \tag{D4}$$

which is manifestly real. From the peeling behavior analyzed in Sec. III, it is possible to determine the decay in r of every term contained in M_{IIRe}. This is a lengthy but straightforward calculation. On the other hand, the r decay of the terms $\partial_r\bar{ð}^2\Lambda$ and $\bar{ð}ð\partial_r\bar{ð}^2\Lambda$ of Eq. (D2) is analyzed from Eq. (50) considering that $g^{01} = \nu^2$ is given by

$$g^{01} = 1 + O(\Lambda^2) + \ldots.$$

Nevertheless, the peeling tools applied directly over the terms $\partial_{\bar{\omega}}\bar{ð}^3\Lambda + \partial_\omega ð^3\bar{\Lambda}$ do not give us the r decay unless these terms are written as functions of σ. It is possible to relate these two terms with the free data by an algebraic manipulation of the first and second metricity conditions. First, we equate their spin weight and factors performing $3\partial_{\bar{\omega}}(17)$:

$$3\partial_r\partial_{\bar{\omega}}\bar{ð}\Lambda + 3\partial_{\bar{\omega}}\partial_\omega\Lambda = 3\partial_{\bar{\omega}}M_I \tag{D5}$$

$\partial_\omega(18)$:

$$\partial_r\partial_\omega ð\Lambda - 3\partial_{\bar{\omega}}\partial_\omega\Lambda = \partial_\omega M_{II} \tag{D6}$$

with

$$M_I = 2\left(1 - \frac{1}{4}\partial_r\bar{\Lambda}\partial_r\Lambda\right)ð\ln g^{01} - \partial_r\bar{\Lambda}\partial_{\bar{\omega}}\Lambda \\ - \frac{1}{2}(\partial_r ð\bar{\Lambda} - \partial_{\bar{\omega}}\bar{\Lambda} - \partial_r\Lambda\partial_\omega\bar{\Lambda} - \partial_r\bar{\Lambda}\partial_r\bar{ð}\Lambda)\partial_r\Lambda \tag{D7}$$

and

$$M_{II} = 3\partial_r\Lambda(\partial_\omega\Lambda + h^{+1}).$$

Any manipulation between (D5) and (D6) results in a equivalent metricity condition too. In this case, we perform the sum of (D5) and (par2), obtaining

$$\partial_r(3\partial_{\bar{\omega}}\bar{ð}\Lambda + \partial_\omega ð\Lambda) = 3\partial_{\bar{\omega}}M_I + \partial_\omega M_{II}$$

and integrating over the variable r,

$$3\partial_{\bar\omega}\bar{ð}\Lambda+\partial_\omega ð\Lambda=4\dot\sigma_B+\int_\infty^r(3\partial_{\bar\omega}M_I+\partial_\omega M_{II})dr'. \tag{D8}$$

We see that this equation is a $\check{s}=2$ expression and we apply $\bar{ð}^2$ to Eq. (D8) to turn it into a $\check{s}=0$ one. Carrying out the corresponding permutations using the full relations (A2), (C1), (C2) and the substitution of $\bar{ð}^2\Lambda=ð^2\bar\Lambda$, the resulting $\bar{ð}^2(D8)$ can be expressed in the following manner:

$$3\partial_{\bar\omega}\bar{ð}^3\Lambda+\partial_\omega ð^3\bar\Lambda-6\partial_v\bar{ð}^2\Lambda+12\partial_r\bar{ð}^2\Lambda=4\bar{ð}^2\dot\sigma_B+N, \tag{D9}$$

where

$$\begin{aligned}N=&-\bar{ð}^2\int_\infty^r(3\partial_{\bar\omega}M_I+\partial_\omega M_{II})dr'-3[\partial_v,\bar{ð}]^1\bar{ð}\Lambda\\&+[\partial_\omega,\bar{ð}]^1ð\bar{ð}\Lambda+\bar{ð}[\partial_\omega,\bar{ð}]^1ð\Lambda+3\bar{ð}[\partial_{\bar\omega},\bar{ð}]^1\bar{ð}\Lambda\\&+3[\partial_{\bar\omega},\bar{ð}]^1\bar{ð}^2\Lambda+6[\partial_r,\bar{ð}]^1\bar{ð}\Lambda+4[\partial_\omega,\bar{ð}]^1\Lambda.\end{aligned}$$

Although Eq. (D9) is a complex equation, we keep the real part of it since we want to rewrite the terms $\partial_{\bar\omega}\bar{ð}^3\Lambda+\partial_\omega ð^3\bar\Lambda$ which are evidently real. Hence, adding to (D9) its complex conjugate, we obtain

$$\begin{aligned}&\partial_{\bar\omega}\bar{ð}^3\Lambda+\partial_\omega ð^3\bar\Lambda-3\partial_v\bar{ð}^2\Lambda+6\partial_r\bar{ð}^2\Lambda\\&\quad=\bar{ð}^2\dot\sigma_B+ð^2\dot{\bar\sigma}_B+N_{\mathrm{Re}}\end{aligned} \tag{D10}$$

with

$$N_{\mathrm{Re}}=\frac{1}{4}(N+\bar N), \tag{D11}$$

representing second- and upper-order terms in Λ. Adding (D10) and (D2), the following result is obtained

$$\begin{aligned}&\partial_v\bar{ð}^2\Lambda+10\partial_r\bar{ð}^2\Lambda+\frac{1}{2}\bar{ð}ð\partial_r\bar{ð}^2\Lambda\\&\quad=\bar{ð}^2\dot\sigma_B+ð^2\dot{\bar\sigma}_B+N_{\mathrm{Re}}-\frac{1}{2}M_{IIRe},\end{aligned}$$

which is Eq. (54) in the main text. After a carefully study of the peeling behavior of the terms $N_{\mathrm{Re}}-\frac{1}{2}M_{IIRe}$, we find that the higher-order term in Λ of Eq. (54) behaves like $(N_{\mathrm{Re}}-\frac{1}{2}M_{IIRe})\to\dot\sigma_B\dot{\bar\sigma}$ in the asymptotic region. Commuting $\bar{ð}^2$ and ∂_v using the full commutation relation (C1) and integrating over the u variable (in the gauge where Λ, σ_B, and its derivatives vanishes when $v\to-\infty$), we can write

$$(\bar{ð}^2\Lambda)_{r\to\infty}=\bar{ð}^2\sigma_B+ð^2\bar\sigma_B+\int_{-\infty}^v\dot\sigma_B\dot{\bar\sigma}_B$$

which corresponds to Eq. (57) in the main text.

APPENDIX E: TRANSCRIPTION OF PREVIOUS RESULTS

In this appendix, we explicitly give Eq. (15) obtained in [1], which relates the variable Λ with the free data,

$$ð^2\bar{ð}^2Z=\bar{ð}^2\sigma_R+ð^2\bar\sigma_R+\frac{1}{2}\int_{-\infty}^v\mathcal{N}du', \tag{E1}$$

with $\mathcal{N}$ given by

$$\begin{aligned}\mathcal{N}=&\bar\Lambda_{,0}\left(\Lambda_{,1}-\Lambda_{,0}-(\bar{ð}\Lambda)_{,-}+\int_\infty^R\frac{1}{2}S+K_{,-}dR'\right)+\Lambda_{,0}\left(\bar\Lambda_{,1}-\bar\Lambda_{,0}-(ð\bar\Lambda)_{,+}+\int_\infty^R\frac{1}{2}\bar S+\bar K_{,+}dR'\right)\\&+\bar{ð}^2\int_\infty^R\frac{1}{2}S+K_{,-}dR'+ð^2\int_\infty^R\frac{1}{2}\bar S+\bar K_{,+}dR'+\frac{1}{2}ð\bar{ð}^2(\bar\Lambda_{,1}\Lambda_{,-}+\Lambda_{,1}(ð\bar\Lambda)_{,1})\\&-\bar{ð}^3\left(\frac{\Lambda_{,1}}{4}(3(\bar\Lambda_{,1}\Lambda_{,-}-\Lambda_{,1}(ð\bar\Lambda)_{,1}+2\Lambda_{,1}\bar{ð}\ln\Omega)-K)\right)-\bar{ð}^2(\Lambda_{,-}\bar\Lambda_{,-}+\Lambda_{,1}(ð\bar\Lambda)_{,-})-\frac{1}{2}ð\bar{ð}^2K\\&-ð\bar{ð}(\Lambda_{,-}\bar\Lambda_{,+}+\Lambda_{,1}(ð\bar\Lambda)_{,+})-ð(-\bar\Lambda_{,+}(\Lambda_{,0}-\Lambda_{,1}-(\bar{ð}\Lambda)_{,-})+(ð\bar\Lambda)_{,+}((\bar{ð}\Lambda)_{,1}-\Lambda_{,+})\\&-\bar\Lambda_{,1}((\bar{ð}\Lambda)_{,0}-(\bar{ð}\Lambda)_{,1})+\bar K-(\bar{ð}\Lambda)_{,+}(ð\bar\Lambda)_{,1})+2\bar\Lambda_{,0}(\bar{ð}\Lambda)_{,-}+2ð(\bar\Lambda)_{,0}(\bar{ð}\Lambda)_{,1}+2\bar{ð}(\bar\Lambda_{,0}\Lambda_{,-}+\Lambda_{,1}(ð\bar\Lambda)_{,0})\end{aligned} \tag{E2}$$

with

$$\begin{aligned}S\equiv&-3L_{,+}+K_{,-}-\frac{1}{2}(3\bar{ð}L+ðK+\Lambda_{,+}^2+2\Lambda_{,1}(\bar{ð}\Lambda)_{,+}+\Lambda_{,-}\bar\Lambda_{,-}+\Lambda_{,1}(ð\bar\Lambda)_{,-}-(ð\Lambda)_{,-}\bar\Lambda_{,1}+(\bar{ð}\Lambda)_{,1}^2\\&+(ð\Lambda)_{,1}(\bar{ð}\Lambda)_{,1})_{,1}\end{aligned} \tag{E3}$$

$$K\equiv4\left(1-\frac{1}{4}\Lambda_{,1}\bar\Lambda_{,1}\right)ð\ln\Omega+\frac{1}{2}\Lambda_{,1}\bar\Lambda_{,1}(\bar{ð}\Lambda)_{,1}-\Lambda_{,-}\bar\Lambda_{,1}+\frac{1}{2}\Lambda_{,1}\bar\Lambda_{,-}+\frac{1}{2}\Lambda_{,1}^2\bar\Lambda_{,+}+\frac{1}{2}\Lambda_{,1}(ð\bar\Lambda)_{,1} \tag{E4}$$

and

$$L \equiv -\Lambda_{,1}\Lambda_{,+} - \frac{1}{2}\Lambda_{,1}\bar{\eth}\Lambda_{,1} - \Lambda_{,1}^2\bar{\eth}\ln\Omega. \tag{E5}$$

The above equation should be equivalent to (58) obtained in our work, but it is not easy to check whether or not they are actually the same.

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