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The interrelations between mathematics and philosophy in Leibniz´s thought

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> Les Mathematiciens ont autant besoin d'estre philosophes, que les philosophes d'estre mathematiciens.
>
> Leibniz to Malebranche, 13/23 March 1699(A II, 3, 539)

ABSTRACT

This paper consists of three main sections. In the first section we consider how early attempts at understanding the relationship between mathematics and philosophy in Leibniz's thought often suffered from being made within the framework of grand reconstructions guided by intellectual trends such as the search for "the ideal of system". In the second section, we proceed to recount Leibniz's first encounter with contemporary mathematics during his four years of study in Paris presenting some of the earliest mathematical successes which he made there. We argue that recently published letters and papers reveal how deeply intertwined his youthful mathematical reflexions were with important philosophical insights that, in turn, acted as guiding ideas for his mathematical research. Nowhere, we suggest, is this relation more evident than in Leibniz's early work on series and on the art of invention. Finally, in the third section we situate the central themes of the essays of present volume within the new understanding of the interrelations between philosophy and mathematics in Leibniz's thought briefly indicated in the opening section.

Key Words

MATHEMATICS AND PHILOSOPHY, IDEAL OF SYSTEM, MATHEMATICAL NOVICE, INFINITE SERIES, ARS INVENIENDI, UNIVERSAL CHARACTER, PHILOSOPHICAL INSIGHT, MATHEMATICAL PRACTICE, CASE-STUDIES

A mixture of philosophical and mathematical reflections and deliberations

The aim of this collection is to explore the ways in which mathematics and philosophy (metaphysics and broader philosophical questions) are interrelated in the letters and papers of Gottfried Wilhelm Leibniz. Taking up one of his most notable expressions, the essays collected in this volume are all in some way concerned with "a curious mixture of philosophical and mathematical thought" which characterizes Leibniz´s reflections and deliberations.[1] One of our principal aims in editing the present volume is to address the interrelations between mathematics and philosophy as far as possible without drawing on grand reconstructions which in the past all too often were based on insufficient evidence or what scholars conceived of as ad hoc programmatic stances, a typical example being Leibniz's so easily misunderstood pronouncement: "My metaphysics is all mathematics, so to speak, or could become so".[2] The difficulties presented by such reconstructions were already apparent when they emerged at the beginning of the last Century, during the second "Leibniz Renaissance" (the first having occurred in the 18th Century). Commentators such as Léon Brunschvicg followed an approach already adopted by the Neo-Kantian philosopher Cassirer, and searching to defend him from attacks by Russell and Couturat[3], criticized those who tended to confuse Leibniz's merely programmatic pronouncements with the position or rather positions which he actually maintained, which Brunschvicg termed his "real logic":

> We do not have the right to claim that Leibniz's philosophy is, properly stated, unambiguously and without ulterior motive, a *panlogism*. It would necessitate, in effect, that the relation of the predicate to the subject be achieved. In fact, the principles of 'the real logic, or a certain general analysis independent of algebra', as Leibniz put it in a letter to Malebranche, bring us back from traditional logic to differential calculus. The alternative expressed here was not completely satisfying for Leibniz in respect of his philosophical ambitions: for him, just as in the case of geometry for Descartes, differential calculus was only the most convincing 'sample' of his method, and he never gave up the project of a system of universal logic, in which the new mathematics would enter as a particular case. This is beyond doubt, but it only concerns, once more, the dream of what leibnizianism should be according to Leibniz – a dream condemned to be lost in the clouds of a tireless imagination and that for two centuries were believed to be without fruit.[4]

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[1] In an exchange with Basnage de Bauval, Leibniz revealed his intention to publish his correspondance with Arnauld and advanced what was to be expected from the content of his letters in these terms:"Il y aura un melange curieux de pensées philosophiques et Mathematiques qui auront peut-estre quelque fois la grace de la nouveauté"; Leibniz to Basnage de Bauval,3/13 January 1696(A II, 3, 121)

[2] Leibniz to L´Hospital, 27 December 1694 (A III, 6, 253) : "Ma metaphysique est toute Mathematique pour dire ainsi, ou la pourroit devenir".

[3] See Russell (1903).

[4] Brunschvicg (1912), 204. Unless otherwise stated, all the translations are ours.

But despite such criticism, Brunschvicg himself (as a reflection of his time) offered his own reconstruction. He was convinced that it was possible to start from a coherent set of theses thus setting the ground for what he conceived of as Leibniz´s "mathematical philosophy", while accepting that tensions and even inconsistencies might possibly remain. As a matter of fact, the use of such reading strategies was not uncommon until fairly recently amongst scholars seeking to elucidate from a variety of intellectual perspectives the way in which mathematics and philosophy are interrelated in Leibniz's thought.[5] To a certain extent, the assumptions underlying such reconstructions often prevented the study of the interrelations between mathematics and philosophy in their own right. A further difficulty with such approaches to the study of Leibniz´s thought was that it motivated scholars to make sometimes arbitrary choices in his mathematical and philosophical writings without any consideration of the time and material context of production. This tendency comes to light paradigmatically in the selection of unpublished material practiced by past editors. As a matter of fact, it was precisely there where the problem started.[6] While B. Russell´s attempt at systematic reconstruction flatly ignored Leibniz´s mathematical contributions, it is noteworthy that Cassirer and Brunschvicg, as reflected in the passage quoted above, mainly focused on the elaboration of the differential calculus taking it to be essential to understanding the interrelations between mathematics, physics, and metaphysics.[7] On the other hand, Couturat was motivated by G.Peano´s references to Leibniz "logical insights" and anticipations to search amongst his unpublished notes for Leibniz's many experiments with "formal calculi" and other programmatic sketches related to his goal to design new working tools - which Leibniz called "characteristics" – as well as any material deemed relevant to the vision of a universal grammar, and universal mathematics with logic as the sustaining link.[8]

[5] Concerning Leibniz scholarship in the 20th Century, see Albert Heinekamp (1989) who distinguished three main lines of study: first, the view that focuses on the ideal of system ("à la recherche du vrai système leibnizien"); second, the defense of the "structuralist" reading ("les interprétations structuralistes"); third, the view that denies any systematic structure in Leibniz´s philosophy ("refus du caractère systématique de la philosophie leibnizienne") which, according to Heinekamp, begins to be present only in the 80´s. The first line of reading may be regarded as the most widely represented amongst scholars interested in studying Leibniz from the perspective of the interrelations between mathematics and philosophy. Amongst French scholars, Serres (1968) and Belaval (1960) may be mentioned as cases where the indirect impact of mid-twentieth Century foundational philosophy of mathematics and logic can be detected. One could also mention the work of G.-G. Granger (1981), who emphasizes the epistemic value of Leibniz´s guiding ideas at the basis of his mathematical contributions (vis-à-vis the work of other great 17th Century contributions to mathematical analysis) but also sees Leibniz´ mathematical work as a possible anticipation of modern non-standard analysis. For a contextual study of the development of formal logic in the late 19th and early 20th Century and the exact role played by Leibniz´s work as a possible anticipation of modern approaches in logic and mathematics, see Peckhaus (1997). Despite revealing historical studies, the ´logicist´ trend is still represented explicitly in recent times, for instance, by Sasaki (2003), 405, who goes so far as to speak of "Leibniz's 'logicist-formalist' philosophy of mathematics".

[6] As Couturat (1903), Preface, already argued, previous editors selected from the Leibniz´s *Nachlass* the most relevant pieces to be published according to their specific intellectual interest; unavoidably, similar objections could be made against the editor of *Opuscules et fragment inédits*.

[7] See Russell´s Preface to the second edition of his book on Leibniz (1937); in composing his original book, Russell conceded that he ignored all material relevant to Leibniz´s mathematical studies and contributions, but still insisted that his "interpretation of Leibniz´s philosophy is still the same" as in 1900.

[8] See Couturat (1903), Preface, and Peckhaus (1997).

As noted, such lines of research by proceeding selectively led not only to the introduction of arbitrary divisions in Leibniz´s writings, often ignoring chronological order, but sometimes even entailed opposing readings of one and the same section of his works. For instance, the very same texts on *analysis situs* could be interpreted either along the lines of conceptual analysis (by commentators such as Cassirer) or along the lines of formal calculus and logical theory of relations (by commentators such as Couturat).

A last difficulty presented by this time-honored approach was its pretention to propose a picture of Leibniz's philosophy as a whole. As Dietrich Mahnke emphasized already in the early 1920s, it left readers with the unfortunate impression of facing a choice between different ´paintings´ of Leibniz, depending on whether or not mathematics was involved in the drawn portrait. Typical examples were, on the one hand, the project to which Mahnke gave the name "universal mathematics", as dealt with in various forms by Couturat, Cassirer, and Brunschvicg and, on the other hand, the so-called "metaphysics of individuation" which he identified with commentators such as Kabitz, Sickel, and Baruzi.[9] Once again, the elements at the basis of all these interpretations are to be found in Leibniz's writings, as well as in his rich and extensive correspondence.

The present collection of essays aims to elucidate how these different aspects in Leibniz´s thought relate to each other, evolving over time as his thinking unfolds. With this aim in mind, the papers in this volume take advantage of two fortunate circumstances. First, we are today in the privileged position of being able to take a fresh look at material which has long been available in conjunction with those letters and papers most recently published by the Academy edition. With the benefit of a considerable extended textual basis we propose to look at Leibniz´s mathematical practice while at the same time exploring his goals and the underlying values and ideas that guided his problem-solving activities. For example, we examine his notes and interactions with others in the process of studying mathematics in Paris under the guidance of Huygens, but we are also interested in exploring how his mathematical experience evolved, transforming his earlier philosophical views. For Leibniz, thinking unfolds and takes place in time, a fact which is beautifully reflected in his writings. The second fortunate circumstance that motivates scholarly research on the interrelations between mathematics and philosophy in Leibniz's thought relates to today´s growing interest in broadening the perspective of philosophy of mathematics, so that it engages historical case-studies. The new emphasis on the history of mathematical practice emphasizes how such practice is intertwined with philosophical ideas. The notion of a specific area of study called "philosophy of mathematics" began to develop only in the early 20th Century as an enterprise whose main concern was to deal with growing worries about foundational issues in mathematics. This logicist project left no room for historical case studies and the institutional contextualization of mathematical practice. Instead, it focused on deductive rigor, the elaboration of predicate logic, and the axiomatic method. Leaving behind such stringent formal concerns, the field has been opening up to include the study of the work of the research mathematician, and how that work interacts with philosophical ideas and other

[9] However, even Mahnke tried to rescue the idea of system by proposing a view which was conceived as a synthesis of both leading interpretations at his time in his book *Leibnizens Synthese von Universalmathematik und Individualmetaphysik* (Mahnke 1925).

cultural ingredients in broader historical context. This is the most welcome setting to return to the study of Leibniz, the research mathematician, who insisted upon the need to think philosophically while immersed in mathematical practice.

Encountering Mathematics in Paris

Although Leibniz had good political reasons for travelling to Paris in March 1672, it was the intellectual culture and above all the presence of some of the then greatest mathematical minds in Europe which persuaded him to prolong his stay, interrupted by a short visit to London, until October 1676.[10] In a letter written some two years after he had returned to Germany in order to take up his position as court counsellor and librarian in Hanover, he talks of devoting himself with an “almost limitless passion” to mathematics during those four heady years in the French capital.[11]

Leibniz’s initiation to mathematics is of course associated primarily with Christiaan Huygens. On numerous occasions in later life he expresses his considerable intellectual debt to the Dutch savant.[12] However, it was some time after Leibniz’s arrival in Paris before the two men actually met. Until late summer 1672, Leibniz was preoccupied with official tasks which his patron Johann Christian von Boineburg had assigned to him: the Egyptian plan, which Leibniz had himself devised in order to divert Louis XIV’s military ambitions away from Europe, and the recovery of Boineburg’s French rent and pension. Nonetheless, by September Leibniz had been introduced to Antoine Arnauld and Pierre de Carcavi, and soon thereafter there were encounters with the astronomers Giovanni Cassini and Ole Rømer.[13] This was the challenging intellectual environment he had long desired:

> Paris is a place where it is difficult to distinguish oneself: one finds the most capable men of the time in every kind of scientific endeavour and much effort and a little robustness is necessary in order to establish one’s reputation.[14]

It was not until the autumn that Leibniz was able to meet with Huygens for the first time. For the Dutch savant, effectively entrusted by Colbert with the planning and organization of the Académie Royale des Sciences, this was not a meeting with an absolute stranger. Leibniz was

[10] Leibniz to Duke Johann Friedrich autumn 1679(A II, 1 (2006), 761); Leibniz to Fabri, beginning of 1677(A II, 1 (2006), 442); Leibniz to Conring, 24 August 1677 (A II, 1 (2006), 563).
[11] Leibniz to the Pfalzgräfin Elisabeth, November 1678(A II, 1 (2006), 66)1.
[12] See for example Leibniz, *De solutionibus problematic catenarii vel funicularis in Actis Junii A. 1691. aliisque a Dn. I. B. propositis*(GM V, 255-8, 255); *Historia et origo calculi differentialis* (GM V, 398); Leibniz to Huygens, first half of October 1690 (A III, 4, 598); Leibniz to Remond, 10 January 1714 (GP III, 606): “Il est vray que je n’entray dans les plus profondes [sc. mathematiques] qu’apres avois conversé avec M. Hugens à Paris”.
[13] See Antognazza (2009), 140-1.
[14] Leibniz to Duke Johann Friedrich, 21 January 1675 (A I, 1, 491-2): “Paris est un lieu, ou il est difficile de se distinguer: on y trouve les plus habiles hommes du temps, en toutes sortes des sciences, et il faut beaucoup de travail, et un peu de solidité, pour y establir sa reputation”. See also Leibniz to Gallois, first half of December 1677 (A III, 2, 293-4) ; Leibniz to Bignon, 9/19 October 1693 (A I, 10, 590).

already becoming known in the Republic of Letters as a man of prodigious learning, who besides possessing exceptional knowledge in law and philosophy was "mathematically very inclined, and well versed in physics, medicine, and mechanics".[15] But, more specifically, Huygens's attention had been drawn to the promising young man from Germany almost a year and a half before they actually met. The Bremen-born secretary of the Royal Society, Henry Oldenburg, eager to promote the growth of the new science in Germany, had spoken enthusiastically of Leibniz in his letters. In his most recent communication, he referred to Leibniz's two tracts on motion, the *Hypothesis physica nova* and the *Theoria motus abstracti*, both of which with his help had been reprinted in London under the auspices of the Royal Society in 1671. Oldenburg's description of Leibniz was clearly intended to serve as an introduction:

> He seems of no ordinary intelligence, but is one who has examined minutely what great men, both ancient and modern, have had to say about Nature, and finding that plenty of difficulties remain, has set to work to resolve them. I cannot tell you how far he has succeeded, but I dare affirm that his ideas deserve consideration.[16]

Knowing full well that Leibniz had first been motivated to write on the theory of motion after he had read the laws of motion published in the *Philosophical Transactions* by John Wallis, Christopher Wren, and Huygens himself, Oldenburg proceeded to quote a passage from Leibniz questioning the conformity of the laws presented by Huygens and Wren to the abstract concepts of motion.

The mathematical novice

It is important to recognize that the young man initiated in mathematics in the autumn of 1672 was, as Oldenburg emphasized, steeped in both ancient and modern philosophy, while having a sound knowledge of jurisprudence and Protestant and Catholic theology. By contrast, as far as mathematics was concerned, Leibniz brought with him little more than what he had been able to glean from introductory textbooks of Harsdörffer or Cardano and from the mathematical exploits of Thomas Hobbes – an author he had read avidly while he was in Mainz. Although he described the two tracts on motion of his youth on one occasion to Nicolas Malebranche as "the beginnings of his mathematical studies"[17], he would later

[15] Boineburg to Conring, 22 April 1670, Gruber (1745), II, 1286-7: "Leibnizio literae tuae maximo sunt solatio. Est iuvenis 24 annorum, Lipsiensis, Juris Doctor: imo doctus supra quam vel dici potest, vel credi, Philosophiam omnem percallet, veteris et novae felix ratiocinator. Scribendi facultate apprime armatus. Mathematicus, rei naturalis, medicinae, mechanicae omnis sciens et percupidus; assiduus et ardens".

[16] Oldenburg to Huygens, 28 March 1671, Hall and Hall (1965-86) VII, 537-8/538-9: "Il ne semble pas un Esprit du commun, mais qui ait esplusché ce que les grands hommes, anciens et modernes ont commenté sur la Nature, et trouvant bien de difficultez qui restent, travaillé d'y satisfaire. Je ne vous scaurois pas dire comment il y ait reussi; j'oseray pourtant affirmer que ses pensees meritent d'estre considerées." See also Oldenburg to Huygens, 8 November 1670, Hall and Hall (1965-86) VII, 239-40/241-2.

[17] Leibniz to Malebranche, end of January 1693 (A II, 2, 659): "Au commencement de mes etudes mathematiques je me fis une theorie du movement absolu, où supposant qu'il n'y avoit rien dans le

generally dismiss them precisely because of their lack of sophistication in exact science. When he arrived in Paris, Leibniz was to all intents and purposes a mathematical novice.

> The desire to do justice to the favourable opinion which people had of me led me by good fortune to find new ways of analysis and to make discoveries in mathematics, although I had scarcely thought about this science before I came to France, for philosophy and jurisprudence had previously been the object of my studies from which I produced a number of essays.[18]

It is probable that the first meeting between Huygens and Leibniz took place in the Dutch savant's rooms in the Royal Library in Paris. During the course of their exchange, Leibniz mentioned with the remarkable boldness typical of his youth that he had discovered a method for summing infinite series. This method was the fruit of investigations into the Euclidean axiom "The whole is greater than its part", to which his attention had been drawn in Mainz, after reading the first part of Hobbes's *De corpore*.[19] In chapter eight, Hobbes argues that *Totum esse maius parte*, like all geometrical axioms, must be demonstrable.[20] Already then during his service at the court of Johann Philipp von Schönborn, Leibniz had considered *Totum esse maius parte* to be reducible to the only two types of unproved truths which he considered admissible, namely definitions and identities. By the time he met Huygens he had not only succeeded in producing a syllogistic proof that every part of a given magnitude is smaller than the whole, but also, using the principle of identity, he had developed his main theorem that the summation of consecutive terms of a series of differences could be carried out over an infinite number of terms – assuming only that the expected total sum approaches a finite limit.

Early successes in Paris

After listening to Leibniz's youthful deliberations, Huygens decided to put him to the test and asked him to determine the sum of the infinite series of reciprocal triangular numbers.[21]

$1/1 + 1/3 + 1/6 + 1/10 +$ etc.

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corps que l'étendüe et l'impenetrabilité, je fis des regles du mouvement absolu que je croyois veritables, et j'esperois de les pouvoir concilier avec les phenomenes par le moyen du systeme des choses."

[18] Leibniz to Pellisson-Fontanier, 7 May 1691 (A I, 6, 195-6): "L'envie de me rendre digne de l'opinion favorable qu'on avoit de eue de moy, m'avoit fait faire quelques decouvertes dans les Mathematiques, quoyque je n'eusse gueres songé à cette science, avant que j'estois venu en France, la philosophie et la jurisprudence ayant esté auparavant l'objet de mes études dont j'avois donné quelques essais." See also Leibniz to Duke Johann Friedrich, 29 March 1679 (A I, 2, 155); Leibniz to Duke Ernst August, early 1680? (A I, 3, 32); Leibniz to Foucher, 1675 (A II, 1 (2006), 389); *De numeris characteristicis ad linguam universalem constituendam* (A VI, 4, 266).

[19] See Leibniz, *Historia et origo calculi differentialis* (GM V, 395).

[20] I, 8, §25; Hobbes (1651), 72.

[21] See Hofmann (1974), 15.

The result of this summation was already known to him, but he had not yet put this into print. Huygens also suggested that Leibniz consult two books which he had previously cited, but had not read: Wallis's *Arithmetica infinitorum* and the *Opus geometricum* of Grégoire de Saint-Vincent.

Developing a principle found in the *Opus Geometricum*, that the line segments representing terms of the geometrical progression must be considered to start from the same place, Leibniz recognized that the differences of consecutive terms are proportional to the original series.

A—|——|————|——————————————B

1/27 2/27   2/9   C   2/3

From here can be read off

$2/3 + 2/9 + 2/27 + \ldots = 1$

Or, more generally

$1/t + 1/t^2 + 1/t^3 + \ldots = 1/(t - 1)$

Decisively, Leibniz was able to show how conceptually a general method could be applied. Thus, by taking AB = 1, AC = 1/2, AD = 1/3, AE = 1/4, he achieved the relation

$1/1.2 + 1/2.3 + 1/3.4 + 1/4.5 + \ldots = 1$

and then, multiplying by 2, produced the result which Huygens had sought, namely

$1/1 + 1/3 + 1/6 + 1/10 + \ldots = 2$

Writing to Oldenburg on 16/26 April 1673, Leibniz does not seek to hide his joy at this early success:

> But by my method I find the sum of the whole series continued to infinity, 1/3, 1/6, 1/10, 1/15, 1/21, 1/28 etc.; indeed, I do not believe this to have been laid before the public previously for the reason that the very noble Huygens first proposed this problem to me, with respect to triangular numbers, and I solved it generally for numbers of all kinds much to the surprise of Huygens himself.[22]

Nor did Leibniz stop here, but also succeeded in obtaining the sum of the reciprocals of pyramidal numbers as well as the sum of reciprocal trigono-trigonal numbers.

[22] Leibniz to Oldenburg, 26 April 1673 (A III, 1, 83-9,88): "At ego totius seriei in infinitum continuatae summam invenio methodo mea: 1/3 1/6 1/10 1/15 1/21 1/28 etc. in infinitum; quod jam publice propositum esse, vel ideo non credidi, quia a Nobilissimo Hugenio mihi primum propositum est hoc problema in numeris triangularibus; ego vero id non in triangularibus tantum, sed et pyramidalibus etc. et in universum in omnibus ejus generis numeris solvi ipso Hugenio mirante".

D = 1 + 1/5 + 1/15 + 1/35 + 1/70 + … = 4/3

The exuberance which Leibniz felt at achieving such early success – and being able to impress Huygens at the same time – can be gauged from the language he employed in what he evidently hoped would be his first mathematical publication, having already seen two letters to Oldenburg on his theory of motion published in the *Philosophical Transactions*. Most articles which appeared in the new scientific journals of the second half of the seventeenth century took the form of letters to the editor. It was therefore perfectly natural for Leibniz to set out some of his newly achieved mathematical results in a long letter to Jean Gallois, editor of the *Journal des Sçavans* and secretary of the Académie Royale des Sciences.[23] Unfortunately for Leibniz, and no doubt unbeknown to him at the time, the French journal temporarily ceased publication on 12 December 1672, that is to say, around the time his letter was sent. By the time publication was resumed on 1 January 1674, Leibniz's contribution would have been considered out of date, not least in view of the author's mathematical development during the intervening twelve months.

Mathematical and philosophical deliberations on infinity

The *Accessio ad arithmeticam infinitorum*, as the letter to Gallois was entitled, provides evidence of the remarkable growth in Leibniz's understanding of the nature of concept of infinity compared to the views he had set out little over a year earlier in his *Theoria motus abstracti*. Whereas there he had approached the continuum ontologically, seeking to reconcile infinite divisibility with the actual existence of parts by postulating points in such a way that they could be conceived as constitutive entities, he now appeals to the argumentative force provided by genuine mathematical proofs, such as those he had shown to Huygens, where there is an infinite progression within finite limits.

> He namely who is led by the senses will persuade himself that there cannot be a line of such shortness, that it contains not only an infinite number of points, but also an infinite number of lines (as an infinite number of actually separated parts) having a finite relation to what is given, unless demonstrations compel this.[24]

Part of what Leibniz sets out to achieve in the *Accessio* is to demonstrate that infinite number is impossible. Employing a strategy used in numerous other contemporary letters and papers, he develops his postion in contrast to the position put forward by Galileo in the *Discorsi e dimostrazioni matematiche*, where infinite number, understood as the number of all numbers, is compared to unity. Galileo argued that every number into infinity had its own square, its own cube, and so on, and that therefore there must be as many squares and cubes as there are roots or integers, which however is impossible. The Pisan mathematician famously concludes

[23] See Bos (1978), 61.

[24] Leibniz for Gallois, end of 1672 (A II, 1 (2006), 342): "Quis enim sensu duce persuaderet sibi, nullam dari posse lineam tantae brevitatis, quin in ea sint non tantum infinita puncta, sed et infinitae lineae (ac proinde partes a se invicem separatae actu infinitae) rationem habentes finitam ad datam; nisi demonstrationes cogerent."

from this that quantitative relations such as those of equality or "greater than" or "less than" do not apply when it comes to the infinite. That is to say, Galileo effectively negated the validity of the axiom *Totum esse maius parte* with respect to infinite numbers.

Leibniz compared Galileo's conclusion to Grégoire's negation of the validity of the axiom in horn angles in his *Opus geometricum*. In both cases, Leibniz found that it was a mistaken concept of infinity which had led to denying the universality of the axiom: "that this axiom should fail is impossible, or, to say the same thing in other words, the axiom never fails except in the case of null or nothing".[25] Precisely the universal validity of the axiom leads to the conclusion that infinite number is impossible, "it is not one, not a whole, but nothing". Employing an argument which is also found in contemporary algebraic studies, Leibniz is able to proclaim that not only is 0 + 0 = 0, but also 0 – 0 = 0. Consequently, an infinity which is produced from all units or which is the sum of all must in his view be regarded quite simply as nothing, about which, therefore, "nothing can be known or demonstrated, and which has no attributes".[26]

Alongside providing evidence of the relative sophistication of Leibniz's mathematical work by the end of 1672, the *Accessio* provides the earliest example of the intimate relation between philosophy and mathematics in his thought.[27] Right at the beginning, he asserts that the method of indivisibles is to be ranked among those things capable of vindicating the incorporeality of the mind. This assertion refers on the one hand to the geometrical method of Cavalieri, Torricelli and Roberval which had since been arithmetized by Wallis and on the other hand to one of the philosophical doctrines of his youth, namely that the immortality of the soul could be guaranteed through its location in a geometrical point. Nor was this remark at the beginning of his Paris sojourn simply a remnant of the philosophy he had brought with him from Mainz. Even in the *Système nouveau* (1695), where Leibniz considers the nature of the communication of substances and of the union between substance and body, he sees points as providing the ontological interface between the various spheres, distinguishing thereby what he calls "metaphysical points" from those of physics and mathematics.

Traces of a general *Ars Inveniendi*

Leibniz brings philosophical incisiveness to mathematics, analyzing concepts which contemporary mathematicians without his philosophical bent were inclined to use unreflectively. "For me the mark of imperfect knowledge," he writes to Malebranche, "is when the subject has properties of which one has not yet been able to provide a

[25] *Ibid*, 349: "at Axioma illud fallere impossibile est, seu quod idem est, Axioma illud nunquam, ac non nisi in Nullo seu Nihilo fallit, Ergo Numerus infinitus est impossibilis, non unum, non totum, sed Nihil."
[26] Leibniz, *Mathematica* (A VII, 1, 657): "Nam 0 + 0 = 0. Et 0 – 0 = 0. Infinitum ergo ex omnibus unitatibus conflatum, seu summa omnium esr nihil, de quo scilicet nihil potest cogitari aut demonstrari, et nulla sunt attributa." See also *De bipartitionibus numerorum eorumque geometricis interpretationibus* (A VII, 1, 227).
[27] See Beeley (2009).

demonstration".[28] He cites the examples of the concept of a straight line employed by the geometers without having a sufficiently clear idea of what the concept involves, and of the notion of extension in respect of bodies, which clearly presupposes that there is something extended or repeated.

Conversely, Leibniz ascribed to mathematics an essential role in extending the limits of human knowledge in the context of his philosophical program of *ars inveniendi*. Shortly after he had left Paris for his new post in Hanover, he wrote that he valued mathematics solely because one could find in it "traces of a general art of invention".[29] Admittedly, Leibniz often described mathematics and indeed philosophy in terms of means to a particular end. But his evaluation of mathematics in respect to discovering new truths reflected in part the relatively recent emergence of mathematical analysis as a discipline, complementing the traditional model of a rigorously deductive science with which the concept of geometrical method had long been identified. Put simply, mathematics could now be considered to encompass both analysis and synthesis according to the ancient model of scientific method.[30] Moreover, these two basic paths to new knowledge would be further enhanced and vastly extended by the implementation of a suitable, that is to say exact system of symbols which would mirror not only the structure of concepts but also thought itself, which could thereby be effectively replaced by a symbolic calculus.

The importance of such a calculus is formulated explicitly in his remarks on George Dalgarno's *Ars signorum*, probably written after his return to Paris following his first visit to London in 1673. Seeking to proceed further than contemporary exponents of artificial languages, he describes his universal character as being among "the most suitable instruments of the human mind, having namely an invincible power of invention, or retention, and judgment. Then this will accomplish in all matters of things, which arithmetical and algebraic symbols accomplish in mathematics."[31]

Building on his early fascination with the art of combinations, Leibniz recognized that a synthetic or deductive model proceeding systematically from simple elements, representing the alphabet of human thought[32], would not only serve as a suitable means of presenting existing knowledge, but also of acquiring entirely new knowledge. In this way, *ars*

[28] Leibniz to Malebranche, end of January 1693 (A II, 2, 661): "La marque d'une connoissance imparfaite chez moy, est, quand le sujet a des proprietiés, dont on ne peut encor donner la demonstration."

[29] Leibniz to the Pfalzgräfin Elisabeth?, November 1678 (A II, 1 (2006), 662): "Mais pour moy je ne cherissois les Mathematiques, que par ce que j'y trouvois les traces de l'art d'inventer en general […]." See also Leibniz to Duke Johann Friedrich, February 1679 (A II, 1 (2006), 684).

[30] See Leibniz, *De arte characteristica inventoriaque analytica* (A VI, 4, 321): "Duobus maxime modis homines inventores fieri deprehendo, per Synthesin scilicet sive Combinationem et per analysin; utrumque autem vel facultati natura usuve comparatae, vel methodo debere." See also *ibid*. (A VI, 4, 329).

[31] Leibniz, *Zur Ars signorum von George Dalgarno* (A VI, 3, 170): "sed vera Characteristica Realis, qualis a me concipitur, inter [ap]tissima humanae Mentis instrumenta censeri deberet, [invin]cibilem scilicet vim habitura et ad inveniendum, et ad retinendum et ad dijudicandum. Illud enim efficient in omni material, quod characteres Arithmetici et Algebraici in Mathematica." See also Antognazza (2009), 162.

[32] See Leibniz, *De alphabeto cogitationum humanarum* (A VI, 4, 271-2).

combinatoria could be understood properly as an important part of the art of invention. In his letter to Jean Gallois of December 1678, he writes:

> I am more and more convinced of the utility and reality of this general science, and I see that few people have grasped its scope. But in order to make this science easier and so to speak sensible, I want to employ the characteristic of which I have spoken to you a number of times, and of which algebra and arithmetic are just samples. This characteristic consists in a certain writing or language, (for whoever has the one may have the other) which corresponds perfectly to the relations of our thoughts. This science will be quite different from everything which one has planned up to now. For the most important part has been overlooked, which is that the characters of this writing must be conducive to discovery and judgment as they are in algebra and arithmetic.[33]

Evidently, one of the by-products of Leibniz's early work on mathematics, and particularly algebra, during his stay in Paris was to recognize the full potential for deriving a symbolic calculus in order to extend human knowledge. On occasion Leibniz describes his *characteristica universalis* as a "universal algebra", with whose help it would in his view be just as easy to make discoveries in ethics, physics or mechanics as it is in geometry.[34] An essential part of this consideration is that the rigor of mathematics will also apply here, enabling us to have no less certainty about God and the mind than about figures and numbers. In this way, Leibniz suggests, inventing machines would be no more difficult than constructing a problem in geometry. He expresses the full promise of universal character in this context in his letter to Oldenburg of 28 December 1675:

> This algebra (of which we deservedly make so much) is only part of that general system. It is an outstanding part, in that we cannot err even if we wish to, and in that truth is as it were delineated for us as though with the aid of a sketching machine. But I am truly willing to recognize that whatever algebra furnishes to us of this sort is the fruit of a superior science which I am accustomed to call either Combinatory or Characteristic, a science very different from either of those which might at once prong to one's mind on hearing those words […] I cannot here describe its nature in a few words, but I am emboldened to say that nothing can be imagined which is more effective for the perfection of the human mind.[35]

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[33] Leibniz to Gallois, 19 December 1678 (A III, 2, 570): "Je suis confirmé de plus en plus de l'utilité et de la realité de cette science generale, et je voy que peu de gens en ont compris l'étendue. Mais pour la rendre plus facile et pour ainsi dire sensible; je pretends de me servir de la Characteristique, dont je vous ay parlé quelques fois, et dont l'Algebre et l'Arithmetique ne sont que des échantillons. Cette Characteristique consiste dans une certaine ecriture ou langue, (car qui a l'une peut avoir l'autre) qui rapporte parfaitement les relations des nos pensées. Ce charactere seroit tout autre que tout ce qu'on a projetté jusqu'icy. Car on a oublié le principal qui est que les caracteres de cette écriture doivvent servir à l'invention et au jugement, comme dans l'algebre et dans l'arithmetique".

[34] Leibniz to Mariotte, July 1676 (A II, 1 (2006), 424): "ce seroit pour ainsi dire une algebre universelle, et il seroit aussi aisé d'inventer en morale, physique ou mechanique, qu'en Geometrie".

[35] Leibniz to Oldenburg, [18]/28 December 1675 (A III, 1, 331): "Haec algebra, quam tanti facimus merito, generalis illius artificii non nisi pars est. Id tamen praestat, ut errare ne possimus quidem, si velimus, et ut veritas quasi picta velut machinae ope in charta expressa deprehendatur. Ego vero

But, by reading this kind of declarations, one should also keep in mind Brunschvicg's lucid warning and not confuse "the dream of what leibnizianism should be according to Leibniz" with his "real logic". Indeed the same letter to Oldenburg begins with an important *caveat*: "we seem to think of many things (though confusedly) which nevertheless imply contradiction". Here again the motivation comes from mathematics, the basic example of contradictory notion mentioned being precisely the one presented in the *Accessio ad Arithmeticam infinitorum*: "the number of all numbers" (A II, 1, 393). This a typical example of a joining together of apparently simple ideas (unit and addition), which produces an impossible object (the sum of all units or "number of all numbers"). As emphasized by the *De synthesi et analysi universali*, one must then take care that "the combinations do not become useless through the joining-together of incompatible concepts". If the *universal character*, based on the constitution of an "alphabet of human thoughts" and the full development of an *ars combinatoria*, is the goal to obtain, one should not forget that it implies nothing less than a complete analysis of human thoughts. Before reaching this goal, which may well be inaccessible to finite human beings, one has to be very cautious with symbolic manipulations, keeping in mind that they must be complemented by demonstrations of possibility: "one must be especially careful, in setting up real definitions, to establish their possibility, that is, to show that the concepts from which they are formed are compatible with each other".[36] Since the main field in which Leibniz developed such an "analysis of thoughts" and such a work on definitions was precisely mathematics, this will be enough to indicate the complexity of the interrelations between the various domains under consideration.

Presentation of the collection of essays

As should be clear from the historical sketch proposed above, mathematics and philosophy evolved *in tandem,* fruitfully interacting in Leibniz's work, influencing each other in multifarious ways throughout the different periods of his intellectual life. Yet relatively few studies have been devoted to the investigation of these complex interrelations. One reason may be the fact that Leibniz´s scholarship has for too long been rather compartmentalized, with the study of metaphysics on the one side, and the study of mathematics on the other, each of these pursuits involving technicalities of its own which would require it to be placed within the context of the time. One could also invoke the changing perceptions in the history of mathematics itself, which in the last thirty years has moved away from "internalist" accounts advocated by the founders of the discipline. The availability or rather lack of availability of most of Leibniz's mathematical papers of course did not help. Until fairly recently, commentators were largely reliant on articles which Leibniz published during his lifetime or the few texts which in intervening years found their way into print. Over the last twenty years things have changed dramatically for the better. Progress in the edition of the Academy Edition of Leibniz's letters and papers has made available to readers many of the previously

agnosco, quicquid in hoc genere praebet algebra, non nisi superioris scientiae beneficium esse, quam nunc combinatoriam, nunc characteristicam appellare soleo, longe diversam ab illis, quae auditis his vocibus statim alicui in mentem venire possent".

[36] Leibniz, *De synthesi et analysi universali seu Arte inveniendi et judicandi*: (A VI, 4, 540).

unpublished drafts or letters long hidden from public view. Material edited in Series VII (Mathematical Papers) as well as in Series III (Mathematical and Scientific Correspondence), not forgetting Series I (General and Political Correspondence), Series II (Philosophical Correspondence), and Series VI (Philosophical Papers) shows just how closely related Leibniz's metaphysical and mathematical deliberations sometimes were.

Together with the newly available material, the papers in this collection also take advantage of the growing interest amongst philosophers and historians of mathematics in addressing the work of the research mathematician, his mathematical practice in specific institutional contexts, often in exchange with others. Thus the scholar enters the workshop of the mathematician to explore underlying values, guiding ideas, methods and working tools, a strategy which in the case of the mathematician-philosopher Leibniz seems most promising. In his paper, Philip **Beeley** invites us to meet Leibniz, the philosopher mathematician who could not help but think as a mathematical philosopher. The paper shows Leibniz´s great concern to account for the explanatory power of the mathematical sciences as applied to our understanding of the natural world, an interest that can be traced to earlier writings from the Mainz Period (before arriving in Paris). Leibniz's ultimate motivation was his recognition of the usefulness of the mathematical sciences with a view to the improvement of the human condition. The deep interconnection Leibniz saw between theory and practice inspired him to discuss mathematical tools such as the notion of "negligible error" used in justifying infinitesimal techniques in connection with practical matters and its applicability in the natural world. In his discussion of "negligible error," Leibniz revisits his early interest in Archimedean ideas further developed by his later mathematical studies, a conjunction which is not divorced from its special place in the search for wisdom. Discussions such as this and related issues reveal that the dialogue between philosophy and mathematics was not just a novelty brought about by his mathematical studies in Paris (1672-1676).

The emphasis on pragmatic considerations in Leibniz´s mathematical practice allows us to trace an important evolution in his thought. Careful scholarship reveals that earlier versions of this great project of an *ars combinatoria*, which if fully realized would have led to establishing an "alphabet of human thoughts", and which a very young Leibniz once assumed was objectively possible, were abandoned. In his paper "The difficulty of being simple", David **Rabouin** shows that with the start of his studies in Paris, Leibniz was motivated seriously to question the feasibility of such a project. In particular, the study of mathematics played a decisive role in this evolution. The *Accessio ad arithmeticam infinitorum* and the demonstration of the impossibility of a "number of all numbers", as already noted, as well as his work on the "arithmetic quadrature of the circle" culminating in another demonstration of impossibility, and the study in number theory, gave Leibniz new insight into crucial questions about the possibility (and impossibility) of notions. Accordingly, the form that an "analysis of human thoughts" should take evolved considerably during this period; Leibniz´s mathematical practice transformed his way of engaging with mathematical concepts.

The question of why mathematics not only applies to the natural world but also helps us to find explanations of natural phenomena was also of great importance to Leibniz. He sought a middle pathway between Bacon's empiricism and the rationalism of Descartes as he framed his conception of scientific method. As Emily **Grosholz** argues in "Leibniz and the Philosophical Analysis of Time," he came to think that mathematics and experience were limited approaches to the study of nature when taken in isolation, and thus should be considered *in tandem*. Leibniz calls upon metaphysics, in particular the principles of Continuity and Sufficient Reason, to play a harmonizing role, as he sought to answer the question about how the two scientific activities (theoretical analysis and empirical compilation) should be combined in practice. She argues in particular that metaphysical principles play a substantive role in his account of time. Another remarkable aspect that comes to the fore in Grosholz´s study is a conception of scientific research which involves a set of values, perhaps the most important of which is the idea that the use of mathematics applied to nature requires careful philosophical reflection.

The complete and carefully designed study of the *Quadratura Arithmetica circuli ellipseos et hyperbolae* (1675/76) which Leibniz himself originally intended to publish, was meticulously edited by Eberhard Knobloch and first published in 1993. This edition offered a welcome occasion for a revival of interest in the study and assessment of Leibniz's views on infinitesimals, including Leibniz´s use and interpretation of the role of "syncategorematical" expressions.[37] As the historian of mathematics Henk Bos (2001) showed in his study of the role of exactness in Descartes´ work on geometry, discernible just under the surface of mathematical working tools lie implicit epistemic values that operate in mathematical practice, but often are never made explicit by the mathematician. Thus, later scholarly debates concerning the relevant values cannot be easily settled. In his essay "Analyticité, équipollence et théorie des courbes chez Leibniz", Eberhard **Knobloch** likewise approaches Leibniz's mathematical writings by studying the way in which he conceived of the relationship between "geometricity" and "analyzability". He also emphasizes the way that Leibniz´s thought evolved throughout his mathematical research. For instance, Leibniz starts out by borrowing notions from Cartesian geometry, but reworks them while progressively transforming their use and meaning. As an illuminating example of this process, Knobloch discusses the Leibnizian notion of "equipollence" which reveals itself as one of the key tools for expanding the range of objects (curves) that can be treated mathematically by using his new methods.

Epistemic values also play a key role in Leibniz's invention of the differential calculus. The philosophical project of a "general character", which turned out to be one of Leibniz´s most fruitful guiding ideas, was central to the search for a symbolic calculus able to express techniques stemming from infinitesimal analysis in an economical way. This may be part of the reason why Leibniz was often unconcerned about acknowledging results previously established by other mathematicians. In his essay on "Leibniz as second inventor", Siegmund **Probst** delivers a careful investigation, based on recently edited material, of the relationship between Leibniz and his predecessors, especially Isaac Barrow and Pietro Mengoli. Although some results were the outcome of Leibniz's intensive study of the relevant sources of the time often overlap, Probst argues, the Hanoverian philosopher-mathematician Leibniz was probably more concerned with the introduction of new methods and a new kind of access to those results, which only a symbolic calculus operating at a higher level of abstraction could

[37] Concerning this issue, see the material gathered in Jesseph and Goldenbaum (2008).

provide.[38] To take up Leibniz's own triumphant words: "Most of the theorems of the geometry of indivisibles which are to be found in the works of Cavalieri, Vincent, Wallis, Gregory, and Barrow are immediately evident from the calculus".[39]

The concept of infinity and its historical adjunct, the concept of continuity, constitutes in many ways an important focus of the meeting of mathematics and philosophy in Leibniz. Metaphysical deliberations on the infinitely small and the infinitely large abound in his letters and papers. Indeed the concept of the continuum effectively constitutes a thread through the whole of his philosophical thought from the *Hypothesis physica nova* of his youth through to the doctrine of monads of his maturity. Although he tells us already in *De quadratura arithmetica circuli* that metaphysical considerations in respect of the infinite are of no consequences when mathematical rigour can be shown to obtain, he nonetheless recognizes that precisely the concept of *infinite parvum* cannot of itself be above philosophical analysis if it is to serve its function of explaining the applicability of the infinitesimal calculus to those natural phenomena which are its object.[40]

Since Leibniz developed and promoted infinitesimal analysis and since also he claimed to be an ardent supporter of the existence of actual infinite in nature[41], one might think that he was furthermore an ardent supporter of actual infinite entities in mathematics. But this is not what the sources tell us. Quite on the contrary, Leibniz regularly insists on the fact that he does not believe in actual infinite in mathematics. This raises many questions which have long remained hidden in reconstructive approaches and which only now are being raised. First, what is exactly his view, or perhaps better, what were his views, on the ontological status of the infinite in mathematics, be it the infinitely large or the infinitely small? Second, how can we reconcile two apparently incompatible theses according to which Leibniz on the one hand supported the existence of an actual infinite in nature and on the other hand denied its

[38] For a discussion of the different "levels of abstraction" in Leibniz´s mathematical practice, see Breger (2008b) and, in particular, in connection with the present idea, see Breger (2008a), 193: "(…) it was only by proving many theorems and gaining experience with the new material that Leibniz arrived at the higher level of abstraction from which he was able to recognize and explicitly formulate the rules of calculus".

[39] Leibniz, *Analyseos tetragonisticae pars tertia* (A VII, 5, 313): "Pleraque theoremata Geometriae indivisibilium quae apud Cavalerium, Vincentium, Wallisium, Gregorium, Barrovium extant statim ex calculo patent".

[40] See Leibniz to Schmidt, 3 August 1694 (A I, 10, 499): "Novum Calculi Analytici genus a me in Geometriam introductam [...] Usum inprimis habet ad ea analysi subjicienda, in quibus quantitates finitae determinantur interveniente aliqua consideratione infiniti, quemadmodum saepe praesertim cum Geometria applicatur ad naturam. Ubique enim infinitum Naturae operationibus involvitur. See also Leibniz to Kochański, 10/20 August 1694 (A I, 10, 513-14); Leibniz to the Electress Sophie for the Duchess Elisabeth Charlotte of Orléans, 28 October 1696 (A I, 13, 85): "Et c'est une chose estrange, qu'on peut calculer avec l'infini comme avec des jettons, et que cependant nos Philosophes et Mathematiciens ont si peu reconnu combien l'infini est mêlé en tout".

[41] Leibniz to Foucher, end of June 1693 (A II, 2, 713) : "Je suis tellement pour l'infini actuel, que au lieu d'admettre que la nature l'abhorre, comme l'on dit vulgairement, je tiens qu'elle l'affecte partout, pour mieux marquer les perfections de son auteur. Ainsi je crois que qu'il n'y a aucune partie de la matiere, qui ne soit, je ne dis pas divisible mais actuellement divisée, et par consequent la moindre particelle doit estre considerée comme un monde plein d'une infinité de creatures differentes."

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existence in mathematics? Is it not the case that we have to accept that there is an infinite number of things in the world? And if so, how can we express this infinity?

In “Leibniz’s Actual Infinite in Relation to his Analysis of Matter”, Richard **Arthur** tackles precisely the last problem mentioned, namely, how to understand why Leibniz denied the existence of an infinite number in mathematics while positing actual infinity in Nature - such as in the infinite division of matter or in the plurality of simple substances. First of all, he sets out to defend Leibniz’s views on the mathematical infinite as a fiction against accusations of inconsistency raised in recent literature. Such claims are often based on the anachronistic point of view of our modern “Cantorian” theory of the infinite. In the remaining part of the paper, Arthur confronts a dilemma already raised by Russell: if infinite plurality is just a fiction, depending on the way we perceive things, then there appears to be no way to assert that there is an infinite plurality of substances or that matter is actually divided into an infinity of parts. If, on the contrary, there is a real, mind-independent, infinite plurality of substances, or infinite plurality of parts of matter, then one must acknowledge infinite pluralities which are not fictions and which would correspond to the actual infinite wholes that Leibniz wants to reject from mathematics. The solution to the dilemma, Arthur argues, is that one should not confuse the plurality itself with its perception as a unity. On this basis, it is possible to understand how the infinite plurality of parts of matter is reconcilable with the infinite plurality of substances, assuming, as Leibniz repeatedly argues, that these parts are real.

In “Comparability and Infinite Multitude in Galileo and Leibniz”, Sam **Levey** returns to the contrasted positions of those two thinkers on the status of “infinite multitude”. Galileo’s paradox, which shows that one infinite multitude can be put in one-to-one correspondence with another even when one is a proper part of the other (such as in the case of natural numbers and their squares), was instrumental in Leibniz’s reflections. In the *Accessio ad arithmeticam infinitorum*, as we already mentioned, he argues against the Pisan Mathematician that infinity should not be compared with unity (which is “equal” to its powers), but with zero or “nothing”. According to Leibniz, this means that there is no such thing as an infinite number, and more generally that a mathematical infinite cannot be considered as a “whole”. Hence emerges a way of saving Euclid’s axiom (“the whole is greater than the part”) which enters as essential ingredient in Galileo’s paradox. This is, however, only one amongst a number of strategies to save the axiom. Another possibility, often time ascribed to Galileo himself, is that that the infinite falls outside of the realm of quantifiable entities (*quanti*). Levey reexamines these interpretations in detail in order to assess the pertinence of Leibniz’s strategy and its strength.

Finally, in “Leibniz on The Elimination of Infinitesimal”, Douglas **Jesseph** studies the status of infinitesimal quantities in Leibniz. Recent literature, inspired by the rediscovery of the *Quadratura arithmetica circuli* has put a lot of emphasis on the so called “syncategorematical” interpretation. According to this view, infinitesimals are “useful fictions” in the sense that they can be eliminated through a paraphrase involving only finite quantities. Following the seminal investigation published by Henk Bos in 1974, Jesseph argues that this is only one amongst two strategies to “find truth in fiction”. He proposes to contrast each strategy as a “syntactic” (or “proof theoretic”) and a “semantic” (or “model theoretic") approach. In the semantic approach, one seeks to show that, even if reference to infinitesimals cannot be eliminated from the mathematical discourse, it will never lead from truth to falsehood. The paper gives an example of these two strategies in Leibniz's texts and seeks to explain why they had to coexist in his mathematical practice.

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