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Simple formal languages and the structure of spacetime

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1. Introduction

The description of any physical system makes always reference to a spacetime. For example, in classical physics one refers to the Galilean spacetime; in special relativity one refers to Minkowski spacetime; in general relativity one refers to a 4-dimensional Lorentzian manifold; in Kaluza-Klein theories and string theories one refers to higher dimensional metric manifolds.

The technique of assuming a smooth manifold to construct the framework for the description of physical systems, is so common that very rarely one stops to think whether it is completely justified. But it is in the study of quantum gravity that this question acquires strong relevance. There exist many indications that make one to think that probably the structure of the spacetime is very different from the smooth one that has been assumed. For example, there are works in which the quantum operator associated to the area of a closed 2-surface has been calculated, and it was found that it has a discrete spectrum [1]. This work has also been extended to the case of volume operator and length operator.

This seems to indicate that to construct the theoretical framework appropriate for the description of quantum gravity, the assumption of a smooth manifold for the basic structure of the spacetime is unjustified; and furthermore, it would probably lead to the wrong direction.

If one accepts the idea that probably the quantum structure of the spacetime is discrete, one is faced with the question of which is the appropriate description of this structure.

Our suggestion for this task is that if one studies a very *small* portion of the spacetime, then the description must necessarily be simple. To specify more what we mean by this statement, let us recall that in other works we have study the implications of some realistic reasonable assumption in the context of the early Universe; the assumption been that [4]:

P 1.1 *It is possible to completely describe a finite system in terms of a finite sentence of a formal language.*

This apparent mild assumption has important physical consequences, as we have shown in previous works [4][3]; but it is also important to remark that this statement is in complete agreement with the discrete expected nature of the spacetime that is suggested by the studies of quantum gravity.

When one takes the above principle, and applies it to *small* systems, like a small portion of the spacetime, one is led to think that not only the sentence of the formal language required to describe the small system would be rather short; but also that the structure of the formal language required would not be too complex; in other words it should be simple.

Therefore in this work we present an approach to the description of the structure of the spacetime, coming from the study of the construction of simple formal languages.

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Since the description of any physical system makes always reference to a spacetime, we consider that the description of the structure of the spacetime is one of the most profound philosophical and physical questions.

In section 2 we present the minimum structure of a simple formal language. The structure of what we call elementary cells, are worked out in section 3. The implicit structure of natural numbers is described in section 4. This structure is included in the language in section 5. A particular case of a four atomic relation is presented in section 6. We reserve the last section 7 for some questions.

2. A simple formal language

To define a formal language [2] we need to determine at least the alphabet, the rule of composition and the set of accepted sentences.

We will assume that we have at our disposal a collection of symbols that form the alphabet; and we will use Latin and Greek letters to denote the elementary symbols.

In order to construct a language it is convenient to have in mind an idea about what the symbols of the alphabet refer to, sometimes also called an interpretation. We will think of each symbol as a relation; in particular given any pair of relations A and B, there is another relation u which connects A with B. Graphically

$$A \xrightarrow{u} B,$$

that is u has a direction, from A to B. So it is observed that u is completely characterized by the pair "A B", in this order. In other words, this defines the constructive relation, or from the point of view of formal languages, the constructive operation.

It is then convenient to assign a symbol to this operation; we will use *, and we will write

$$u : A * B,$$

to indicate that u is the letter that represents (is the elementary symbol of) the relation $A * B$.

Note that the symbols ":" is used to define an assignment.

We will say that A is the first argument of the relation u, and B its second argument.

We see that A, B, and u are all relations; but in the previous considerations we did not care what A (or B) was relating. On the contrary we know that u relates A with B. In this situation we call A and B atomic relations, and u a relation (not atomic). Whenever possible we will try to refer to atomic relations by capital letters and to relations by lower case letters.

Given another relation C, we can relate u with C, or in other words, we can apply the constructive operation * to u and C. Let μ denote this new relation, that is

$$\mu : u * C.$$

In order to be able to represent μ in terms of the atomic relations, we now introduce the use of parenthesis, and write

$$\mu : (A * B) * C,$$

which defines μ as the elementary symbol of the relation that relates $A * B$ with C. The need for the use of parenthesis is obvious from the meaning of the above expression.

Now let υ denote the relation between B and C, and let x denote the relation between A and C. Graphically we have

$$\begin{array}{ccc} & B & \\ u\nearrow & & \searrow v \; . \\ A & \xrightarrow{x} & C \end{array}$$

We now introduce more structure by defining a new relation out of u and υ, the composite relation, or we could refer to the composition operation, denoted by

$$u \cdot \upsilon.$$

Note that the composition operation "$\cdot$" is only defined between relations u and υ, where the second argument of u, coincides with the first argument of υ.

In order to completely define the structure of the composition operation, we demand x to be identical with $u \cdot \upsilon$. In other words, we are introducing a new relation, the identity relation between x and $u \cdot \upsilon$, which is expressed in symbols by

$$x = u \cdot \upsilon, \tag{1}$$

and which means that whenever x appears in an expression, it can be replaced by $u \cdot \upsilon$, and vice versa.

Let us note that we have just introduced another constant of our alphabet, the symbol =. We will assume the transitivity property; that is, if $\alpha = \beta$ and $\beta = \gamma$ then $\alpha = \gamma$. Also the relation "=" has the symmetric property; that is, if $\alpha = \beta$ then $\beta = \alpha$.

If the relation A appears in an expression; for example

$$u : A * B,$$

and we know that

$$A = D,$$

then the new expression

$$u : D * B,$$

is said to be *deduced* from the previous ones.

Let us consider the following relations

$$p : C * D, \quad q: A * D, \quad r : B * D, \tag{2}$$

then

$$(u \cdot \upsilon) \cdot p = x \cdot p = q \tag{3}$$

and

$$u \cdot (\upsilon \cdot p) = u \cdot r = q, \tag{4}$$

so we have

$$(u \cdot \upsilon) \cdot p = u \cdot (\upsilon \cdot p) \tag{5}$$

in other words, the composition operation "$\cdot$" is associative, and therefore we can omit the parenthesis in the above expressions, and write

$$u \cdot \upsilon \cdot p : (u \cdot \upsilon) \cdot p. \tag{6}$$

Note that although the symbols ":" and "=" seems to behave in a similar way, they have quite different meanings. In particular whenever ":" is used no new structure is introduced. On the contrary "=" is considered to be the first structure constructing symbol. The use of "=" between non a priori identical relations changes the structure of the language.

The use of ":" is reserved for giving a new name to what is written to the right of ":", for example $r : B * D$ can be read: "r is the name used for the relation $B * D$" And $u \cdot \upsilon \cdot p : (u \cdot \upsilon) \cdot p$ can be read: "$u \cdot \upsilon \cdot p$ can be used to represent $(u \cdot \upsilon) \cdot p$ in any expression".

From this we have that if $w : u$, then $w = u$.

3. Working out the structure of elementary cells

Let us now deduce the information contained in the primitive structure introduced before by making use of first order constructive operation "*".

Consider first the case of a single atomic relation A.

3.1. Case [A] :

Now we write all possible relations which can be deduced from A. From the constructive relations we obtain

$$i_A : A * A$$

and nothing else.

Then, by making use of the composite relation we obtain the expression

$$i_A \cdot i_A;$$

and it can be immediately deduced that

$$i_A = i_A \cdot i_A. \tag{7}$$

It is also easy to see that no new information can be deduced at this order in this case.

Therefore let us consider now the case of a pair of atomic relations A and B.

3.1. Case [AB]:

Applying now the constructive relation in first order, we obtain

$$i_A : A * A, \quad u : A * B, \quad i_B : B * B, \quad z : B * A, \tag{8}$$

which completes the list of first order relations; as it can be seen from figure 1.

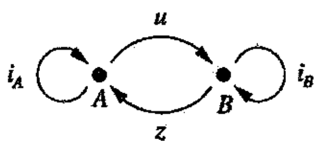

Figure 1: The atomic relations A and B are represented as dots; while lowercase labels refer to the possible relations; which are shown as arrows.

Then, by using the composite operation we obtain the following relations

$$i_A \cdot i_A = i_A \tag{9}$$

$$i_A \cdot u = u \tag{10}$$

$$u \cdot i_B = u \tag{11}$$

$$u \cdot z = i_A \tag{12}$$

$$i_B \cdot i_B = i_B \tag{13}$$

$$i_B \cdot z = z \tag{14}$$

$$z \cdot i_A = z \tag{15}$$

$$z \cdot u = i_B \tag{16}$$

A natural step now would be to consider the case of a triple of atomic relations and to carry out the same calculation; but instead we will continue the study of this system with the introduction of more structure.

How do we introduce more structure? By making certain identifications among some of these relations. So next we will consider one particular case; for reasons of space we can not enumerate the complete series here.

3.2.1. Case [AB; u = z]:

Now let us consider the case in which we assume that $u = z$; which leads to the relations

$$i_A \cdot i_A = i_A \tag{17}$$

$$i_A \cdot u = u \tag{18}$$

$$u \cdot i_B = u \tag{19}$$

$$u \cdot u = i_A \tag{20}$$

$$i_B \cdot i_B = i_B \tag{21}$$

$$i_B \cdot u = u \tag{22}$$

$$u \cdot i_A = u \tag{23}$$

$$u \cdot u = i_B \tag{24}$$

from which it is deduced that $i_B = i_A$.

Then, using the notation

$$I : i_A \quad \text{and} \quad \Gamma : u$$

the set of relations becomes

$$I \cdot I = I \tag{25}$$

$$I \cdot \Gamma = \Gamma \tag{26}$$

$$\Gamma \cdot \Gamma = I \tag{27}$$

$$\Gamma \cdot I = \Gamma. \tag{28}$$

It is observed that interpreting the "·" operation as a multiplication, "I" behaves as an identity and the system agrees with the multiplication table of the Clifford Algebra in one dimension. However we do not have yet the sum operation in our system.

4. Implicit natural structure

After the consideration of the case [AB], it would be natural to study the case [ABC]; and so repeating the same step as we did from the case [A] to that of [AB]. In other words, in our constructive method for different structures, we apply an operation to a previous level of structure to construct the next one. This indicates the presence of the structure of a progression in the formal language; namely:

- A is a symbol
- every symbol has a unique successor and the successor is a symbol

- A is the first symbol
- every symbol except A has a predecessor
- the alphabet is formed by all the successors of A

Therefore we recognize the structure of addition "+" among relations; and the existence of natural numbers; which in our construction are a special kind of relations.

After recognizing the intrinsic structure of natural numbers in the procedure, it is advantageous to generalize to the structure of integer numbers.

It is then convenient to recapitulate and add the structure of integers numbers to the cases discussed in the previous section.

5. Including the structure of integers

The first level of structure is that of case [A], where a single nontrivial relation is obtained,

$$i_A = i_A \cdot i_A.$$

To add the structure of integer numbers we now introduce the operation "×" between numbers and relations such that if "α" is a number and "u" a relation, then "$\alpha \times u$" is a new relation, which will be read "the multiplication of u by α". In particular, integer numbers are considered as atomic relations.

The multiplication operation also satisfy the following property: if v, w and z are relations which satisfy

$$v \cdot w = z,$$

and α and β are two natural numbers then the following relation is deduced

$$(\alpha \times v) \cdot (\beta \times w) = (\alpha \times \beta) \times z.$$

Since integer numbers will be written with lower case Greek letters, it is convenient for notational purposes to omit the appearance of the symbol "×" to denote the multiplication; so the above relation will be written

$$(\alpha v) \cdot (\beta w) = (\alpha \beta) z.$$

It is deduced then that

$$(\alpha \beta) z = (\alpha v) \cdot (\beta w) = (\alpha \beta v) \cdot (w) = (v) \cdot (\alpha \beta w) = (\beta v) \cdot (\alpha w)$$

and so we can omit the parenthesis without falling into ambiguities; that is we can just write

$$\alpha v \cdot \beta w = \alpha \beta \ z.$$

We also introduce the operation "+" between relations such that, if α, β and δ are integer numbers which satisfy

$$\alpha + \beta = \delta,$$

and u is a relation, then

$$\alpha u + \beta u = \beta u + \alpha u = \delta u.$$

In particular we need to define a universal relation 0, which has the property

$$u + 0 = u, \quad (29)$$

$$u + (-u) = 0. \quad (30)$$

Let us remark that at this stage we have not defined yet the sum between two different relations.

Adding this structure to the case [A] does not give more than the structure of the natural numbers. Let us consider then the cases of a pair of atomic relations.

5.1. Case [AB; $u = z$, +, ×] :
The structure in this case also resembles a Clifford Algebra since

$$\Gamma \cdot \Gamma + \Gamma \cdot \Gamma = 2\,I,$$

and

$$\Gamma \cdot \Gamma = I.$$

However we still have not defined the expression "Γ + Γ".

5.2. Case [AB; $u = -z$, +, ×] :
In this case the structure resembles a Clifford Algebra but with the negative signature; namely

$$\Gamma \cdot \Gamma + \Gamma \cdot \Gamma = -2\,I,$$

and

$$\Gamma \cdot \Gamma = -I.$$

6. Structure of a cell with four atomic relations

6.1. Case [ABCD]

Using the constructing relation in this case one obtains the structure shown in figure 2.

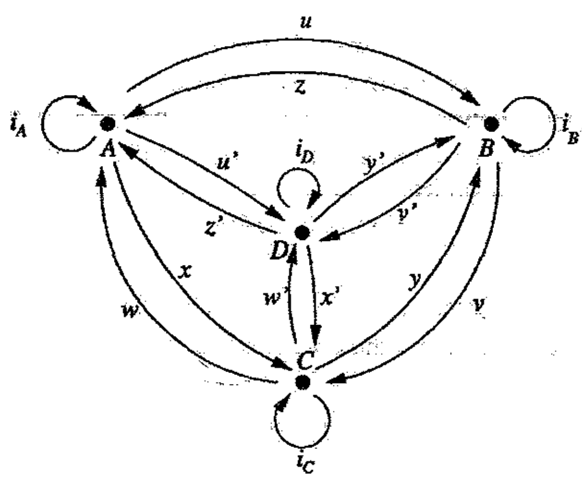

Figure 2. The atomic relations A,B,C and D are represented by the dots; and the arrows show the possible connecting relations.

Applying the composite operation gives further natural relations.
Let us next consider a particular case.

6.2. Case [ABCD; $u = -z, v = -y, w = -x, u' = z', v' = y', w' = x'$, +, ×] :
Having now six pairs of atomic points, we are supposed to set six fundamental relations. Let us consider the structure constructed out of the relations

$$u = -z, \qquad u' = z', \tag{31}$$

$$v = -y, \quad v' = y', \tag{32}$$

$$w = -x, \quad w' = x'. \tag{33}$$

Applying these relations and using the identifications

$$\gamma_{x0} = x' \cdot x \cdot y = -x' \cdot z, \tag{34}$$

$$\gamma_{y0} = y' \cdot y \cdot z = -y' \cdot x, \tag{35}$$

$$\gamma_{z0} = z' \cdot z \cdot x = -z' \cdot y; \tag{36}$$

one obtains $\gamma_{x0} = \gamma_{y0} = \gamma_{z0}$; so we will use the notation $\gamma_0 = \gamma_{x0} = \gamma_{y0} = \gamma_{z0}$.

It is observed that all relations are generated by four fundamental ones, namely: γ_0, $\gamma_1 = x$, $\gamma_2 = y$ and $\gamma_3 = z$. There is an identity I; and using the notation a, b = 0, 1, 2, 3, it is easy to see that they satisfy the equation

$$\boxed{\gamma_a \cdot \gamma_b + \gamma_b \cdot \gamma_a = 2I\,\eta_{ab}}; \tag{37}$$

where η_{ab} has the form of the Minkowskian metric, that is $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$; in other words, γ_a are the generators of the Clifford algebra of the Minkowski metric.

The structure of the case under consideration then implies the metric structure at a point; and we also recognize then the spinor structure of the Minkowskian four dimensional spacetime at a point.

It is important to remark that the relativistic description of elementary particles, like the electron, is made in terms precisely of this structure.

7. Some questions

In graph 2 there are 16 relations represented by arrows. The number of all possible pairs is $16^2 = 256$. The equations in which a product between two relations is made equal to another relation are associated to triples of relations; which for the case [ABCD] amounts to a possible total of $16^3 = 4096$ equations. If we allow to introduce the structure of integers, as explained before, this number immediately jumps to infinity.

One natural question to ask is whether we have just been lucky in introducing an approach to the study of simple formal languages that has conduced us to precisely the structure of a Lorentzian spacetime at a point out of a priori very small probability, if we were just playing at random.

In order to assert that we have not been lucky we should answer some questions. For example; we have used our intuition to go through a path that we thought was fruitful; but: is there a principle that can be stated and serve as a guide for the construction of simple formal languages; which are useful for the description of the fundamental structure of the spacetime?

Other questions: What is the most economical way to describe the connection among neighboring cells that capture the structure of a 4-dimensional manifold? What is the meaning of other possible structure for case [ABCD]? What is the meaning of the structure [ABCDE]? And so on.[1]

Acknowledgments

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Note

[1] Length constraints have forced us to maintain the content of this article to a minimum, however, we intend to publish a full length version of this work in the public preprint //xxx.lanl.gov/ site.

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