# On Dirichlet problems with singular nonlinearity of indefinite sign ${ }^{*+\ddagger}$ 

T. Godoy, U. Kaufmann, ${ }^{\S}$<br>FaMAF, Universidad Nacional de Córdoba, (5000) Córdoba, Argentina

February 28, 2018


#### Abstract

Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^{N}, N \geq 1$, let $K, M$ be two nonnegative functions and let $\alpha, \gamma>0$. We study existence and nonexistence of positive solutions for singular problems of the form $-\Delta u=K(x) u^{-\alpha}-\lambda M(x) u^{-\gamma}$ in $\Omega, u=0$ on $\partial \Omega$, where $\lambda>0$ is a real parameter. We mention that as a particular case our results apply to problems of the form $-\Delta u=m(x) u^{-\gamma}$ in $\Omega, u=0$ on $\partial \Omega$, where $m$ is allowed to change sign in $\Omega$.


## 1 Introduction

Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^{N}, N \geq 1$, and let $0 \leq K, M \in$ $L^{p}(\Omega)$ for some $p \geq 2$ if $N=1$ and $p>N$ when $N \geq 2$. Our aim in this article is to consider existence and nonexistence of solutions for singular problems of the form

$$
\begin{cases}-\Delta u=K(x) u^{-\alpha}-\lambda M(x) u^{-\gamma} & \text { in } \Omega  \tag{1.1}\\ u>0 & \text { in } \Omega \\ u=0 & \text { on } \partial \Omega\end{cases}
$$

where $\alpha, \gamma>0$ and $\lambda>0$ is a real parameter.
Concerning the results in this paper, in Section 3 we shall study (1.1) when $N=1$. More precisely, in Theorem 3.1 we shall give sufficient conditions for the existence of solutions in the case $\alpha=\gamma$, and as a consequence

[^0]we shall derive existence results when $\alpha>\gamma$ in Corollary 3.3, A further result without any relation between $\alpha$ and $\gamma$ is presented in Theorem 3.4, while in Theorem 3.5 we prove necessary conditions on $\lambda, K$ and $M$ in order to have solutions for (1.1) (see also Remark [3.6, where we also characterize the set of $\lambda^{\prime}$ s such that (1.1) admits a solution). We remark that as a particular case ( $\operatorname{taking} K:=m^{+}, M:=m^{-}, \alpha=\gamma$ and $\lambda=1$ ) we are able to deal with problems of the form
\[

$$
\begin{cases}-\Delta u=m(x) u^{-\gamma} & \text { in } \Omega  \tag{1.2}\\ u>0 & \text { in } \Omega \\ u=0 & \text { on } \partial \Omega,\end{cases}
$$
\]

where $m$ may change in $\Omega$ (see Corollary 3.7).
In Section 4 we shall consider the case $N \geq 2$. We point out that although all the results obtained for the one-dimensional problem have their counterpart in $N$ dimensions, the conditions derived in Section 3 are sharper or provide more information than the ones available when $N>1$. Sufficient conditions are stated in Theorems 4.1 and 4.3 and Corollary 4.2, and in Theorem 4.4 some necessary conditions are presented. The particular case (1.2) is covered in Corollary 4.6. Let us mention that these theorems shall be obtained employing some estimates on linear problems combined with the sub and supersolution method (see Theorem 2.3 below) together with Schauder's fixed point theorem.

In order to relate our results to others already existing, let us first mention that problems like (1.2) have been extensively studied both when $m>0$ (see e.g. [6], [19], [12] and its references) and also if $m$ is nonnegative (e.g. [15], [7, [18]), but to our knowledge no theorems are known when $m$ is allowed to change sign in $\Omega$. In fact, even when (1.2) is sublinear (that is, $\gamma \in(-1,0))$ and one-dimensional, these kind of problems are quite intriguing and involved and, as far as we know, only recently existence of (strictly) positive solutions has started being studied in detail when $m$ changes sign in $\Omega$ (see [13], [16], [14] and [10]; and [17] for the $p$-laplacian). Let us also mention that nonnegative solutions of these semilinear problems have been studied carefully in [2].

On the other side, problem (1.1) with $\alpha=0$ and $M \equiv 1$ was treated in for instance [8], [5], while in [12] it was considered assuming that $K>0$ (under stronger regularity assumptions), but we could not find any results in the literature in the case where $K$ vanishes in a subset of $\Omega$. We refer the reader to the nice review papers [9] and [11] for further references, applications and historical remarks concerning these types of singular elliptic problems, and for specifically one-dimensional singular problems we refer to the book [1] and the references therein.

## 2 Preliminaries and auxiliary results

Let $\Omega \subset \mathbb{R}^{N}$ be a smooth bounded domain. For $h \in L^{q}(\Omega), q>1$, let $\phi \in W^{2, q}(\Omega) \cap W_{0}^{1, q}(\Omega)$ be the unique solution of

$$
\begin{cases}-\Delta \phi=h(x) & \text { in } \Omega  \tag{2.1}\\ \phi=0 & \text { on } \partial \Omega,\end{cases}
$$

and let us write $\left.\mathcal{S}: L^{q} \Omega\right) \rightarrow W^{2, q}(\Omega)$ for the solution operator of (2.1).
The two following lemmas provide some useful lower and upper bounds for $\mathcal{S}(h)$ when $h$ is nonnegative. We set

$$
\delta_{\Omega}(x):=\operatorname{dist}(x, \partial \Omega)
$$

Lemma 2.1. Suppose $\Omega:=(a, b)$, and let $0 \leq h \in L^{q}(\Omega)$ with $q>1$. Then in $\bar{\Omega}$ it holds that

$$
\begin{gather*}
\mathcal{S}(h) \geq \frac{1}{b-a}\left(\int_{a}^{b} h(t) \delta_{\Omega}(t) d t\right) \delta_{\Omega}:=\underline{c} \delta_{\Omega} \quad \text { and }  \tag{2.2}\\
\mathcal{S}(h) \leq \frac{1}{b-a} \max \left(\int_{a}^{b}(t-a) h(t) d t, \int_{a}^{b}(b-t) h(t) d t\right) \delta_{\Omega}:=\bar{c} \delta_{\Omega} \tag{2.3}
\end{gather*}
$$

Moreover, the inequalities (2.2) and (2.3) are sharp in the sense that if $c_{1} \delta_{\Omega} \leq \mathcal{S}(h) \leq c_{2} \delta_{\Omega}$ in $\Omega$ for some $c_{1}, c_{2}>0$, then $c_{1} \leq \underline{c}$ and $\bar{c} \leq c_{2}$.

Proof. Let $\phi:=\mathcal{S}(h)$. It is easy to verify that (even if $h$ changes sign in $\Omega$ )

$$
\begin{gather*}
\phi(x)=\frac{x-a}{b-a} \int_{a}^{b} \int_{a}^{y} h(t) d t d y-\int_{a}^{x} \int_{a}^{y} h(t) d t d y= \\
\frac{x-a}{b-a} \int_{a}^{b}(b-t) h(t) d t-\int_{a}^{x}(x-t) h(t) d t . \tag{2.4}
\end{gather*}
$$

Also, if $t_{1}, t_{2} \in \Omega$ with $t_{1}<t_{2}$ we may integrate over $\left(t_{1}, t_{2}\right)$ (see e.g. [3], Theorem 8.2) to obtain that

$$
\phi^{\prime}\left(t_{1}\right)-\phi^{\prime}\left(t_{2}\right)=-\int_{t_{1}}^{t_{2}} \phi^{\prime \prime}(t) d t=\int_{t_{1}}^{t_{2}} h(t) d t \geq 0
$$

and therefore we find that $\phi$ is concave in $\Omega$.
Since $\delta_{\Omega}(x)=\min (x-a, b-x)$, using (2.4) we now observe that

$$
\begin{aligned}
& \phi\left(\frac{a+b}{2}\right)=\frac{1}{2} \int_{a}^{b}(b-t) h(t) d t-\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right) h(t) d t= \\
& =\frac{1}{2}\left(\int_{a}^{\frac{a+b}{2}}(t-a) h(t) d t+\int_{\frac{a+b}{2}}^{b}(b-t) h(t) d t\right)=\underline{c} \delta_{\Omega}\left(\frac{a+b}{2}\right)
\end{aligned}
$$

and hence the concavity of $\phi$ yields that $\phi \geq \underline{c} \delta_{\Omega}$ in $\bar{\Omega}$.
On the other hand, from (2.4) we also get that

$$
\phi^{\prime}(a)=\int_{a}^{b} \frac{b-t}{b-a} h(t) d t, \quad \phi^{\prime}(b)=\int_{a}^{b}\left(\frac{b-t}{b-a}-1\right) h(t) d t
$$

Furthermore, if $h \not \equiv 0$, since $h \geq 0$ it holds that $\phi^{\prime}(b)<0<\phi^{\prime}(a)$ and thus recalling the definition of $\bar{c}$ again by the concavity of $\phi$ we deduce that $\phi \leq \bar{c} \delta_{\Omega}$ in $\bar{\Omega}$.

Since the final assertion of the lemma is clearly true this concludes the proof.
Lemma 2.2. Let $0 \leq h \in L^{q}(\Omega)$ with $q>N$. Then in $\bar{\Omega}$ it holds that

$$
\begin{equation*}
c_{\Omega}\left(\int_{\Omega} h(x) \delta_{\Omega}(x) d x\right) \delta_{\Omega} \leq \mathcal{S}(h) \leq C_{\Omega}\|h\|_{L^{q}(\Omega)} \delta_{\Omega} \tag{2.5}
\end{equation*}
$$

for some $c_{\Omega}, C_{\Omega}>0$ depending only on $\Omega$.
Proof. The first inequality in (2.5) appeared first in an unpublished work by Morel and Oswald, and there is a nice proof in the paper of Brezis and Cabré, [4], Lemma 3.2. On the other side, set $\phi:=\mathcal{S}(h)$, let $x \in \Omega$ and pick $y \in \partial \Omega$ such that $|y-x|=\delta_{\Omega}(x)$. Since by the Sobolev imbedding theorems $\left.\mathcal{S}: L^{q} \Omega\right) \hookrightarrow C^{1}(\bar{\Omega})$ and hence $\|\nabla \phi \mid\|_{L^{\infty}(\Omega)} \leq C_{\Omega}\|h\|_{L^{q}(\Omega)}$ for some $C_{\Omega}>0$ depending only on $\Omega$, from the mean value theorem we find that

$$
|\phi(x)|=|\phi(x)-\phi(y)| \leq\||\nabla \phi|\|_{L^{\infty}(\Omega)} \delta_{\Omega}(x) \leq C_{\Omega}\|h\|_{L^{q}(\Omega)} \delta_{\Omega}(x)
$$

which in turn gives the second inequality in (2.5).
Let $f: \Omega \times(0, \infty) \rightarrow \mathbb{R}$ be a Carathéodory function (that is, $f(\cdot, \xi)$ is measurable for all $\xi \in(0, \infty)$ and $f(x, \cdot)$ is continuous for a.e. $x \in \Omega)$. We consider now singular problems of the form

$$
\begin{cases}-\Delta u=f(x, u) & \text { in } \Omega  \tag{2.6}\\ u=0 & \text { on } \partial \Omega\end{cases}
$$

in a suitable sense. We say that $v \in W_{l o c}^{1,2}(\Omega) \cap C(\bar{\Omega})$ is a subsolution (in the sense of distributions) of (2.6) if $v>0$ in $\Omega, v=0$ on $\partial \Omega$, and

$$
\int_{\Omega}\langle\nabla v, \nabla \varphi\rangle \leq \int_{\Omega} f(x, v) \varphi \quad \text { for all } 0 \leq \varphi \in C_{c}^{\infty}(\Omega)
$$

Similarly, $w \in W_{l o c}^{1,2}(\Omega) \cap C(\bar{\Omega})$ is a supersolution of (2.6) if $w>0$ in $\Omega$, $w=0$ on $\partial \Omega$, and

$$
\int_{\Omega}\langle\nabla w, \nabla \varphi\rangle \geq \int_{\Omega} f(x, w) \varphi \quad \text { for all } 0 \leq \varphi \in C_{c}^{\infty}(\Omega)
$$

For the sake of completeness we state the following existence result (for a proof, see [20], Theorem 4.1).

Theorem 2.3. Suppose there exist $v, w \in C^{1}(\Omega)$ sub and supersolutions respectively of (2.6), satisfying that $v \leq w$ in $\Omega$. Suppose also that there exists $g \in L_{\text {loc }}^{2}(\Omega)$ such that $|f(x, \xi)| \leq g(x)$ for a.e. $x \in \Omega$ and all $\xi \in$ $[v(x), w(x)]$. Then there exists $u \in C^{1}(\Omega) \cap C(\bar{\Omega})$ solution (in the sense of distributions) of (2.6) with $v \leq u \leq w$, that is,

$$
\int_{\Omega}\langle\nabla u, \nabla \varphi\rangle=\int_{\Omega} f(x, u) \varphi \quad \text { for all } \varphi \in C_{c}^{\infty}(\Omega)
$$

Remark 2.4. When $0 \leq K \in L^{p}(\Omega)$ with $K \not \equiv 0$ and $p>N$ one can readily check that (1.1) admits arbitrarily large supersolutions. Indeed, let $\psi:=\mathcal{S}(K)>0$ and define also $\sigma \in(0,1)$ by $\sigma:=1 /(1+\alpha)$. We have that $\psi^{\sigma} \in W_{l o c}^{2, p}(\Omega) \cap C(\bar{\Omega}), \psi^{\sigma}=0$ on $\partial \Omega$, and a simple computation shows that for all $\lambda>0$ and every $c \geq \sigma^{-1 /(1+\alpha)}$ it holds that

$$
\begin{gathered}
-\Delta\left(c \psi^{\sigma}\right)=-c \sigma \psi^{\sigma-1} \Delta \psi-c \sigma(\sigma-1) \psi^{\sigma-2}|\nabla \psi|^{2} \geq \\
-c \sigma \psi^{\sigma-1} \Delta \psi=c \sigma \psi^{\sigma-1} K(x) \geq K(x)\left(c \psi^{\sigma}\right)^{-\alpha} \geq \\
K(x)\left(c \psi^{\sigma}\right)^{-\alpha}-\lambda M(x)\left(c \psi^{\sigma}\right)^{-\gamma} \quad \text { in } \Omega^{\prime}
\end{gathered}
$$

for every $\Omega^{\prime} \subset \subset \Omega$, and therefore $c \psi^{\sigma}$ is a supersolution of (1.1).

## 3 The one-dimensional case

In this section we shall assume that

$$
\Omega:=(a, b) \quad \text { for some }-\infty<a<b<\infty .
$$

For $1<p \leq \infty$, we define as usual $p^{\prime}$ by $1 / p+1 / p^{\prime}=1$ if $p$ is finite and $p^{\prime}=1$ if $p=\infty$. For $0 \leq h \in L^{p}(\Omega)$ we also introduce the following notation:

$$
\begin{equation*}
h_{a}(x):=(x-a) h(x), \quad h_{b}(x):=(b-x) h(x) . \tag{3.1}
\end{equation*}
$$

Let us note that $\left\|h_{a}\right\|_{L^{p}(\Omega)} \leq(b-a)\|h\|_{L^{p}(\Omega)}$ if $p$ is finite and that $\left\|h_{a}\right\|_{L^{\infty}(\Omega)}=$ $(b-a)\|h\|_{L^{\infty}(\Omega)}$, and analogously for $h_{b}$. We also set

$$
P^{\circ}:=\text { interior of the positive cone of } C^{1}(\bar{\Omega}) .
$$

Theorem 3.1. Let $0 \leq K, M \in L^{2}(\Omega)$ with $K \not \equiv 0$. Assume $\alpha=\gamma$ and let $M_{a}$ and $M_{b}$ be given by (3.1).
(i) If $M \in L^{p}(\Omega)$ with $p \geq 2, \gamma \in(0,(p-1) / p)$ and

$$
\begin{gather*}
\max \left(\left\|M_{a}\right\|_{L^{p}(\Omega)},\left\|M_{b}\right\|_{L^{p}(\Omega)}\right) \leq c_{\gamma, p, a, b} \frac{\left(\int_{a}^{b} K(t) \delta_{\Omega}(t) d t\right)^{1+\gamma}}{\left(\int_{a}^{b} K(t) d t\right)^{\gamma}}  \tag{3.2}\\
c_{\gamma, p, a, b}:=\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{\left(1-\gamma p^{\prime}\right)^{1 / p^{\prime}}}{(b-a)^{\gamma+1 / p^{\prime}}}
\end{gather*}
$$

then (1.1) has a solution $u \in C^{1}(\Omega) \cap C(\bar{\Omega})$ for all $\lambda \leq 1$; and $u \in W^{2, q}(\Omega) \cap$ $P^{\circ}, q>1$, whenever $K \delta_{\Omega}^{-\gamma} \in L^{r}(\Omega)$ with $r>1$.
(ii) If

$$
\begin{equation*}
\max \left(\int_{a}^{b} M_{a}(t) d t, \int_{a}^{b} M_{b}(t) d t\right)<\int_{a}^{b} K(t) \delta_{\Omega}(t) d t \tag{3.3}
\end{equation*}
$$

then there exists $\gamma_{0}>0$ such that the problem (1.1) has a solution $u \in$ $W^{2, q}(\Omega) \cap P^{\circ}, q>1$, for all $\gamma \in\left(0, \gamma_{0}\right]$ and all $\lambda \leq 1$.

Proof. Taking into account Theorem 2.3 and Remark 2.4 we note that it is enough to build a subsolution for (1.1). Furthermore, clearly any solution of (1.1) is a subsolution of (1.1) with $\underline{\lambda}$ in place of $\lambda$ whenever $\underline{\lambda} \leq \lambda$ and so in order to prove the theorem we may assume that $\lambda=1$.

Let us prove (i). Since (1.1) with $\alpha=\gamma$ is homogeneous (i.e. it has a solution for $K$ and $M$ if and only if it has one for $c K$ and $c M$ for any $c>0$ ), we shall prove (i) for $\tau K$ and $\tau M$, where

$$
\tau:=\frac{2}{(b-a) \int_{a}^{b} K(t) d t}
$$

Let $K_{a}$ and $K_{b}$ be given by (3.1). Since $\delta_{\Omega} \leq(b-a) / 2$ in $\Omega$ we observe that by (2.3)

$$
\mathcal{S}(\tau K) \leq \frac{\tau}{b-a} \max \left(\int_{a}^{b} K_{a}(t) d t, \int_{a}^{b} K_{b}(t) d t\right) \delta_{\Omega} \leq \tau \delta_{\Omega} \int_{a}^{b} K(t) d t \leq 1
$$

in $\Omega$. Let $\gamma \in(0,(p-1) / p)$, and let us now define

$$
\begin{gathered}
\mathcal{M}_{p}:=\max \left(\left\|M_{a}\right\|_{L^{p}(\Omega)},\left\|M_{b}\right\|_{L^{p}(\Omega)}\right), \quad \beta:=\frac{\tau}{b-a} \\
r:=\left(\beta\left\|\delta_{\Omega}^{-\gamma}\right\|_{L^{p^{\prime}(\Omega)}} \mathcal{M}_{p} \gamma\right)^{1 /(\gamma+1)}, \quad d:=r \delta_{\Omega} \\
\mathcal{C}:=\{v \in C(\bar{\Omega}): d \leq v \leq \tau \mathcal{S}(K) \text { in } \Omega\}
\end{gathered}
$$

A simple computation shows that

$$
\left\|\delta_{\Omega}^{-\gamma}\right\|_{L^{p^{\prime}}(\Omega)}=2^{\gamma} \frac{(b-a)^{1 / p^{\prime}-\gamma}}{\left(1-\gamma p^{\prime}\right)^{1 / p^{\prime}}}
$$

and so (3.2) says that

$$
\begin{equation*}
\frac{(b-a)^{2 \gamma}}{2^{\gamma}}\left\|\delta_{\Omega}^{-\gamma}\right\|_{L^{p^{\prime}}(\Omega)} \mathcal{M}_{p} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{\left(\int_{a}^{b} K(t) \delta_{\Omega}(t) d t\right)^{1+\gamma}}{\left(\int_{a}^{b} K(t) d t\right)^{\gamma}} \tag{3.4}
\end{equation*}
$$

Therefore, taking into account (2.2) and (3.4) we find that

$$
\begin{gather*}
\tau \mathcal{S}(K) \geq \beta\left(\int_{a}^{b} K(t) \delta_{\Omega}(t) d t\right) \delta_{\Omega} \geq \\
\beta\left(\left\|\delta_{\Omega}^{-\gamma}\right\|_{L^{p^{\prime}}(\Omega)} \mathcal{M}_{p}\left(\int_{a}^{b} K(t) d t\right)^{\gamma} \frac{(b-a)^{2 \gamma}(\gamma+1)^{\gamma+1}}{2^{\gamma} \gamma^{\gamma}}\right)^{1 /(\gamma+1)} \delta_{\Omega}= \\
\left(\beta\left\|\delta_{\Omega}^{-\gamma}\right\|_{L^{p^{\prime}(\Omega)}} \mathcal{M}_{p} \frac{(\gamma+1)^{\gamma+1}}{\gamma^{\gamma}}\right)^{1 /(\gamma+1)} \delta_{\Omega}=r \frac{\gamma+1}{\gamma} \delta_{\Omega} \geq d \quad \text { in } \Omega . \tag{3.5}
\end{gather*}
$$

In particular, $\mathcal{C} \neq \emptyset$. On the other hand, utilizing (2.3) we also deduce that

$$
\begin{gather*}
\tau \mathcal{S}\left(M d^{-\gamma}\right) \leq \beta r^{-\gamma} \max \left(\int_{a}^{b} M_{a}(t) \delta_{\Omega}^{-\gamma}(t) d t, \int_{a}^{b} M_{b}(t) \delta_{\Omega}^{-\gamma}(t) d t\right) \delta_{\Omega} \leq \\
\beta r^{-\gamma}\left\|\delta_{\Omega}^{-\gamma}\right\|_{L^{p^{\prime}}(\Omega)} \mathcal{M}_{p} \delta_{\Omega}=\frac{1}{\gamma} r \delta_{\Omega} \quad \text { in } \Omega \tag{3.6}
\end{gather*}
$$

$\left(\mathcal{S}\left(M d^{-\gamma}\right)\right.$ is well defined since $M d^{-\gamma} \in L^{s}(\Omega)$ for some $s>1$ because $M \in L^{p}(\Omega)$ with $p \geq 2$ and $\gamma p^{\prime}<1$.) For $v \in \mathcal{C}$ we now set $u:=$ $\tau \mathcal{S}\left(K-M v^{-\gamma}\right):=\mathcal{T}(v)$. Recalling (3.5), (3.6) and that $v \geq d$ we derive that

$$
\tau \mathcal{S}(K) \geq u \geq \tau \mathcal{S}\left(K-M d^{-\gamma}\right) \geq\left(r \frac{\gamma+1}{\gamma}-\frac{1}{\gamma} r\right) \delta_{\Omega}=d \quad \text { in } \Omega
$$

and thus $u \in \mathcal{C}$. Moreover, since $\gamma<1 / p^{\prime}$ one can see that $v \rightarrow K-M v^{-\gamma}$ is continuous from $C(\bar{\Omega})$ into $L^{s}(\Omega)$ for some $s>1$, and then it is easy to check employing the Sobolev imbedding theorems that $\mathcal{T}: \mathcal{C} \rightarrow \mathcal{C}$ is a continuous compact operator. Hence, Schauder's fixed point theorem gives some $v \in \mathcal{C}$ solution of

$$
\begin{cases}-v^{\prime \prime}=K(x)-M(x) v^{-\gamma} & \text { in } \Omega  \tag{3.7}\\ v>0 & \text { in } \Omega \\ v=0 & \text { on } \partial \Omega\end{cases}
$$

Furthermore, $v \in C^{1}(\bar{\Omega})$, and since $v \leq 1$ (because $v \leq \mathcal{S}(\tau K) \leq 1$ ) it follows from (3.7) that $v$ is a subsolution of (1.1) for $\lambda=1$. Therefore recalling Remark 2.4 and Theorem 2.3 we obtain some $u \in C^{1}(\Omega) \cap C(\bar{\Omega})$ solution of (1.1). Moreover, if $K \delta_{\Omega}^{-\gamma} \in L^{r}(\Omega)$ for some $r>1$, since we also have that $M \delta_{\Omega}^{-\gamma} \in L^{s}(\Omega)$ with $s>1$ and $u \geq c \delta_{\Omega}$ for some $c>0$, by standard regularity arguments we conclude that $u \in W^{2, q}(\Omega) \cap P^{\circ}, q>1$. This ends the proof of (i).

In order to prove (ii) we proceed similarly. Without loss of generality we assume that $\mathcal{S}(K) \leq 1$ in $\Omega$. We now utilize (2.2) to get that

$$
\mathcal{S}(K) \geq \frac{1}{b-a}\left(\int_{a}^{b} K(t) \delta_{\Omega}(t) d t\right) \delta_{\Omega}:=\frac{c_{K}}{b-a} \delta_{\Omega} \quad \text { in } \Omega
$$

Also, by (3.3) we may fix $\varepsilon \in(0,1)$ such that

$$
\varepsilon \leq \min \left(\frac{2}{c_{K}}, \frac{2\left(c_{K}-\mathcal{M}_{1}\right)}{2 c_{K}+1}\right),
$$

where $\mathcal{M}_{1}$ is defined as in (i). Let $d:=\frac{\varepsilon}{b-a} c_{K} \delta_{\Omega}$, and note that $d \leq 1$ in $\Omega$. We next choose $\gamma_{0}>0$ such that for all $\gamma \in\left(0, \gamma_{0}\right]$ it holds that $M d^{-\gamma} \in L^{s}(\Omega)$ with $s>1$. Making $\gamma_{0}$ smaller if necessary, from (2.3) and Lebesgue's dominated convergence theorem we obtain that for such $\gamma$

$$
\begin{aligned}
\mathcal{S}\left(M d^{-\gamma}\right) \leq & \frac{1}{b-a} \max \left(\int_{a}^{b} M_{a}(t) d^{-\gamma}(t) d t, \int_{a}^{b} M_{b}(t) d^{-\gamma}(t) d t\right) \delta_{\Omega} \leq \\
& \frac{1}{b-a}\left(\mathcal{M}_{1}+\frac{\varepsilon}{2}\right) \delta_{\Omega} \leq \frac{1-\varepsilon}{b-a} c_{K} \delta_{\Omega} \quad \text { in } \Omega .
\end{aligned}
$$

Define now $\mathcal{C}:=\{v \in C(\bar{\Omega}): d \leq v \leq \mathcal{S}(K)$ in $\Omega\}$, and for $v \in \mathcal{C}$ let $u:=\mathcal{S}\left(K-M v^{-\gamma}\right)$. For $\gamma \in\left(0, \gamma_{0}\right]$ we have that

$$
\mathcal{S}(K) \geq u \geq \mathcal{S}\left(K-M d^{-\gamma}\right) \geq \frac{1}{b-a}\left(c_{K}-(1-\varepsilon) c_{K}\right) \delta_{\Omega} \quad \text { in } \Omega
$$

and thus $u \in \mathcal{C}$. Now the proof of (ii) can be continued exactly as the proof of (i).
Remark 3.2. Let us mention that one can verify that the inequality (3.3) is better than (3.2), but on the other hand in (ii) there is no lower estimate for $\gamma_{0}$. Let us also note that if for instance $\gamma<1 / 2$ then $K \delta_{\Omega}^{-\gamma} \in L^{r}(\Omega)$ for some $r>1$ (in fact, if $K \in L^{p}(\Omega)$ with $p \geq 2$, then $K \delta_{\Omega}^{-\gamma} \in L^{r}(\Omega)$ with $r>1$ when $p(1-\gamma)>1)$.

Corollary 3.3. Let $K, M$ and $\gamma$ be as in Theorem 3.1, and let $\alpha>\gamma$. Then there exists $\lambda_{0}>0$ such that the problem (1.1) has a solution $u \in$ $C^{1}(\Omega) \cap C(\bar{\Omega})$ for all $\lambda \leq \lambda_{0}$; and $u \in W^{2, q}(\Omega) \cap P^{\circ}, q>1$, whenever $K \delta_{\Omega}^{-\alpha} \in L^{r}(\Omega)$ with $r>1$.

Proof. As in the above theorem it suffices to construct a subsolution for (1.1). Let $u$ be the solution of (1.1) for $\lambda=1$ and $\alpha=\gamma$ provided by either Theorem3.1(i) or (ii). We choose $0<\varepsilon \leq \min \left(1,1 /\|u\|_{\infty}\right)$. Now, for every $\alpha>\gamma$ we get that

$$
\begin{gathered}
-(\varepsilon u)^{\prime \prime}=\left(\varepsilon^{1+\gamma} K(x)-\varepsilon^{1+\gamma} M(x)\right)(\varepsilon u)^{-\gamma} \leq \\
K(x)(\varepsilon u)^{-\alpha}-\varepsilon^{1+\gamma} M(x)(\varepsilon u)^{-\gamma} \quad \text { in } \Omega
\end{gathered}
$$

and hence $\varepsilon u$ is a subsolution of (1.1) for $\lambda_{0}:=\varepsilon^{1+\gamma}$.
In the next theorem we complement the results contained in Theorem 3.1 (ii) and Corollary 3.3, without imposing any relation between $\alpha$ and $\gamma$. We set

$$
\mathcal{M}:=\{x \in \Omega: M(x)>0\} .
$$

Theorem 3.4. Let $\alpha, \gamma>0$, let $0 \leq K, M \in L^{2}(\Omega)$ with $M \in C(\Omega)$ and let $K_{a}, K_{b}, M_{a}, M_{b}$ be given by (3.1). Suppose that $\overline{\mathcal{M}} \subset \Omega$ and that (3.3) holds. Then there exists

$$
\begin{equation*}
\lambda_{0} \geq\left(\frac{\operatorname{dist}(\mathcal{M}, \partial \Omega)}{b-a}\right)^{\gamma} \frac{\left(\int_{a}^{b} K \delta_{\Omega}-\max \left(\int_{a}^{b} M_{a}, \int_{a}^{b} M_{b}\right)\right)^{\gamma}}{\left(\frac{1}{2}\left(\max \left(\int_{a}^{b} K_{a}, \int_{a}^{b} K_{b}\right)-\int_{a}^{b} M \delta_{\Omega}\right)\right)^{\alpha \frac{1+\gamma}{1+\alpha}}} \tag{3.8}
\end{equation*}
$$

such that for all $\lambda \leq \lambda_{0}$ the problem (1.1) has a solution $u \in C^{1}(\Omega) \cap C(\bar{\Omega})$; and $u \in W^{2, q}(\Omega) \cap P^{\circ}, q>1$, whenever $K \delta_{\Omega}^{-\alpha}, M \delta_{\Omega}^{-\gamma} \in L^{r}(\Omega)$ with $r>1$.

Proof. Let $\phi:=\mathcal{S}(K-M)$. Applying Lemma 2.1 to $\mathcal{S}(K)$ and $\mathcal{S}(M)$ we see that in $\Omega$

$$
\begin{aligned}
\phi & \geq \frac{1}{b-a}\left(\int_{a}^{b} K(t) \delta_{\Omega}(t) d t-\max \left(\int_{a}^{b} M_{a}(t) d t, \int_{a}^{b} M_{b}(t) d t\right)\right) \delta_{\Omega}, \\
\phi & \leq \frac{1}{b-a}\left(\max \left(\int_{a}^{b} K_{a}(t) d t, \int_{a}^{b} K_{b}(t) d t\right)-\int_{a}^{b} M(t) \delta_{\Omega}(t) d t\right) \delta_{\Omega} .
\end{aligned}
$$

We note that in particular it follows from (3.3) that $\phi>0$ in $\Omega$. Let us now define

$$
\begin{gathered}
\bar{\mu}:=\left(\frac{1}{2}\left(\max \left(\int_{a}^{b} K_{a}, \int_{a}^{b} K_{b}\right)-\int_{a}^{b} M \delta_{\Omega}\right)\right)^{\alpha} \\
\underline{\beta}:=\left(\frac{\operatorname{dist}(\mathcal{M}, \partial \Omega)}{b-a}\left(\int_{a}^{b} K \delta_{\Omega}-\max \left(\int_{a}^{b} M_{a}, \int_{a}^{b} M_{b}\right)\right)\right)^{\gamma} .
\end{gathered}
$$

Let $\mu \geq \bar{\mu}$ and $\beta \leq \underline{\beta}$ and set $\varepsilon:=\mu^{-1 /(1+\alpha)}$. Since $\phi \leq \bar{\mu}^{1 / \alpha}$ in $\Omega$ and $\phi \geq \underline{\beta}^{1 / \gamma}$ in $\mathcal{M}$ we have

$$
\begin{aligned}
-(\varepsilon \phi)^{\prime \prime} & =\mu^{-1} \varepsilon^{-\alpha} K(x)-\varepsilon^{1+\gamma} \varepsilon^{-\gamma} M(x) \leq \\
K(x)(\varepsilon \phi)^{-\alpha} & -\left(\frac{\beta}{\mu^{(1+\gamma) /(1+\alpha)}}\right) M(x)(\varepsilon \phi)^{-\gamma} \quad \text { in } \Omega
\end{aligned}
$$

and so $\varepsilon \phi$ is a subsolution of (1.1) for $\lambda=\beta \mu^{-(1+\gamma) /(1+\alpha)}$. Since (3.8) is clearly true the theorem follows.

The next theorem states necessary conditions on $\lambda, K$ and $M$ in order for (1.1) to admit solutions in the case $\alpha \leq \gamma$.
Theorem 3.5. Let $0<\alpha \leq \gamma$ and let $K_{a}$ and $K_{b}$ be given by (3.1). Suppose for some $\lambda>0$ (1.1) has a solution $u \in W^{2, q}(\Omega), q>1$. Then

$$
\lambda<\left(\frac{(\alpha+1)}{2} \max \left(\int_{a}^{b} K_{a}(t) d t, \int_{a}^{b} K_{b}(t) d t\right)\right)^{\frac{\gamma-\alpha}{\alpha+1}} \frac{\int_{a}^{b} K(t) \delta_{\Omega}(t) d t}{\int_{a}^{b} M(t) \delta_{\Omega}(t) d t}
$$

Proof. Let $u>0$ be a solution of (1.1) for some $\lambda>0$ and pick $\sigma:=\alpha+1$. We get that

$$
\begin{gathered}
-\left(u^{\sigma}\right)^{\prime \prime}=-\sigma u^{\sigma-1} u^{\prime \prime}-\sigma(\sigma-1) u^{\sigma-2}\left(u^{\prime}\right)^{2} \leq \\
-\sigma u^{\sigma-1} u^{\prime \prime} \leq \sigma u^{\sigma-1} K(x) u^{-\alpha}=\sigma K(x) \quad \text { in } \Omega
\end{gathered}
$$

and so, since $\mathcal{S}$ is a positive operator, recalling (2.3) we find that

$$
\begin{aligned}
0<u^{\sigma} \leq & \sigma \mathcal{S}(K) \leq \frac{\sigma}{b-a} \max \left(\int_{a}^{b} K_{a}(t) d t, \int_{a}^{b} K_{b}(t) d t\right) \delta_{\Omega} \leq \\
& \frac{\sigma}{2} \max \left(\int_{a}^{b} K_{a}(t) d t, \int_{a}^{b} K_{b}(t) d t\right) \quad \text { in } \Omega
\end{aligned}
$$

Therefore, it follows that

$$
\|u\|_{\infty} \leq\left(\frac{(\alpha+1)}{2} \max \left(\int_{a}^{b} K_{a}(t) d t, \int_{a}^{b} K_{b}(t) d t\right)\right)^{1 /(\alpha+1)}
$$

Let $\varepsilon:=1 /\|u\|_{\infty}$. On the other side, since $\alpha \leq \gamma$ and $\sigma-1=\alpha$ we also have that

$$
\begin{gathered}
-\left((\varepsilon u)^{\sigma}\right)^{\prime \prime} \leq-\sigma(\varepsilon u)^{\sigma-1}(\varepsilon u)^{\prime \prime} \leq \\
\sigma(\varepsilon u)^{\sigma-1}\left(\varepsilon^{1+\alpha} K(x)-\lambda \varepsilon^{1+\gamma} M(x)\right)(\varepsilon u)^{-\alpha}= \\
\sigma\left(\varepsilon^{1+\alpha} K(x)-\lambda \varepsilon^{1+\gamma} M(x)\right) \quad \text { in } \Omega .
\end{gathered}
$$

Hence the positivity of $\mathcal{S}$ now tells us that $\mathcal{S}\left(\varepsilon^{1+\alpha} K-\lambda \varepsilon^{1+\gamma} M\right)>0$ and thus $\lambda<\varepsilon^{-(\gamma-\alpha)} \mathcal{S}(K) / \mathcal{S}(M)$. Furthermore, since

$$
\mathcal{S}(K)\left(\frac{a+b}{2}\right)=\frac{1}{2} \int_{a}^{b} K(t) \delta_{\Omega}(t) d t
$$

(see the proof of Lemma 2.1) and an analogous statement is valid for $\mathcal{S}(M)$, the theorem follows employing the upper bound for $\|u\|_{\infty}$ derived in the first part of the proof.

Remark 3.6. Given $M, K, \alpha$ and $\gamma$, let $\Lambda:=\{\lambda>0:$ (1.1) has a solution $\}$. If $\Lambda \neq \emptyset$ (for instance if (3.3) holds) then either $\Lambda=\left(0, \lambda_{0}\right)$ or $\left(0, \lambda_{0}\right]$ for some $0<\lambda_{0} \leq \infty$. Indeed, define $\lambda_{0}:=\sup _{\lambda \in \Lambda}$, and now this follows from Theorem [2.3, Remark 2.4 and the fact any solution of (1.1) is a subsolution of (1.1) with $\underline{\lambda}$ in place of $\lambda$ whenever $\underline{\lambda} \leq \lambda$. Let us note that if $\alpha \leq \gamma$ the above theorem says that $\lambda_{0}<\infty$.

Let now $m: \Omega \rightarrow \mathbb{R}$ be a function that may change $\operatorname{sign}$ in $\Omega$. We write as usual $m=m^{+}-m^{-}$with $m^{+}:=\max (m, 0)$ and $m^{-}:=\max (-m, 0)$. As a direct consequence of Theorems 3.1 and 3.5 we obtain the

Corollary 3.7. Let $0 \not \equiv m \in L^{2}(\Omega)$ and let $m_{a}^{-}$and $m_{b}^{-}$be given by (3.1). (i) If $m^{-} \in L^{p}(\Omega)$ with $p \geq 2, \gamma \in(0,(p-1) / p)$ and

$$
\begin{gathered}
\max \left(\left\|m_{a}^{-}\right\|_{L^{p}(\Omega)},\left\|m_{b}^{-}\right\|_{L^{p}(\Omega)}\right) \leq c_{\gamma, p, a, b} \frac{\left(\int_{a}^{b} m^{+}(t) \delta_{\Omega}(t) d t\right)^{1+\gamma}}{\left(\int_{a}^{b} m^{+}(t) d t\right)^{\gamma}}, \quad \text { where } \\
c_{\gamma, p, a, b}:=\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{\left(1-\gamma p^{\prime}\right)^{1 / p^{\prime}}}{(b-a)^{\gamma+1 / p^{\prime}}},
\end{gathered}
$$

then (1.2) has a solution $u \in C^{1}(\Omega) \cap C(\bar{\Omega})$; and $u \in W^{2, q}(\Omega) \cap P^{\circ}, q>1$, whenever $K \delta_{\Omega}^{-\gamma} \in L^{r}(\Omega)$ with $r>1$.
(ii) If

$$
\max \left(\int_{a}^{b} m_{a}^{-}(t) d t, \int_{a}^{b} m_{b}^{-}(t) d t\right)<\int_{a}^{b} m^{+}(t) \delta_{\Omega}(t) d t
$$

then (1.2) has a solution $u \in W^{2, q}(\Omega) \cap P^{\circ}, q>1$, for all $\gamma \in\left(0, \gamma_{0}\right]$ and some $\gamma_{0}>0$.
(iii) If (1.2) has a solution $u \in W^{2, q}(\Omega), q>1$, then

$$
\int_{a}^{b} m^{-}(t) \delta_{\Omega}(t) d t<\int_{a}^{b} m^{+}(t) \delta_{\Omega}(t) d t
$$

## 4 The $N$-dimensional problem

We consider now the case of a smooth bounded domain $\mathbb{R}^{N}, N \geq 2$. We shall denote

$$
\begin{gathered}
B_{R}:=\left\{x \in \mathbb{R}^{N}:|x|<R\right\} \\
\omega_{N-1}:=\text { surface area of the unit sphere } \partial B_{1} \text { in } \mathbb{R}^{N}, \\
\operatorname{diam}(\Omega):=\sup _{x, y \in \Omega}|x-y| \\
P^{\circ}:=\text { interior of the positive cone of } C^{1}(\bar{\Omega})
\end{gathered}
$$

Since all the proofs in this section are natural adaptations of the proofs in the one-dimensional case, we shall only give the minor changes that are needed. For simplicity in the first theorem we shall assume that $M \in L^{\infty}(\Omega)$.

Theorem 4.1. Let $0 \leq K \in L^{p}(\Omega), p>N$, and let $0 \leq M \in L^{\infty}(\Omega)$. Assume $\alpha=\gamma$ and let $c_{\Omega}$ and $C_{\Omega}$ be given by Lemma 2.2. If $\gamma \in(0,1 / N)$ and

$$
\begin{gather*}
\|M\|_{L^{\infty}(\Omega)}<c_{\Omega, \gamma, N} \frac{\left(\int_{\Omega} K(x) \delta_{\Omega}(x) d x\right)^{1+\gamma}}{\|K\|_{L^{p}(\Omega)}^{\gamma}}, \quad \text { where }  \tag{4.1}\\
c_{\Omega, \gamma, N}:=\left(\frac{c_{\Omega}}{C_{\Omega}}\right)^{1+\gamma} \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\left(\frac{2}{\operatorname{diam}(\Omega)}\right)^{\gamma} \frac{1}{\left\|\delta_{\Omega}^{-\gamma}\right\|_{L^{N}(\Omega)}}
\end{gather*}
$$

then (1.1) has a solution $u \in C^{1}(\Omega) \cap C(\bar{\Omega})$ for all $\lambda \leq 1$; and $u \in W^{2, q}(\Omega) \cap$ $P^{\circ}, q>N$, whenever $K \delta_{\Omega}^{-\gamma} \in L^{r}(\Omega)$ with $r>N$. In particular, if $\Omega:=B_{R}$ then

$$
\begin{equation*}
c_{\Omega, \gamma, N} \geq\left(\frac{c_{\Omega}}{C_{\Omega}}\right)^{1+\gamma} \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{1}{R}\left(\frac{1-\gamma N}{\omega_{N-1}}\right)^{1 / N} . \tag{4.2}
\end{equation*}
$$

Proof. For $\gamma \in(0,1 / N)$ we pick $q \in(N, 1 / \gamma)$ and we set

$$
\begin{gathered}
\tau:=\frac{2}{C_{\Omega}\|K\|_{L^{p}(\Omega)} \operatorname{diam}(\Omega)}, \quad \mu:=\tau C_{\Omega}\|M\|_{L^{\infty}(\Omega)}\left\|\delta_{\Omega}^{-\gamma}\right\|_{L^{q}(\Omega)}, \\
\beta:=\tau c_{\Omega} \int_{\Omega} K(x) \delta_{\Omega}(x) d x, \quad r:=(\mu \gamma)^{1 /(\gamma+1)}, \quad d:=r \delta_{\Omega}, \\
\mathcal{C}:=\{v \in C(\bar{\Omega}): d \leq v \leq \tau \mathcal{S}(K) \text { in } \Omega\} .
\end{gathered}
$$

Taking into account Lemma 2.2 we find that $\mathcal{S}(\tau K) \leq 1$ in $\Omega$ and also

$$
\begin{gathered}
\mathcal{S}(\tau K) \geq \beta \delta_{\Omega} \quad \text { and } \\
\mathcal{S}\left(\tau M d^{-\gamma}\right) \leq \tau\|M\|_{L^{\infty}(\Omega)} \mathcal{S}\left(d^{-\gamma}\right) \leq \mu r^{-\gamma} \delta_{\Omega} \quad \text { in } \Omega .
\end{gathered}
$$

On the other hand, we may fix $q$ close enough to $N$ so that (4.1) implies that

$$
\mu \leq \beta^{1+\gamma} \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} .
$$

For $v \in \mathcal{C}$ define now $u:=\tau \mathcal{S}\left(K-M v^{-\gamma}\right)$. Taking into account the aforementioned facts we derive that

$$
\begin{gathered}
\tau \mathcal{S}(K) \geq u \geq \tau \mathcal{S}\left(K-M d^{-\gamma}\right) \geq\left(\beta-\mu r^{-\gamma}\right) \delta_{\Omega}= \\
(\mu \gamma)^{1 /(\gamma+1)}\left(\frac{\beta}{(\mu \gamma)^{1 /(\gamma+1)}}-\frac{1}{\gamma}\right) \delta_{\Omega} \geq(\mu \gamma)^{1 /(\gamma+1)}\left(\frac{\gamma+1}{\gamma}-\frac{1}{\gamma}\right) \delta_{\Omega}=d
\end{gathered}
$$

in $\Omega$ and hence $u \in \mathcal{C}$. Furthermore, since $\gamma<1 / N$ and $v \geq d$ it holds that $v \rightarrow K-M v^{-\gamma}$ is continuous from $C(\bar{\Omega})$ into $L^{s}(\Omega)$ for some $s>N$, and then the proof can be continued as the proof of Theorem 3.1(i).

Finally, if $\Omega:=B_{R}$, using polar coordinates one can see that

$$
\begin{gathered}
\left\|\delta_{\Omega}^{-\gamma}\right\|_{L^{N}(\Omega)}^{N}=\omega_{N-1} \int_{0}^{R} r^{N-1}(R-r)^{-\gamma N} d r \leq \\
\omega_{N-1} R^{N-1} \int_{0}^{R} r^{-\gamma N} d r=\frac{\omega_{N-1}}{1-\gamma N} R^{N(1-\gamma)}
\end{gathered}
$$

which in turn yields (4.2).
With the same proof as in the previous section we deduce the

Corollary 4.2. Let $K, M$ and $\gamma$ be as in Theorem 4.1, and let $\alpha>\gamma$. Then there exists $\lambda_{0}>0$ such that the problem (1.1) has a solution $u \in$ $C^{1}(\Omega) \cap C(\bar{\Omega})$ for all $\lambda \leq \lambda_{0}$; and $u \in W^{2, q}(\Omega) \cap P^{\circ}, q>N$, whenever $K \delta_{\Omega}^{-\alpha} \in L^{r}(\Omega)$ with $r>N$.

Let $\mathcal{M}:=\{x \in \Omega: M(x)>0\}$. We also have
Theorem 4.3. Let $\alpha, \gamma>0$ and let $0 \leq K, M \in L^{p}(\Omega)$, $p>N$, and $M \in C(\Omega)$. Let $c_{\Omega}$ and $C_{\Omega}$ be given by Lemma 2.2. Suppose that $\overline{\mathcal{M}} \subset \Omega$ and that

$$
\|M\|_{L^{p}(\Omega)}<\frac{c_{\Omega}}{C_{\Omega}} \int_{\Omega} K(x) \delta_{\Omega}(x) d x
$$

Then there exists

$$
\lambda_{0} \geq(\operatorname{dist}(\mathcal{M}, \partial \Omega))^{\gamma} \frac{\left(c_{\Omega} \int_{\Omega} K \delta_{\Omega}-C_{\Omega}\|M\|_{L^{p}(\Omega)}^{\gamma}\right)^{\gamma}}{\left(\frac{\operatorname{diam}(\Omega)}{2}\left(C_{\Omega}\|K\|_{L^{p}(\Omega)}-c_{\Omega} \int_{\Omega} M \delta_{\Omega}\right)\right)^{\alpha \frac{1+\gamma}{1+\alpha}}}
$$

such that for all $\lambda \leq \lambda_{0}$ the problem (1.1) has a solution $u \in C^{1}(\Omega) \cap C(\bar{\Omega})$; and $u \in W^{2, q}(\Omega) \cap P^{\circ}, q>N$, whenever $K \delta_{\Omega}^{-\alpha}, M \delta_{\Omega}^{-\gamma} \in L^{r}(\Omega)$ with $r>N$.

Proof. The proof follows as in Theorem 3.4 defining now

$$
\begin{aligned}
& \bar{\mu}:=\left(\frac{\operatorname{diam}(\Omega)}{2}\left(C_{\Omega}\|K\|_{L^{p}(\Omega)}-c_{\Omega} \int_{\Omega} M \delta_{\Omega}\right)\right)^{\alpha} \\
& \underline{\beta}:=\left(\operatorname{dist}(\mathcal{M}, \partial \Omega)\left(c_{\Omega} \int_{\Omega} K \delta_{\Omega}-C_{\Omega}\|M\|_{L^{p}(\Omega)}\right)\right)^{\gamma}
\end{aligned}
$$

Theorem 4.4. Let $0<\alpha \leq \gamma$ and let $C_{\Omega}$ be given by Lemma 2.2. Suppose for some $\lambda>0$ (1.1) has a solution $u \in W^{2, q}(\Omega), q>N$. Then

$$
\begin{equation*}
\lambda<\left(\frac{\operatorname{diam}(\Omega)}{2} C_{\Omega}(\alpha+1)\|K\|_{L^{p}(\Omega)}\right)^{\frac{\gamma-\alpha}{\alpha+1}} \inf _{\Omega}\left(\frac{\mathcal{S}(K)}{\mathcal{S}(M)}\right) \tag{4.3}
\end{equation*}
$$

Proof. Suppose $u$ is a solution of (1.1) for some $\lambda>0$. Employing Lemma 2.2 and the positivity of $\mathcal{S}$ and arguing as in Theorem 3.5 we can prove that

$$
\begin{gathered}
\|u\|_{\infty} \leq\left(\frac{\operatorname{diam}(\Omega)}{2} C_{\Omega}(\alpha+1)\|K\|_{L^{p}(\Omega)}\right)^{1 /(\alpha+1)} \quad \text { and } \\
\lambda<\|u\|_{\infty}^{\gamma-\alpha} \inf _{\Omega}\left(\frac{\mathcal{S}(K)}{\mathcal{S}(M)}\right)
\end{gathered}
$$

and this gives (4.3).
Remark 4.5. Let us note that the statement in Remark 3.6 is clearly also valid for the $N$-dimensional problem.

Corollary 4.6. Let $m \in L^{p}(\Omega)$ with $p>N$ and $m^{-} \in L^{\infty}(\Omega)$. Let $c_{\Omega}$ and $C_{\Omega}$ be given by Lemma 2.2.
(i) If $\gamma \in(0,1 / N)$ and

$$
\begin{gathered}
\left\|m^{-}\right\|_{L^{\infty}(\Omega)}<c_{\Omega, \gamma, N} \frac{\left(\int_{\Omega} m^{+}(x) \delta_{\Omega}(x) d x\right)^{1+\gamma}}{\left\|m^{+}\right\|_{L^{p}(\Omega)}^{\gamma}}, \quad \text { where } \\
c_{\Omega, \gamma, N}:=\left(\frac{c_{\Omega}}{C_{\Omega}}\right)^{1+\gamma} \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\left(\frac{2}{\operatorname{diam}(\Omega)}\right)^{\gamma} \frac{1}{\left\|\delta_{\Omega}^{-\gamma}\right\|_{L^{N}(\Omega)}}
\end{gathered}
$$

then (1.2) has a solution $u \in C^{1}(\Omega) \cap C(\bar{\Omega})$; and $u \in W^{2, q}(\Omega) \cap P^{\circ}, q>N$, whenever $m^{+} \delta_{\Omega}^{-\gamma} \in L^{r}(\Omega)$ with $r>N$. In particular, if $\Omega:=B_{R}$ then

$$
c_{\Omega, \gamma, N} \geq\left(\frac{c_{\Omega}}{C_{\Omega}}\right)^{1+\gamma} \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{1}{R}\left(\frac{1-\gamma N}{\omega_{N-1}}\right)^{1 / N}
$$

(ii) If (1.2) has a solution $u \in W^{2, q}(\Omega)$ with $q>N$, then $\mathcal{S}(m)>0$ in $\Omega$.

## References

[1] R. Agarwal, D. O'Regan, Singular Differential and Integral Equations with Applications, Kluwer Academic Publishers, Dordrecht, 2003.
[2] C. Bandle, M. Pozio, A. Tesei, The asymptotic behavior of the solutions of degenerate parabolic equations, Trans. Amer. Math. Soc. 303 (1987), 487-501.
[3] H. Brezis, Functional analysis, Sobolev spaces and partial differential equations. Universitext. Springer, New York, 2011.
[4] H. Brezis, X. Cabré, Some simple nonlinear PDE's without solutions, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 1 (1998), 223-262.
[5] Y. Choi, A. Lazer, P. McKenna, Some remarks on a singular elliptic boundary value problem, Nonlinear Anal. 32 (1998), 305-314.
[6] M. Crandall, P. Rabinowitz, L. Tartar, On a Dirichlet problem with a singular nonlinearity, Comm. Partial Differential Equations 2 (1977), 193-222.
[7] M. del Pino, A global estimate for the gradient in a singular elliptic boundary value problem, Proc. Roy. Soc. Edinburgh Sect. A 122 (1992), 341-352.
[8] J. Diaz, J. Morel, L. Oswald, An elliptic equation with singular nonlinearity, Comm. Partial Differential Equations 12 (1987), 1333-1344.
[9] J. Hernández, F. Mancebo, Singular elliptic and parabolic equations, M. Chipot (ed.) et al., Handbook of differential equations: Stationary partial differential equations. Vol. III. Amsterdam: Elsevier/North Holland. Handbook of Differential Equations, 317-400 (2006).
[10] J. Hernández, F. Mancebo, J. Vega, On the linearization of some singular, nonlinear elliptic problems and applications, Ann. Inst. H. Poincaré Anal. Non Linéaire 19 (2002), 777-813.
[11] J. Hernández, F. Mancebo, J. Vega, Nonlinear singular elliptic problems: recent results and open problems, in: Nonlinear Elliptic and Parabolic Problems, Progr. Nonlinear Differential Equations Appl., Vol. 64, Birkhäuser, Basel, 2005, pp. 227-242.
[12] J. Hernández, F. Mancebo, J. Vega, Positive solutions for singular nonlinear elliptic equations, Proc. Roy. Soc. Edinburgh Sect. A 137 (2007), 41-62.
[13] T. Godoy, U. Kaufmann, On strictly positive solutions for some semilinear elliptic problems, NoDEA Nonlinear Differ. Equ. Appl. 20 (2013), 779-795.
[14] T. Godoy, U. Kaufmann, Existence of strictly positive solutions for sublinear elliptic problems in bounded domains, Adv. Nonlinear Stud. 14 (2014), 353-359.
[15] S. Gomes, On a singular nonlinear elliptic problem, SIAM J. Math. Anal. 17 (1986), 1359-1369.
[16] U. Kaufmann, I. Medri, Strictly positive solutions for one-dimensional nonlinear elliptic problems, Electron. J. Differential Equations 2014, No. 126, 13 pp.
[17] U. Kaufmann, I. Medri, Strictly positive solutions for one-dimensional nonlinear problems involving the p-Laplacian, Bull. Austral. Math. Soc. 89 (2014), 243-251.
[18] A. Lair, A. Shaker, Classical and weak solutions of a singular semilinear elliptic problem, J. Math. Anal. Appl. 211 (1997), 371-385.
[19] A. Lazer, P. McKenna, On a singular nonlinear elliptic boundary-value problem, Proc. Amer. Math. Soc. 111 (1991), 721-730.
[20] N. H. Loc, K. Schmitt, Applications of sub-supersolution theorems to singular nonlinear elliptic problems, Adv. Nonlinear Stud. 11 (2011), 493-524.


[^0]:    *2000 Mathematics Subject Clasification. 35J75, 35J25, 35B09, 34B16.
    ${ }^{\dagger}$ Keywords and phrases. Singular elliptic problems, indefinite nonlinearities, positive solutions.
    ${ }^{\ddagger}$ Partially supported by Secyt-UNC.
    ${ }^{\S}$ E-mail addresses. godoy@mate.uncor.edu (T. Godoy), kaufmann@mate.uncor.edu (U. Kaufmann, Corresponding Author).

