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Spherical Functions: The Spheres Vs. The Projective Spaces

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Abstract. In this paper we establish a close relationship between the spherical functions of the n-dimensional sphere $S^n \simeq \mathrm{SO}(n+1)/\mathrm{SO}(n)$ and those of the n-dimensional real projective space $P^n(\mathbb{R}) \simeq \mathrm{SO}(n+1)/\mathrm{O}(n)$. In fact, for n odd a function on $\mathrm{SO}(n+1)$ is an irreducible spherical function of some type $\pi \in \hat{\mathrm{SO}}(n)$ if and only if it is an irreducible spherical function of some type $\gamma \in \hat{\mathrm{O}}(n)$. When n is even this is also true for certain types, and in the other cases we exhibit a clear correspondence between the irreducible spherical functions of both pairs $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ and $(\mathrm{SO}(n+1), \mathrm{O}(n))$. Summarizing, to find all spherical functions of one pair is equivalent to do so for the other pair.
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1. Introduction.

The theory of spherical functions dates back to the classical papers of É. Cartan and H. Weyl; they showed that spherical harmonics arise in a natural way from the study of functions on the n-dimensional sphere $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$. The first general results in this direction were obtained in 1950 by Gelfand who considered zonal spherical functions of a Riemannian symmetric space G/K.

The general theory of scalar valued spherical functions of arbitrary type, associated to a pair (G, K) with G a locally compact group and K a compact subgroup, goes back to Godement and Harish-Chandra. Later, in [Tir77] the attention was focused on the underlying matrix valued spherical functions defined as solutions of an integral equation; see Definition 1.1. These two notions are related by the operation of taking traces.

A first thorough study of irreducible spherical functions of any K-type was accomplished in the seminal work of the complex projective plane $P^2(\mathbb{C}) = \mathrm{SU}(3)/\mathrm{U}(2)$ in [GPT02]. Later, this study was developed in the complex projective spaces $P^n(\mathbb{C})$ for certain K-types; see [PT12].

Let us remember the definition of spherical function. Let G be a locally compact unimodular group and let K be a compact subgroup of G. Let $\hat{K}$

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denote the set of all equivalence classes of complex finite-dimensional irreducible representations of K; for each $\delta \in \hat{K}$, let ξ_δ denote the character of δ, $d(\delta)$ the degree of δ, i.e. the dimension of any representation in the class δ, and $\chi_\delta = d(\delta)\xi_\delta$. We shall choose once and for all the Haar measure dk on K normalized by $\int_K dk = 1$.

We shall denote by V a finite-dimensional vector space over the field $\mathbb{C}$ of complex numbers and by $\mathrm{End}(V)$ the space of all linear transformations of V into V. Whenever we refer to a topology on such a vector space we shall be talking about the unique Hausdorff linear topology on it.

Definition 1.1. A spherical function Φ on G of type $\delta \in \hat{K}$ is a continuous function on G with values in $\mathrm{End}(V)$ such that

i) $\Phi(e) = I$ (I= identity transformation).

ii) $\Phi(x)\Phi(y) = \int_K \chi_\delta(k^{-1})\Phi(xky)\, dk$, for all $x, y \in G$.

A spherical function $\Phi : G \longrightarrow \mathrm{End}(V)$ is called *irreducible* if V has no proper subspace invariant by $\Phi(g)$ for all $g \in G$. The reader can find a number of general results in [Tir77] and [GV88].

It is known that the compact connected symmetric spaces of rank one are of the form $X \simeq G/K$, where G and K are as follows:

i) $G = \mathrm{SO}(n+1)$, $K = \mathrm{SO}(n)$, $X = S^n$.

ii) $G = \mathrm{SO}(n+1)$, $K = \mathrm{O}(n)$, $X = P^n(\mathbb{R})$.

iii) $G = \mathrm{SU}(n+1)$, $K = \mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$, $X = P^n(\mathbb{C})$.

iv) $G = \mathrm{Sp}(n+1)$, $K = \mathrm{Sp}(n) \times \mathrm{Sp}(1)$, $X = P^n(\mathbb{H})$.

v) $G = F_{4(-52)}$, $K = \mathrm{Spin}(9)$, $X = P^2(Cay)$.

The zonal (i.e. of trivial K-type) spherical functions on $X \simeq G/K$ are the eigenfunctions of the Laplace-Beltrami operator that only depend on the distance $d(x, o)$, $x \in X$, where o is the origin of X. In each case we call them $\varphi_0, \varphi_1, \varphi_2, \dots$, with $\varphi_0 = 1$, and let $\varphi_j^*(\theta)$ be the corresponding function induced on $[0, L]$ by φ_j, where L is the diameter of X. These functions turn out to be Jacobi polynomials

$$\varphi_j^*(\theta) = c_j\, P_j^{(\alpha,\beta)}(\cos \lambda\theta),$$

with c_j defined by the condition $\varphi_j(0) = 1$ and λ, α and β depending on the pair (G, K). By renormalization of the distance we can assume $\lambda = 1$ and $L = \pi$, (cf. [Hel65, p. 171]). Now we quote the following list from [Koo73, p. 239]:

i) $G/K \simeq S^n$: $\alpha = (n-2)/2$, $\beta = (n-2)/2$.

ii) $G/K \simeq P^n(\mathbb{R})$: $\alpha = (n-2)/2$, $\beta = -1/2$.

iii) $G/K \simeq P^n(\mathbb{C})$: $\alpha = n-1$, $\beta = 0$.

iv) $G/K \simeq P^n(\mathbb{H})$: $\alpha = 2n-1$, $\beta = 1$.

v) $G/K \simeq P^2(Cay)$: $\alpha = 7$, $\beta = 3$.

Therefore, at first sight, the zonal spherical functions on the sphere and on the real projective space seem to be two completely different families (to clarify this point see the Appendix). But we prove in this paper that to know all the spherical functions associated to the n-dimensional sphere is equivalent to know them for the n-dimensional real projective space. Precisely, we state a direct relation between matrix valued spherical functions of the pair $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ and of the pair $(\mathrm{SO}(n+1), \mathrm{O}(n))$. In first place we prove that, for n even the spherical functions of the sphere and the spherical functions of the real projective space are the same, i.e., a function Φ on $\mathrm{SO}(n+1)$ is an irreducible spherical function of type $\pi \in \hat{\mathrm{SO}}(n)$ if and only if there exists $\gamma \in \hat{\mathrm{O}}(n)$ such that the function Φ is a spherical function of type γ. When n is odd there are some particular cases in which one has the same situation as when n is even, and we show that these cases are easily distinguished by looking at the highest weight of the corresponding $\mathrm{SO}(n)$-types. For the generic cases we show how every irreducible spherical function of the projective space is explicitly related with two spherical functions of the sphere; see Theorem 3.3.

An immediate consequence of this paper is obtained by combining it with [PTZ12], where all irreducible spherical functions of the pair $(\mathrm{SO}(4),\ \mathrm{SO}(3))$ are studied and exhibited. Therefore, by applying Theorem 3.1 we also know all spherical functions of the pair $(\mathrm{SO}(4), \mathrm{O}(3))$, which as functions on $\mathrm{SO}(4)$ are the same as those of the pair $(\mathrm{SO}(4), \mathrm{SO}(3))$.

2. Preliminaries

Spherical functions of type $\delta \in \hat{K}$ arise in a natural way upon considering representations of G. If $g \mapsto \tau(g)$ is a continuous representation of G, say on a finite-dimensional vector space E, then

$$P_\delta = \int_K \chi_\delta(k^{-1})\tau(k)\, dk$$

is a projection of E onto $P_\delta E = E(\delta)$. The function $\Phi : G \longrightarrow \mathrm{End}(E(\delta))$ defined by

$$\Phi(g)a = P_\delta \tau(g)a, \quad g \in G,\ a \in E(\delta),$$

is a spherical function of type δ. In fact, if $a \in E(\delta)$ we have

$$\begin{aligned}\Phi(x)\Phi(y)a = P_\delta\tau(x)P_\delta\tau(y)a &= \int_K \chi_\delta(k^{-1})P_\delta\tau(x)\tau(k)\tau(y)a\, dk\\ &= \left(\int_K \chi_\delta(k^{-1})\Phi(xky)\, dk\right) a.\end{aligned}$$

If the representation $g \mapsto \tau(g)$ is irreducible then the spherical function Φ is also irreducible. Conversely, any irreducible spherical function on a compact group G arises in this way from a finite-dimensional irreducible representation of G.

Now we recall some known facts ([GW09, §5.5.5]) about how one obtains the irreducible finite-dimensional representations of $\mathrm{O}(n)$ from the irreducible finite-dimensional representations of $\mathrm{SO}(n)$, in order to deeply understand our main results: Theorems 3.1, 3.2, 3.3.

Let us take $a \in \mathrm{O}(n)$ depending on n:

$$a = \mathrm{diag}(1, \dots, 1, -1), \qquad \text{if } n \text{ is even,}$$
$$a = \mathrm{diag}(-1, \dots, -1), \qquad \text{if } n \text{ is odd.}$$

And let ϕ be the involutive automorphism of $\mathrm{SO}(n)$ defined by

$$\phi(k) = aka,$$

for all $k \in \mathrm{SO}(n)$. Notice that when n is odd ϕ is trivial and $\mathrm{O}(n) = \mathrm{SO}(n) \times F$, where $F = \{1, a\}$. Therefore in this case the irreducible finite-dimensional representations of $\mathrm{O}(n)$ are of the form $\gamma = \pi \otimes 1$ or $\gamma = \pi \otimes \epsilon$ where π is an irreducible finite-dimensional representation of $\mathrm{SO}(n)$ and ϵ is the nontrivial character of F. Thus we have the following theorem.

Theorem 2.1. If n is odd $\mathrm{O}(n) = \mathrm{SO}(n) \times F$, and $\hat{\mathrm{SO}}(n) \times \hat{F}$ can be identified with the unitary dual of $\mathrm{O}(n)$, under the bijection $([\pi], 1) \mapsto [\pi \otimes 1]$ and $([\pi], \epsilon) \mapsto [\pi \otimes \epsilon]$.

If n is even we have $\mathrm{O}(n) = \mathrm{SO}(n) \rtimes F$. Let us denote by V_π the vector space associated to $\pi \in \hat{\mathrm{SO}}(n)$, then set $V_{\pi_\phi} = V_\pi$ and let $\pi_\phi : \mathrm{SO}(n) \to \mathrm{End}(V_{\pi_\phi})$ be the irreducible representation of $\mathrm{SO}(n)$ given by

$$\pi_\phi = \pi \circ \phi.$$

In this situation, in the following two subsections we shall consider two cases: $\pi_\phi \sim \pi$ and $\pi_\phi \nsim \pi$.

2.1. When π_ϕ is equivalent to π.

Take $A \in \mathrm{GL}(V)$ such that $\pi_\phi(k) = A\pi(k)A^{-1}$ for all $k \in \mathrm{SO}(n)$. Then

$$\pi(k) = \pi(a(aka)a) = \pi_\phi(aka) = A\pi(aka)A^{-1} = A\pi_\phi(k)A^{-1} = A^2\pi(k)A^{-2}.$$

Therefore, by Schur's Lemma, we have $A^2 = \lambda I$. By changing A by $\sqrt{\lambda^{-1}}A$ we may assume that $A^2 = I$. Let ϵ_A be the representation of F defined by

$$\epsilon_A(a) = A.$$

Now we define $\gamma = \pi \cdot \epsilon_A : \mathrm{O}(n) \to \mathrm{GL}(V)$ by

$$\gamma(kx) = \pi(k)\epsilon_A(x), \tag{1}$$

and it is easy to verify that γ is an irreducible representation of $\mathrm{O}(n)$. Moreover, if B is another solution of $\pi_\phi(k) = B\pi(k)B^{-1}$ for all $k \in \mathrm{SO}(n)$, and $B^2 = I$, then $B = \pm A$. In fact, by Schur's Lemma, $B = \mu A$ and $\mu^2 = 1$. In the set of all such pairs (π, A) we introduce the equivalence relation $(\pi, A) \sim (T\pi T^{-1}, TAT^{-1})$, where T is any bijective linear map from V onto another vector space, and set $[\pi, A]$ for the equivalence class of (π, A).

Proposition 2.2. When n is even $\mathrm{O}(n) = \mathrm{SO}(n) \rtimes F$. Let us assume that $\pi_\phi \sim \pi$. If $\gamma = \pi \cdot \epsilon_A$, then γ is an irreducible representation of $\mathrm{O}(n)$. Moreover, $\gamma' = \pi \cdot \epsilon_{-A}$, is another irreducible representation of $\mathrm{O}(n)$ not equivalent to γ. Therefore, the set $\{[\pi, A] : \pi_\phi(k) = A\pi(k)A^{-1}, A^2 = I\}$ can be included in $\hat{\mathrm{O}}(n)$ via the map $[\pi, A] \mapsto [\pi \cdot \epsilon_A]$.

Proof. The only thing that we really need to prove is that $\gamma' \nsim \gamma$. In fact, if $\gamma' = T\gamma T^{-1}$ for some $T \in \mathrm{GL}(V)$, then, since $\gamma'_{|\mathrm{SO}(n)} = \gamma_{|\mathrm{SO}(n)} = \pi$, Schur's Lemma implies that $T = \lambda I$. Hence $-A = \gamma'(a) = T\gamma(a)T^{-1} = \gamma(a) = A$ is a contradiction. ∎

2.2. When π_ϕ is not equivalent to π.
Assume that n is even and that $\pi_\phi \nsim \pi$. Let us consider the $\mathrm{SO}(n)$-module $V_\pi \times V_\pi$ and define

$$\gamma(k)(v, w) = (\pi(k)v, \pi_\phi(k)w), \quad \gamma(ka)(v, w) = (\pi(k)w, \pi_\phi(k)v), \tag{2}$$

for all $k \in \mathrm{SO}(n)$ and $v, w \in V_\pi$. Then it is easy to verify that $\gamma : \mathrm{O}(n) \to \mathrm{GL}(V_\pi \times V_\pi)$ is a representation of $\mathrm{O}(n)$.

Proposition 2.3. Assume that n is even and $\pi \in \widehat{\mathrm{SO}}(n)$, then $\mathrm{O}(n) = \mathrm{SO}(n) \rtimes F$. Moreover, if $\pi_\phi \nsim \pi$ and we define $\gamma : \mathrm{SO}(n) \times F \to \mathrm{GL}(V_\pi \times V_\pi)$ as in (2), then γ is an irreducible representation of $\mathrm{O}(n)$. Also $\gamma' : \mathrm{SO}(n) \times F \to \mathrm{GL}(V_\pi \times V_\pi)$, defined by

$$\gamma'(k)(v, w) = (\pi_\phi(k)v, \pi(k)w), \qquad \gamma'(ka)(v, w) = (\pi_\phi(k)w, \pi(k)v),$$

for all $k \in \mathrm{SO}(n)$ and $v, w \in V_\pi$, is an irreducible representation of $\mathrm{O}(n)$, but it is equivalent to γ. Therefore, the set $\{\{[\pi], [\pi_\phi]\} : [\pi] \in \widehat{\mathrm{SO}}(n)\}$ can be included in $\hat{\mathrm{O}}(n)$ via the map $\{[\pi], [\pi_\phi]\} \mapsto [\gamma]$.

Theorem 2.4. Assume that n is even. We split $\hat{\mathrm{O}}(n)$ into two disjoint sets: (a) $\{[\gamma] : \gamma_{|\mathrm{SO}(n)}$ irreducible$\}$ and (b) $\{[\gamma] : \gamma_{|\mathrm{SO}(n)}$ reducible$\}$.

(a) If $[\gamma]$ is in the first set and $\pi = \gamma_{|\mathrm{SO}(n)}$, then

$$\pi_\phi(k) = \pi(aka) = \gamma(aka) = \gamma(a)\pi(k)\gamma(a),$$

for all $k \in \mathrm{SO}(n)$. Therefore $\pi_\phi \sim \pi$ and γ is equivalent to the representation $\pi \cdot \epsilon_A$ constructed from π and $A = \gamma(a)$ in (1).

(b) If $[\gamma]$ is in the second set, let W be the representation space of γ. Let $V_\pi < W$ be an irreducible $\mathrm{SO}(n)$-module. Then $W = V_\pi \oplus V_{\pi_\phi}$ as $\mathrm{SO}(n)$-modules, and γ is equivalent to the representation γ' defined on $V_\pi \times V_\pi$ by (2).

2.3. The highest weights of π and π_ϕ.

When n is even it would be very useful to know when $\pi \in \hat{\mathrm{SO}}(n)$ is equivalent to π_ϕ. In that direction we prove a very simple criterion in terms of the highest weight of π.

For a given $\ell \in \mathbb{N}$, we know from [VK92] that the highest weight of an irreducible representation π of $\mathrm{SO}(2\ell)$ is of the form $\mathbf{m}_\pi = (m_1, m_2, m_3, \dots, m_\ell) \in \mathbb{Z}^\ell$, with

$$m_1 \geq m_2 \geq m_3 \geq \cdots \geq m_{\ell-1} \geq |m_\ell|.$$

We state the following simple result, which relates the highest weights of π and π_ϕ.

Theorem 2.5. If $\mathbf{m}_\pi = (m_1, m_2, m_3, \dots, m_\ell)$ is the highest weight of $\pi \in \hat{\mathrm{SO}}(2\ell)$ then $\mathbf{m}_{\pi_\phi} = (m_1, m_2, m_3, \dots, -m_\ell)$ is the highest weight of π_ϕ.

The matrices I_{ki}, $1 \leq i < k \leq 2\ell$, with -1 in the place (k, i), 1 in the place (i, k) and everywhere else zero, form a basis of the Lie algebra $\mathfrak{so}(2\ell)$. The linear span

$$\mathfrak{h} = \langle I_{21}, I_{43}, \dots, I_{2\ell,2\ell-1} \rangle_{\mathbb{C}}$$

is a Cartan subalgebra of $\mathfrak{so}(2\ell, \mathbb{C})$.

Now consider

$$H = i(x_1 I_{21} + \cdots + x_\ell I_{2\ell,2\ell-1}) \in \mathfrak{h},$$

and let $\epsilon_j \in \mathfrak{h}^*$ be defined by $\epsilon_j(H) = x_j$, $1 \leq j \leq \ell$. Then for $1 \leq j < k \leq \ell$, the following matrices are root vectors of $\mathfrak{s}o(2\ell, \mathbb{C})$:

$$\begin{aligned} X_{\epsilon_j+\epsilon_k} &= I_{2k-1,2j-1} - I_{2k,2j} - i(I_{2k-1,2j} + I_{2k,2j-1}), \\ X_{-\epsilon_j-\epsilon_k} &= I_{2k-1,2j-1} - I_{2k,2j} + i(I_{2k-1,2j} + I_{2k,2j-1}), \\ X_{\epsilon_j-\epsilon_k} &= I_{2k-1,2j-1} + I_{2k,2j} - i(I_{2k-1,2j} - I_{2k,2j-1}), \\ X_{-\epsilon_j+\epsilon_k} &= I_{2k-1,2j-1} + I_{2k,2j} + i(I_{2k-1,2j} - I_{2k,2j-1}). \end{aligned} \tag{3}$$

We choose the following set of positive roots

$$\Delta^+ = \{\epsilon_j + \epsilon_k, \epsilon_j - \epsilon_k : 1 \leq j < k \leq \ell\}$$

so that

$$\mathbf{m}_\pi = m_1\epsilon_1 + m_2\epsilon_2 + \ldots + m_\ell\epsilon_\ell.$$

Now we can prove Theorem 2.5.

Proof. First we prove that the highest weight vector v_π of the representation π is also a highest weight vector of π_ϕ: For every root vector $X_{\epsilon_j \pm \epsilon_k}$ with $1 \leq j < k < \ell$ we have that $\mathrm{Ad}(a)X_{\epsilon_j\pm\epsilon_k} = X_{\epsilon_j\pm\epsilon_k}$. And, when $k = \ell$ we have that $\mathrm{Ad}(a)X_{\epsilon_j\pm\epsilon_\ell} = X_{\epsilon_j\mp\epsilon_\ell}$. Hence, if we denote by $\dot\pi$ and $\dot\pi_\phi$ the representations of the complexification of $\mathfrak{so}(2\ell)$ corresponding to π and π_ϕ, respectively, we have $\dot\pi \circ \mathrm{Ad}(a) = \dot\pi_\phi$ and thus

$$\dot\pi_\phi(X_{\epsilon_j\pm\epsilon_k})v_\pi = \dot\pi(\mathrm{Ad}(a)X_{\epsilon_j\pm\epsilon_k})v_\pi = \dot\pi(X_{\epsilon_j\pm\epsilon_k})v_\pi = 0,$$

for $1 \leq j < k < \ell$. When $k = \ell$ we have

$$\dot{\pi}_\phi(X_{\epsilon_j \pm \epsilon_\ell})v_\pi = \dot{\pi}(\mathrm{Ad}(a)X_{\epsilon_j \pm \epsilon_\ell})v_\pi = \dot{\pi}(X_{\epsilon_j \mp \epsilon_\ell})v_\pi = 0.$$

Therefore v_π is a highest weight vector of π_ϕ.

Notice that $\mathrm{Ad}(a)I_{2j,2j-1} = I_{2j,2j-1}$ for $1 \leq j < \ell$ and that $\mathrm{Ad}(a)I_{2\ell,2\ell-1} = -I_{2\ell,2\ell-1}$, then

$$\dot{\pi}_\phi(iI_{2j,2j-1})v_\pi = \dot{\pi}(\mathrm{Ad}(a)iI_{2j,2j-1})v_\pi = \dot{\pi}(iI_{2j,2j-1})v_\pi = m_j v_\pi,$$

for $1 \leq j < \ell$. When $k = \ell$ we have

$$\dot{\pi}_\phi(iI_{2\ell,2\ell-1})v_\pi = \dot{\pi}(\mathrm{Ad}(a)iI_{2\ell,2\ell-1})v_\pi = -\dot{\pi}(iI_{2\ell,2\ell-1})v_\pi = -m_\ell v_\pi.$$

Hence the highest weight of π_ϕ is

$$\mathbf{m} = (m_1, m_2, \ldots, m_{\ell-1}, -m_\ell). \qquad \blacksquare$$

Corollary 2.6. An irreducible representation π of $\mathrm{SO}(2\ell)$, $\ell \in \mathbb{N}$, of highest weight $\mathbf{m} = (m_1, m_2, \ldots, m_\ell)$ is equivalent to π_ϕ if and only if $m_\ell = 0$.

Remark 2.1. The referee considered worth noting the following proof of Theorem 2.5 in terms of fundamental weights and the Dynkin diagram of $\mathfrak{so}(2\ell, \mathbb{C})$. The simple roots are

$$\epsilon_1 - \epsilon_2, \ldots, \epsilon_{\ell-1} - \epsilon_\ell, \epsilon_{\ell-1} + \epsilon_\ell.$$

These correspond to nodes of the Dynkin diagram. Conjugation by a corresponds to the non-trivial outer automorphism of $\mathfrak{so}(2\ell, \mathbb{C})$, which in turn corresponds to the non-trivial symmetry of the Dynkin diagram. Hence $\mathrm{Ad}(a)^*(\epsilon_i - \epsilon_{i+1}) = \epsilon_i - \epsilon_{i+1}$ if $1 \leq i \leq \ell - 2$ and $\mathrm{Ad}(a)^*(\epsilon_{\ell-1} - \epsilon_\ell) = \epsilon_{\ell-1} + \epsilon_\ell$. Therefore $\mathrm{Ad}(a)^*(\epsilon_i) = \epsilon_i$ if $1 \leq i \leq \ell - 1$ and $\mathrm{Ad}(a)^*(\epsilon_\ell) = -\epsilon_\ell$.

Since conjugation by a also preserves the set of positive roots, $\pi(a)$ transforms a dominant vector of V_π of weight $(m_1, \ldots, m_{\ell-1}, m_\ell)$ into a dominant vector of V_{π_ϕ} of weight $(m_1, \ldots, m_{\ell-1}, -m_\ell)$.

3. Spherical Functions

Let (V_τ, τ) be a unitary irreducible representation of $G = \mathrm{SO}(n+1)$ and (V_π, π) a unitary irreducible representation of $\mathrm{SO}(n)$.

Let us assume that n is odd. Then $\mathrm{O}(n) = \mathrm{SO}(n) \times F$ and the irreducible unitary representations of $\mathrm{O}(n)$ are of the form $\gamma = \pi \otimes 1$ or $\gamma = \pi \otimes \epsilon$. Suppose that π is a sub-representation of $\tau_{|\mathrm{SO}(n)}$. Let us observe that $a \in \mathrm{O}(n)$ as an element of G becomes $-I \in G$. Clearly $\tau(-I) = \pm I$. Take $\gamma = \pi \otimes 1$ if $\tau(-I) = I$ and $\gamma = \pi \otimes \epsilon$ if $\tau(-I) = -I$. Then γ is a sub-representation of $\tau_{|\mathrm{O}(n)}$. Let $\Phi^{\tau,\pi}$ and $\Phi^{\tau,\gamma}$ be, respectively, the corresponding spherical functions of $(G, \mathrm{SO}(n))$ and $(G, \mathrm{O}(n))$.

Theorem 3.1. Assume that n is odd. If $\Phi^{\tau,\pi}(-I) = I$, take $\gamma = \pi \otimes 1$, and if $\Phi^{\tau,\pi}(-I) = -I$, take $\gamma = \pi \otimes \epsilon$. Then $\Phi^{\tau,\pi}(g) = \Phi^{\tau,\gamma}(g)$ for all $g \in G$.

Proof. As $\mathrm{SO}(n)$-modules $V_\tau = V_\pi \oplus V_\pi^\perp$. But since $\tau(a) = \tau(-I) = \pm I$ the decomposition $V_\tau = V_\pi \oplus V_\pi^\perp$ is also an $\mathrm{O}(n)$-decomposition. Hence, the $\mathrm{SO}(n)$-projection P_π onto V_π is equal to the $\mathrm{O}(n)$-projection P_γ onto V_π. Therefore $\Phi^{\tau,\pi}(g) = P_\pi\tau(g)P_\pi = P_\gamma\tau(g)P_\gamma = \Phi^{\tau,\gamma}(g)$, completing the proof. ∎

Let us assume now that n is even, then $\mathrm{O}(n) = \mathrm{SO}(n) \rtimes F$. Suppose that $\pi \in \hat{\mathrm{SO}}(n)$ and that $\pi \sim \pi_\phi$. Then $\gamma = \pi \cdot \epsilon_A$, where $A \in \mathrm{GL}(V_\pi)$ is such that $\pi_\phi = A\pi A^{-1}, A^2 = I$, is an irreducible representation of $\mathrm{O}(n)$ in V_π as we have seen in Proposition 2.3. Now we use this result to obtain the following Theorem.

Theorem 3.2. Assume that n is even. Let $a = \mathrm{diag}(1,\dots,1,-1) \in \mathrm{O}(n)$ be identified with $a = \mathrm{diag}(1,\dots,1,-1,-1) \in \mathrm{SO}(n+1)$. Suppose that π is a sub-representation of $\tau_{|\mathrm{SO}(n)}$ and that $\pi \sim \pi_\phi$. Set $A = \Phi^{\tau,\pi}(a)$, and take $\gamma = \pi \cdot \epsilon_A$. Then $\Phi^{\tau,\pi}(g) = \Phi^{\tau,\gamma}(g)$ for all $g \in G$.

Proof. The first thing we have to prove is that $A \in \mathrm{GL}(V_\pi)$, $\pi_\phi = A\pi A^{-1}$ and $A^2 = I$. This last property follows directly from $a^2 = e$.

For all $k \in \mathrm{SO}(n)$ we have

$$\pi_\phi(k) = \pi(aka) = \tau(aka)_{|V_\pi} = \tau(a)\tau(k)\tau(a)_{|V_\pi}. \tag{4}$$

Therefore $\tau(a)V_\pi$ is a $\mathrm{SO}(n)$-module equivalent to π_ϕ. Since $\pi_\phi \sim \pi$, and by the multiplicity one property of the pair $(\mathrm{SO}(n+1), \mathrm{SO}(n))$, we obtain that $V_\pi = \tau(a)V_\pi$. Therefore $A = \tau(a)_{|V_\pi} \in \mathrm{GL}(V_\pi)$ and $\pi_\phi = A\pi A^{-1}$. Hence V_π is an $\mathrm{O}(n)$-submodule of V_τ and the corresponding representation is $\gamma = \pi \cdot \epsilon_A$. This implies that the $\mathrm{SO}(n)$-projection P_π onto V_π is equal to the $\mathrm{O}(n)$-projection P_γ onto V_π. Therefore $\Phi^{\tau,\pi}(g) = P_\pi\tau(g)P_\pi = P_\gamma\tau(g)P_\gamma = \Phi^{\tau,\gamma}(g)$. Finally we observe that $\Phi^{\tau,\pi}(a) = P_\pi\tau(a)P_\pi = \tau(a)_{|V_\pi}$, completing the proof. ∎

Let us assume that n is even, and take $\pi \in \hat{\mathrm{SO}}(n)$ such that $\pi \nsim \pi_\phi$. Let us consider the $\mathrm{SO}(n)$-module $V_\pi \times V_{\pi_\phi}$ and define $\gamma(k)(v,w) = (\pi(k)v, \pi_\phi(k)w)$, $\gamma(ka)(v,w) = (\pi(k)w, \pi_\phi(k)v)$ for all $k \in \mathrm{SO}(n)$, $v \in V_\pi$ and $w \in V_{\pi_\phi}$. Then γ is an irreducible representation of $\mathrm{O}(n)$ in $V_\pi \times V_{\pi_\phi}$.

Theorem 3.3. Assume that n is even. Let $a = \mathrm{diag}(1,\dots,1,-1) \in \mathrm{O}(n)$ be identified with $a = \mathrm{diag}(1,\dots,1,-1,-1) \in \mathrm{SO}(n+1)$. Suppose that π is a sub-representation of $\tau_{|\mathrm{SO}(n)}$ and that $\pi \nsim \pi_\phi$. Then $\tau(a)V_\pi \sim V_{\pi_\phi}$ as $\mathrm{SO}(n)$-modules and $V_\pi \oplus \tau(a)V_\pi$ is an irreducible $\mathrm{O}(n)$-submodule of V_τ equivalent to the irreducible representation γ in $V_\pi \times V_{\pi_\phi}$ constructed above. Moreover,

$$\Phi^{\tau,\gamma}(g) = \begin{pmatrix} \Phi^{\tau,\pi}(g) & \Phi^{\tau,\pi}(ga) \\ \Phi^{\tau,\pi_\phi}(ga) & \Phi^{\tau,\pi_\phi}(g) \end{pmatrix}$$

for all $g \in G$.

Proof. That $\tau(a)V_\pi \sim V_{\pi_\phi}$ as $\mathrm{SO}(n)$-modules follows from (4). Also, if we make the identification $V_\pi \times V_{\pi_\phi} \sim V_\pi \oplus \tau(a)V_\pi$ via the $\mathrm{SO}(n)$-isomorphism

$(v,w) \mapsto v+\tau(a)w$, and using again that $\pi_\phi(k)w = \tau(a)\tau(k)\tau(a)w$ (see (4)), we have

$$\begin{aligned}\gamma(k)(v,w) &= (\pi(k)v, \pi_\phi(k)w) = (\pi(k)v, \tau(a)\tau(k)\tau(a)w)\\ &\sim \pi(k)v + \tau(k)\tau(a)w = \tau(k)(v+\tau(a)w)\end{aligned}$$

for all $k \in \mathrm{SO}(n)$, and

$$\gamma(a)(v,w) = (w,v) \sim (w+\tau(a)v) = \tau(a)(v+\tau(a)w).$$

This proves that $V_\pi \oplus \tau(a)V_\pi$ as an $\mathrm{O}(n)$-submodule of V_τ is equivalent to the irreducible representation γ in $V_\pi \times V_{\pi_\phi}$. Therefore $P_\gamma = P_\pi \oplus P_{\pi_\phi}$.

Hence, for all $g \in G$ we have,

$$\begin{aligned}\Phi^{\tau,\gamma}(g) &= (P_\pi \oplus P_{\pi_\phi})\tau(g)(P_\pi \oplus P_{\pi_\phi})\\ &= P_\pi\tau(g)P_\pi \oplus P_\pi\tau(g)P_{\pi_\phi} \oplus P_{\pi_\phi}\tau(g)P_\pi \oplus P_{\pi_\phi}\tau(g)P_{\pi_\phi}.\end{aligned}$$

Thus in matrix form we have

$$\Phi^{\tau,\gamma}(g) = \begin{pmatrix}\Phi^{\tau,\pi}(g) & \Phi_{12}(g)\\ \Phi_{21}(g) & \Phi^{\tau,\pi_\phi}(g)\end{pmatrix},$$

where $\Phi_{21}(g) = P_{\pi_\phi}\tau(g)_{|V_\pi}$ and $\Phi_{12}(g) = P_\pi\tau(g)_{|\tau(a)V_\pi}$.

From the identity $\Phi^{\tau,\gamma}(ga) = \Phi^{\tau,\gamma}(g)\tau(a)_{|V_\pi\oplus\tau(a)V_\pi}$ we get

$$\begin{pmatrix}\Phi^{\tau,\pi}(ga) & \Phi_{12}(ga)\\ \Phi_{21}(ga) & \Phi^{\tau,\pi_\phi}(ga)\end{pmatrix} = \begin{pmatrix}\Phi^{\tau,\pi}(g) & \Phi_{12}(g)\\ \Phi_{21}(g) & \Phi^{\tau,\pi_\phi}(g)\end{pmatrix}\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},$$

which is equivalent to $\Phi_{12}(g) = \Phi^{\tau,\pi}(ga)$ and $\Phi_{21}(g) = \Phi^{\tau,\pi_\phi}(ga)$. The theorem is proved. ■

4. Appendix

The irreducible spherical functions of trivial K-type of $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ and $(\mathrm{SO}(n+1), \mathrm{O}(n))$ are, respectively, the zonal spherical functions of S^n and $P^n(\mathbb{R})$. According to our Theorems 3.1 and 3.2 the zonal spherical functions ϕ of $P^n(\mathbb{R})$, as functions on $\mathrm{SO}(n+1)$, coincide with those zonal spherical functions φ of S^n such that $\varphi(-I) = 1$.

As we said in the introduction the zonal spherical functions on the n-dimensional sphere and on the corresponding real projective space are, respectively, given by

$$\varphi_j^*(\theta) = c_j\, P_j^{\left(\frac{n-2}{2},\frac{n-2}{2}\right)}(\cos\theta), \qquad \phi_j^*(\theta) = c_j'\, P_j^{\left(\frac{n-2}{2},-\frac{1}{2}\right)}(\cos\theta),$$

with c_j, c_j' scalars such that $\varphi_j(0) = 1 = \phi_j(0)$, and $0 \le \theta \le \pi$.

In this appendix we explain this apparent inconsistency. To begin with, we notice that in both spaces the metric is chosen normalized by the diameter $L = \pi$. For a given $g \in \mathrm{SO}(n+1)$ we denote by $\theta(g)$ the distance in the sphere between

$g \cdot o$ and the origin o, and analogously we denote by $\theta'(g)$ the distance in the projective space between $g \cdot o$ and the origin o. Thus, it is not difficult to see that

$$\begin{aligned} 2\theta(g) &= \theta'(g), && \text{for } 0 \leq \theta(g) \leq \pi/2, \\ 2\pi - 2\theta(g) &= \theta'(g), && \text{for } \pi/2 \leq \theta(g) \leq \pi. \end{aligned}$$

Therefore we have that

$$\cos(2\theta(g)) = \cos(\theta'(g)), \tag{5}$$

for any $g \in \mathrm{SO}(n+1)$. In the other hand from [AAR00, (3.1.1)] we know that Jacobi polynomials satisfy

$$P_{2k}^{(\alpha,\alpha)}(x) = \frac{k!(\alpha+1)_{2k}}{(2k)!(\alpha+1)_k} P_k^{(\alpha,-1/2)}(2x^2-1). \tag{6}$$

Then, if we put $x = \cos(\theta(g))$ in (6) we obtain

$$P_{2k}^{(\alpha,\alpha)}(\cos(\theta(g))) = \frac{k!(\alpha+1)_{2k}}{(2k)!(\alpha+1)_k} P_k^{(\alpha,-1/2)}(\cos(2\theta(g))),$$

hence, using (5) we have that $\varphi_{2j}^*(\theta(g)) = \phi_j^*(\theta'(g))$ for all $g \in \mathrm{SO}(n+1)$. In other words the following identity between zonal spherical functions holds: $\varphi_{2j} = \phi_j$ as functions on $\mathrm{SO}(n+1)$ for all $j \geq 0$.

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