WEIGHTED INEQUALITIES FOR SOME INTEGRAL OPERATORS WITH ROUGH KERNELS

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ABSTRACT. In this paper we study integral operators with kernels

$$K(x,y) = k_1(x - A_1 y)...k_m(x - A_m y),$$

$k_i(x) = \frac{\Omega_i(x)}{|x|^{n/q_i}}$ where $\Omega_i : \mathbb{R}^n \to \mathbb{R}$ are homogeneous functions of degree zero, satisfying a size and a Dini condition, A_i are certain invertible matrices, and $\frac{n}{q_1} + \dots \frac{n}{q_m} = n - \alpha$, $0 \le \alpha < n$. We obtain the appropriate weighted $L^p - L^q$ estimate, the weighted BMO and weak type estimates for certain weights in $A(p,q)$. We also give a Coifman type estimate for these operators.

1. INTRODUCTION

Let $0 \le \alpha < n$, $1 < m \in \mathbb{N}$. For $1 \le i \le m$, let $1 < q_i < \infty$ such that $\frac{n}{q_1} + \dots + \frac{n}{q_m} = n - \alpha$. We denote by $\Sigma = \Sigma_{n-1}$ the unit sphere in $\mathbb{R}^n$. Let $\Omega_i \in L^1(\Sigma)$. If $x \neq 0$, we write $x' = x/|x|$. We extend this function to $\mathbb{R}^n \setminus \{0\}$ as $\Omega_i(x) = \Omega_i(x')$. Let

$$k_i(x) = \frac{\Omega_i(x)}{|x|^{n/q_i}}. \tag{1.1}$$

In this paper we study the integral operator

$$T_\alpha f(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy, \tag{1.2}$$

with $K(x,y) = k_1(x - A_1 y)...k_m(x - A_m y)$, where A_i, are certain invertible matrices and $f \in L^\infty_{loc}(\mathbb{R}^n)$.

In the case $A_i = a_i I$, $a_i \in \mathbb{R}$, T. Godoy and M. Urciuolo in [6] obtain the $L^p(\mathbb{R}^n, dx) - L^q(\mathbb{R}^n, dx)$ boundedness of this operator for $0 \le \alpha < n$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. In the case that Ω_i are smooth functions, in [12] P. Rocha and M. Urciuolo consider the operator T_α for matrices $A_1, \dots, A_m$ satisfying the following hypothesis

(H) *A_i is invertible and $A_i - A_j$ is invertible for $i \neq j$, $1 \le i, j \le m$.*

They obtain that T_α is bounded from $H^p(\mathbb{R}^n, dx)$ into $L^q(\mathbb{R}^n, dx)$, for $0 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

For $0 \le \alpha < n$ and $1 \le s < \infty$ we define

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$$M_{\alpha,s}f(x)=\sup_B |B|^{\frac{\alpha}{n}}\left(\frac{1}{|B|}\int_B |f(x)|^s\,dx\right)^{\frac{1}{s}}$$

where the supremum is taken along all the balls B such that x belongs to B. We observe that $M=M_{0,1}$, where M is the classical Hardy-Littlewood maximal operator, also for $0<\alpha<n$, $M_\alpha=M_{\alpha,1}$ is the classical fractional maximal operator.

It is well known (see [9]) that if w is a weight (i.e. w is a non negative function and $w\in L^1_{loc}(\mathbb{R}^n,dx)$) then M_α is bounded from $L^p(\mathbb{R}^n,w^p)$ into $L^q(\mathbb{R}^n,w^q)$, for $1<p<\frac{n}{\alpha}$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, if and only if

$$\sup_B\left[\left(\frac{1}{|B|}\int_B w^q\right)^{\frac{1}{q}}\left(\frac{1}{|B|}\int_B w^{-p'}\right)^{\frac{1}{p'}}\right]<\infty, \tag{1.3}$$

where $\frac{1}{p}+\frac{1}{p'}=1$. The class of functions that satisfy (1.3) is called $A(p,q)$.

Throughout this paper we understand that for $p=\infty$, $(\int_E |f|^p)^{\frac{1}{p}}$ stands for $||f\chi_E||_\infty$, for any E is a measurable set . With this in mind we define the class $A(p,q)$ still by (1.3) for all $1\le p\le\infty$ and $1\le q\le\infty$. If A_p, $p\ge 1$, denotes the classical Muckenhoupt class of weights, we note that $w\in A(p,p)$ is equivalent to $w^p\in A_p$. We recall that $A_\infty=\cup_{p\ge1}A_p$. We recall that the statement $w\in A(\infty,\infty)$ is equivalent to $w^{-1}\in A_1$.

In [10] and [11] we consider $\Omega_i\equiv 1$ and weights satisfying the following condition

There exists $c>0$ such that

$$w(A_i x)\le c w(x), \tag{1.4}$$

a.e. $x\in\mathbb{R}^n$, $1\le i\le m$.

We note that if w is a power weight then w satisfies (1.4). Observe that there are another weights that satisfy this condition. For example consider

$$w(x)=\begin{cases}\ln(\frac{1}{|x|}), & \text{if} \quad |x|\le\frac{1}{e},\\ 1, & \text{if} \quad |x|>\frac{1}{e},\end{cases}$$

in [7], it is shown that $w\in A_1$ and it is easy to check that for any $a\in\mathbb{R}-\{0\}$ there exists C_a such that $w(ax)\le C_a w(x)$, a.e. $x\in\mathbb{R}$. In [11] we obtain weighted estimates for this kind of operators and certain weights satisfying (1.4), precisely as for the classical fractional integral operator I_α (for $0<\alpha<n$) or the singular integral operator (for $\alpha=0$), we prove the $L^p(\mathbb{R}^n,w^p)-L^q(\mathbb{R}^n,w^q)$ boundedness of T_α for weights $w\in A(p,q)$, $1<p<\frac{n}{\alpha}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$ and $0\le\alpha<n$.

Given a function $f\in L^1_{loc}(\mathbb{R}^n,dx)$ we define the sharp maximal function by

$$M^\sharp f(x)=\sup_{B\ni x}\frac{1}{|B|}\int\left|f(y)-\frac{1}{|Q|}\int_B|f|\right|dy,$$

and the space

$$BMO=\{f\in L^1_{loc}(\mathbb{R}^n,dx): M^\sharp f\in L^\infty(\mathbb{R}^n,dx)\},$$

the norm in this space is $||f||_*=||M^\sharp f||_\infty$.

There is also a weighted version of BMO, this is denoted $BMO(w)$ that is described by the semi norm

$$|||f|||_w = \sup_B \|w\chi_B\|_\infty \left(\frac{1}{|B|} \int_B \left| f(x) - \frac{1}{|B|} \int_B f \right| dx \right). \tag{1.5}$$

It is easy to check that

$$|||f||| \simeq ||wM^\sharp f||_\infty.$$

In [11] we also obtain the weighted weak type $(1, \frac{n}{n-\alpha})$ estimate for $w \in A(1, \frac{n}{n-\alpha})$ and w satisfying (1.4). We also prove that if $w \in A\left(\frac{n}{\alpha}, \infty\right)$ and w satisfies (1.4) then

$$|||T_\alpha f|||_w \le C \left(\int (|f|\, w)^{\frac{n}{\alpha}} \right)^{\frac{\alpha}{n}}, \tag{1.6}$$

The key argument to obtain the above stated results was the Coifman type estimate (see Theorem 2.1 in [11])

$$\int_{\mathbb{R}^n} |T_\alpha f(x)|^p w(x)\, dx \le C \int_{\mathbb{R}^n} |M_\alpha f(x)|^p w(x)\, dx,$$

$f \in L^\infty_c(\mathbb{R}^n, dx)$, $p > 0$ and $w \in A_\infty$ satisfying (1.4).

For integral operators with rough kernels of the form

$$T_{\Omega,\alpha} f(x) = \int \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)\, dy,$$

in [8], [4] and [13] the authors obtain weighted estimates for $T_{\Omega,0}$ for certain functions Ω homogeneous of degree zero and $\Omega \in L^p(S^{n-1})$ for some $p > 1$. In [2] the authors prove the corresponding weighted results, for $\alpha > 0$. Also in [1] the authors obtain a Coifman type inequality for general fractional integrals operators with kernels satisfying a Hörmander condition given by a Young function. In §2 we describe this condition.

In this paper we consider the operator T_α defined in (1.2) where, for $1 \le i \le m$, k_i is given by (1.1) and the matrices A_i satisfy the hypothesis (H). For $1 \le p \le \infty$ and $\Omega_i \in L^1(\Sigma)$, we define the L^p- modulus of continuity as

$$\varpi_{i,p}(t) = \sup_{|y|\le t} \|\Omega_i(\cdot + y) - \Omega_i(\cdot)\|_{p,\Sigma}.$$

We will make the following hypothesis about the functions Ω_i, $1 \le i \le m$,

(H_1) There exists $p_i > q_i$ such that $\Omega_i \in L^{p_i}(\Sigma)$,

(H_2) $\displaystyle\int_0^1 \varpi_{i,p_i}(t) \frac{dt}{t} < \infty.$

In §2 we obtain a pointwise estimate that relates $(M^\sharp |T_\alpha f|^\delta(x))^{1/\delta}$, for $0 < \delta < 1$, with a fractional maximal function of an appropriate power of f. This estimate is the fundamental key to obtain weighted inequalities for the operator T_α. These inequalities are developed in §3. We give first a Coifman type estimate for these operators that allows us to get the adequate weighted $L^p - L^q$ estimate for certain weights in $A(p,q)$. The results that we obtain in Theorems 3.3 and 3.4 are the analogous of Theorem 1 and 2 in [2]. We also get corresponding weighted BMO and weak type estimates.

Throughout this paper c and C will denote positive constants, not the same at each occurrence.

2. Pointwise estimate

We denote by $|x| \sim R$ the set $\{x \in \mathbb{R}^n : R < |x| \leq 2R\}$ and for $1 \leq r \leq \infty$

$$||f||_{r,|x|\sim R} = \left(\frac{1}{|B(0,2R)|} \int_{B(0,2R)} |f|^r \chi_{|x|\sim R} \right)^{\frac{1}{r}}.$$

In [1] the authors introduce the following definition

Definition 2.1. *Given $0 \leq \alpha < n$ and $1 \leq r \leq \infty$ we say that $k \in H_{r,\alpha}$ if there exist $c \geq 1$ and $C > 0$ such that for all $y \in \mathbb{R}^n$ and $R > c|y|$*

$$\sum_{m=1}^{\infty} (2^m R)^{n-\alpha} ||k(. - y) - k(.)||_{r,|x|\sim 2^m R} \leq C.$$

In Proposition 4.2 of the mentioned paper they prove that that if k_i is as in (1.1) and Ω_i satisfies (H_2) then

$$k_i \in H_{\frac{n}{q_i'},p_i}.$$

Theorem 2.2. *Let $0 \leq \alpha < n$ and let T_α the integral operator defined by (1.2). We suppose that for $1 \leq i \leq m$, the matrices A_i and the functions Ω_i satisfy the hypothesis (H), (H_1) and (H_2). If $s \geq 1$ is defined by $\frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{s} = 1$ then there exists $C > 0$ such that, for $0 < \delta \leq 1$ and $f \in L_c^\infty(\mathbb{R}^n, dx)$*

$$(M^\sharp |T_\alpha f|^\delta (x))^{1/\delta} \leq C \sum_{i=1}^{m} M_{\alpha,s} f(A_i^{-1} x). \tag{2.1}$$

Proof. Let $f \in L_c^\infty(\mathbb{R}^n, dx)$, $f \geq 0$ and $0 < \delta \leq 1$. As in [6] it can be proved that T_α is a bounded operator from $L^p(\mathbb{R}^n, dx)$ into $L^q(\mathbb{R}^n, dx)$, for $1 < p < \frac{n}{\alpha}$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, so $T_\alpha(f) \in L^1_{loc}(\mathbb{R}^n, dx)$ and $M_\delta^\sharp(T_\alpha f)(x)$ is well defined for all $x \in \mathbb{R}^n$. Let $x \in \mathbb{R}^n$ and let $B = B(x_B, R)$ be a ball that contains x, centered at x_B with radius R, and $T_\alpha f(x_B) < \infty$. We write $\widetilde{B} = B(x_B, 4R)$, and for $1 \leq i \leq m$ we also set $\widetilde{B}_i = A_i^{-1}\widetilde{B}$. Let $f_1 = f\, \chi_{\bigcup_{1\leq i\leq m} \widetilde{B}_i}$ and let $f_2 = f - f_1$.

We choose $a = T_\alpha f_2(x_B)$. By Jensen's inequality and from the inequality

$$|t^\delta - s^\delta|^{1/\delta} \leq |t - s|,$$

which holds for any positive t, s,

$$\begin{aligned} \left(\frac{1}{|B|} \int_B |(T_\alpha f)^\delta(y) - a^\delta| dy \right)^{1/\delta} &\leq \left(\frac{1}{|B|} \int_B |T_\alpha f(y) - a| dy \right) \\ &\leq \left(\frac{1}{|B|} \int_B |T_\alpha f_1(y)| dy \right) + \left(\frac{1}{|B|} \int_B |T_\alpha f_2(y) - a| dy \right) \\ &= I + II. \end{aligned}$$

We consider first the case $0 < \alpha < n$.

$$I = \frac{1}{|B|} \int_B |T_\alpha f_1(y)| dy$$
$$\leq \frac{1}{|B|} \int_B \sum_{i=1}^m \int_{\widetilde{B}_i} |K(y,z)| f(z) dz\, dy = \sum_{i=1}^m \frac{1}{|B|} \int_{\widetilde{B}_i} f(z) \int_B |K(y,z)| dy\, dz.$$

If $z \in \widetilde{B}_i$

$$\int_B |K(y,z)| dy \leq \int_{\{y \in B : |y - A_1 z| \leq |y - A_r z|,\ 1 \leq r \leq m\}} |K(y,z)| dy + \cdots + \int_{\{y \in B : |y - A_m z| \leq |y - A_r z|,\ 1 \leq r \leq m\}} |K(y,z)| dy \tag{2.2}$$

For $1 \leq l \leq m$ and $j \in \mathbb{N}$, let

$$C_j^l = \{y \in B : |y - A_l z| \leq |y - A_r z|,\ \ 1 \leq r \leq m \text{ and } |y - A_l z| \sim 2^{-j-1} R\}.$$

We observe that if $y \in B$ then $|y - A_l z| \leq 5R < 8R$. By Hölder's inequality

$$\int_{\{y \in B : |y - A_l z| \leq |y - A_r z|,\ 1 \leq r \leq m\}} |K(y,z)| dy \leq \sum_{j=-3}^{\infty} \int_{C_j^l} |K(y,z)| dy$$
$$\leq C \sum_{j=-3}^{\infty} \left[||k_1(. - A_1 z)\chi_{C_j^l}||_{p_1} \dots ||k_m(. - A_m z)\chi_{C_j^l}||_{p_m} (2^{-j} R)^{n/s} \right]. \tag{2.3}$$

If $p_l < \infty$, then

$$||k_l(. - A_l z)\chi_{C_j^l}||_{p_l} = \left(\int_{2^{-j-1} R \leq |u| \leq 2^{-j} R} \left(\frac{|\Omega_l(u)|}{|u|^{n/q_l}} du \right)^{p_l} \right)^{1/p_l}$$
$$\leq C 2^{\frac{jn}{q_l}} R^{-\frac{n}{q_l}} \left(\int_{2^{-j-1} R \leq |u| \leq 2^{-j} R} |\Omega_l(u)|^{p_l} du \right)^{1/p_l}$$
$$\leq C 2^{\frac{jn}{q_l}} R^{-\frac{n}{q_l}} 2^{\frac{-jn}{p_l}} R^{\frac{n}{p_l}} ||\Omega_l||_{p_l}, \tag{2.4}$$

where the last inequality follows since Ω_l is homogeneous of degree zero. We observe that if $p_l = \infty$ we also have

$$||k_l(. - A_l z)\chi_{C_j^l}||_\infty \leq C 2^{\frac{jn}{q_l}} R^{-\frac{n}{q_l}} ||\Omega_l||_\infty.$$

For $1 \leq r \leq m$, $r \neq l$,we observe that if $y \in C_j^l$ then $|y - A_r z| \geq |y - A_l z| > 2^{-j-1}R$, so if $p_r < \infty$

$$
\begin{aligned}
||k_r(. - A_r z)\chi_{C_j^l}||_{p_r} &\leq \left(\sum_{k\geq 0} \int_{\{2^{-j+k-1}R \leq |u| \leq 2^{-j+k}R\}} \left(\frac{|\Omega_r(u)|}{|u|^{n/q_r}} \right)^{p_r} \right)^{1/p_r} \\
&\leq C \sum_{k\geq 0} 2^{(j-k)\frac{n}{q_r}} R^{-\frac{n}{q_r}} 2^{(-j+k)\frac{n}{p_r}} R^{\frac{n}{p_r}} ||\Omega_r||_{p_r} \\
&\leq C 2^{j\frac{n}{q_i r}} R^{-\frac{n}{q_r}} 2^{-j\frac{n}{p_r}} R^{\frac{n}{p_r}} ||\Omega_r||_{p_r} \sum_{k\geq 0} 2^{k(\frac{n}{p_r} - \frac{n}{q_r})} \\
&\leq C 2^{j\frac{n}{q_r}} R^{-\frac{n}{q_r}} 2^{-j\frac{n}{p_r}} R^{\frac{n}{p_r}} ||\Omega_r||_{p_r},
\end{aligned}
\tag{2.5}
$$

the last inequality follows since $p_r > q_r$. Again if $p_r = \infty$ we get

$$
||k_r(. - A_r z)\chi_{C_j^l}||_\infty \leq C 2^{\frac{jn}{q_r}} R^{-\frac{n}{q_r}} ||\Omega_r||_\infty
$$

Then from (2.3), (2.4) and (2.5) we obtain

$$
\begin{aligned}
&\int_{\{y\in B: |y - A_l z| \leq |y - A_r z|,\ 1\leq r\leq m\}} |K(y,z)| dy \\
&\leq C \sum_{j=-3}^{\infty} 2^{\frac{jn}{q_1}} R^{-\frac{n}{q_1}} 2^{\frac{-jn}{p_1}} R^{\frac{n}{p_1}} ||\Omega_1||_{p_1} \dots 2^{j\frac{n}{q_m}} R^{-\frac{n}{q_m}} 2^{-j\frac{n}{p_m}} R^{\frac{n}{p_m}} ||\Omega_m||_{p_m} (2^{-j}R)^{n/s} \\
&\leq C R^\alpha ||\Omega_1||_{p_1} \dots ||\Omega_m||_{p_m}.
\end{aligned}
\tag{2.6}
$$

So

$$
I \leq C \sum_{i=1}^{m} \frac{R^\alpha}{|B|} \int_{\widetilde{B}_i} f(z) dz \leq C \sum_{i=1}^{m} M_\alpha f(A_i^{-1}x) \leq C \sum_{i=1}^{m} M_{\alpha,s} f(A_i^{-1}x).
$$

On the other hand

$$
\begin{aligned}
II &= \frac{1}{|B|} \int_B |T_\alpha f_2(y) - T_\alpha f_2(x_B)| \, dy \\
&\leq \frac{1}{|B|} \int_B \int_{\left(\bigcup_{1\leq i\leq m} \widetilde{B}_i \right)^c} |K(y,z) - K(x_B,z)| f(z) dz dy \\
&\leq \sum_{l=1}^{m} \frac{1}{|B|} \int_B \int_{Z_l} |K(y,z) - K(x_B,z)| f(z) dz dy,
\end{aligned}
$$

where

$$
Z^l = \left(\bigcup_{1\leq i\leq m} \widetilde{B}_i \right)^c \bigcap \{ z : |x_B - A_l z| \leq |x_B - A_r z|, \text{for } 1 \leq r \leq m \}. \tag{2.7}
$$

We estimate now $|K(y,z)-K(x_B,z)|$ for $y\in B$ and $z\in Z^l$. It is easy to check that

$$\begin{aligned}&|K(y,z)-K(x_B,z)|\\ &\leq \sum_{i=1}^{m}\left[\prod_{r=1}^{i}|k_{r-1}(x_B-A_{r-1}z)||k_i(y-A_iz)-k_i(x_B-A_iz)|\prod_{r=i}^{m}|k_{r+1}(y-A_{r+1}z)|\right]\end{aligned} \tag{2.8}$$

where we define $k_0=k_{m+1}\equiv 1$.
For simplicity we estimate the first summand of (2.8), the other summands follow in analogous way. For $j\in\mathbb{N}$, let $D_j^l=\{z\in Z^l : |x_B-A_lz|\sim 2^{j+1}R\}$. We use Hölder's inequality to get

$$\begin{aligned}&\int_{Z^l}|k_1(y-A_1z)-k_1(x_B-A_1z)|\prod_{r=2}^{m}|k_r(y-A_rz)|f(z)dz\\ &=\sum_{j=1}^{\infty}\int_{D_j^l}|k_1(y-A_1z)-k_1(x_B-A_1z)|\prod_{r=2}^{m}|k_r(y-A_rz)|f(z)dz \qquad (2.9)\\ &\leq\sum_{j=1}^{\infty}||(k_1(y-A_1\cdot)-k_1(x_B-A_1\cdot))\chi_{D_j^l}||_{p_1}\prod_{r=2}^{m}||k_r(y-A_r\cdot)\chi_{D_j^l}||_{p_r}||f\chi_{D_j^l}||_s.\end{aligned}$$

Now, if $p_l<\infty$,

$$\begin{aligned}||k_l(y-A_l\cdot)\chi_{D_j^l}||_{p_l}&=\left(\int_{D_j^l}\frac{|\Omega_l(y-A_lz)|^{p_l}}{|y-A_lz|^{\frac{np_l}{q_l}}}dz\right)^{\frac{1}{p_l}}\\ &\leq C(R2^j)^{-\frac{n}{q_l}}\left(\int_{\{2^jR<|y-A_lz|\leq 2^{j+3}R\}}|\Omega_l(y-A_lz)|^{p_l}dz\right)^{\frac{1}{p_l}} \qquad (2.10)\\ &\leq C(2^jR)^{-\frac{n}{q_l}+\frac{n}{p_l}}\left(\int_{\{1<|u|\leq 8\}}|\Omega_l(u)|^{p_l}du\right)^{\frac{1}{p_l}}\\ &\leq C(2^jR)^{-\frac{n}{q_l}+\frac{n}{p_l}}||\Omega_l||_{p_l}\end{aligned}$$

where the first inequality follows since $|x_B-A_lz|/2\leq|y-A_lz|\leq 2|x_B-A_lz|$. If $p_l=\infty$ we also get

$$||k_l(y-A_l\cdot)\chi_{D_j^l}||_{\infty}\leq C(2^jR)^{-\frac{n}{q_l}}||\Omega_l||_{\infty}$$

For $r\neq l$, we observe that if $z\in D_l^j$ then $|x_B-A_rz|\geq|x_B-A_lz|\geq 2^{j+1}R$, so we decompose $D_j^l=\bigcup_{k\geq j}(D_j^l)_{k,r}$ where

$$(D_j^l)_{k,r}=\{z\in D_j^l : |x_B-A_rz|\sim 2^{k+1}R\} \tag{2.11}$$

If $p_r < \infty$,

$$\begin{aligned} ||k_r(y - A_r \cdot)\chi_{D_j^l}||_{p_r} &= \sum_{k\geq j}^{\infty} \left(\int_{(D_j^l)_{k,r}} |k_r(y - A_r z)|^{p_r} dz \right)^{\frac{1}{p_r}} \\ &\leq C||\Omega_r||_{p_r} \sum_{k\geq j}^{\infty} (2^k R)^{-\frac{n}{q_r} + \frac{n}{p_r}} \\ &\leq C||\Omega_r||_{p_r} (2^j R)^{-\frac{n}{q_r} + \frac{n}{p_r}} \end{aligned} \tag{2.12}$$

where the geometric sums converges since $p_r > q_r$. If $p_r = \infty$,

$$\begin{aligned} ||k_r(y - A_r \cdot)\chi_{D_j^l}||_{\infty} &= \sum_{k\geq j}^{\infty} ||k_r(y - A_r \cdot)\chi_{(D_j^l)_{k,r}}||_{\infty} \\ &\leq C||\Omega_r||_{\infty} (2^j R)^{-\frac{n}{q_r}}. \end{aligned}$$

Now for $l = 1$

$$||(k_1(y - A_1 \cdot) - k_1(x_B - A_1 \cdot))\chi_{D_j^1}||_{p_1} \leq C||(k_1(y - x_B + \cdot) - k_1(\cdot))\chi_{|x|\sim 2^{j+1}R}||_{p_1} \tag{2.13}$$

Since $n/p_2 + \cdots + n/p_m - (n/q_2 + \cdots + n/q_m) = \alpha - n/s - n/p_1 + n/q_1$ then (2.10), (2.12) and (2.13) imply

$$\begin{aligned} &\int_{Z^1} |k_1(y - A_1 z) - k_1(x_B - A_1 z)| \prod_{r=2}^{m} |k_r(y - A_r z)| f(z) dz \\ &\leq C \sum_{j=1}^{\infty} (2^j R)^{\frac{n}{q_1} - \frac{n}{p_1}} ||(k_1(y - x_B + \cdot) - k_1(\cdot))\chi_{|x|\sim 2^{j+1}R}||_{p_1} (2^j R)^{\alpha} \left(\frac{1}{(2^j R)^n} \int_{D_j^1} f^s(z) dz \right)^{\frac{1}{s}} \\ &\leq C M_{\alpha,s} f(A_1^{-1} x) \sum_{j=1}^{\infty} (2^j R)^{\frac{n}{q_1} - \frac{n}{p_1}} ||(k_1(y - x_B + \cdot) - k_1(\cdot))\chi_{|x|\sim 2^{j+1}R}||_{p_1} \\ &\leq C M_{\alpha,s} f(A_1^{-1} x), \end{aligned} \tag{2.14}$$

where the last inequality follows since $k_1 \in H_{\frac{n}{q_1'}, p_1}$.

For $l \neq 1$ we observe that

$$\begin{aligned} ||(k_1(y - A_1 \cdot) - k_1(x_B - A_1 \cdot))\chi_{D_j^l}||_{p_1} &\leq \sum_{k\geq j}^{\infty} ||(k_1(y - A_1 \cdot) - k_1(x_B - A_1 \cdot))\chi_{(D_j^l)_{k,1}}||_{p_1} \\ &\leq C \sum_{k\geq j}^{\infty} (2^k R)^{\frac{n}{p_1} - \frac{n}{q_1}} (2^k R)^{\frac{n}{q_1} - \frac{n}{p_1}} ||(k_1(y - x_B + \cdot) - k_1(\cdot))\chi_{|x|\sim 2^{k+1}R}||_{p_1} \\ &\leq C(2^j R)^{\frac{n}{p_1} - \frac{n}{q_1}}, \end{aligned} \tag{2.15}$$

where the last inequality follows since $p_1 > q_1$ and since $k_1 \in H_{\frac{n}{q_1'}, p_1,}$. So as in the case $l = 1$ we obtain

$$\int_{Z^l} |k_1(y - A_1 z) - k_1(x_B - A_1 z)| \prod_{r=2}^{m} |k_r(y - A_r z)| f(z)dz \le C M_{\alpha,s} f(A_l^{-1} x). \tag{2.16}$$

Then

$$II \le C \sum_{i=1}^{m} M_{\alpha,s} f(A_i^{-1} x).$$

Now we start with the case $\alpha = 0$.

If $p_i = \infty$ for all $1 \le i \le m$, we decompose

$$\begin{aligned}\left(\frac{1}{|B|}\int_B |(T_0 f)^\delta(y) - a^\delta| dy\right)^{1/\delta} &\le \left(\frac{C}{|B|}\int_B (T_0 f_1)^\delta(y) dy\right)^{1/\delta} \\ &\quad + \left(\frac{C}{|B|}\int_B |(T_0 f_2)^\delta(y) - a^\delta| dy\right)^{1/\delta} \\ &= I + II.\end{aligned}$$

To estimate I we observe that

$$|T_0 f(x)| \le C \int |x - A_1 y|^{-\frac{n}{q_1}} \cdots |x - A_m y|^{-\frac{n}{q_m}} f(y) dy = C T f(x). \tag{2.17}$$

In [11] we obtain that the operator T is of weak-type (1,1) with respect to the Lebesgue measure. Thus we take $0 < \delta < 1$ and we use Kolmogorov's inequality (see exercise 2.1.5. p. 91 in [7]) to get

$$\begin{aligned} I &\le \frac{C}{|B|}\int_{\mathbb{R}^n} f_1(y) dy \le \sum_{j=1}^{m} \frac{C}{|B|}\int_{\widetilde{B}_j} f(y) dy \\ &\le C \sum_{j=1}^{m} M f(A_j^{-1} x).\end{aligned}$$

To estimate II, we first use Jensen's inequality and then we proceed just as in the case $0 < \alpha < n$ to get

$$II \le C \sum_{j=1}^{m} M f(A_j^{-1} x),$$

and so the theorem follows in this case.

If $p_i < \infty$ for some $1 \le i \le m$, by Jensen's inequality

$$\begin{aligned}\left(\frac{1}{|B|}\int_B |(T_0 f)^\delta(y) - a^\delta| dy\right)^{1/\delta} &\le \left(\frac{1}{|B|}\int_B |T_0 f(y) - a| dy\right) \\ &\le \left(\frac{1}{|B|}\int_B |T_0 f_1(y)| dy\right) + \left(\frac{1}{|B|}\int_B |T_0 f_2(y) - a| dy\right) \\ &= I + II.\end{aligned}$$

As in [6] it can be proved that T_0 is bounded on $L^p(\mathbb{R}^n, dx)$ for $1 < p < \infty$. So by Hölder's inequality

$$I \leq \left(\frac{1}{|B|} \int_B |T_0 f_1(y)|^p dy \right)^{\frac{1}{p}} \leq C \left(\frac{1}{|B|} \int_{\mathbb{R}^n} |f_1(y)|^p dy \right)^{\frac{1}{p}} \leq C \sum_{j=1}^m M_{0,p} f(A_j^{-1}x).$$

As before, to estimate II we proceed as in the case $0 < \alpha < n$ to get

$$II \leq C \sum_{j=1}^m M_{0,s} f(A_j^{-1}x),$$

If we chose $p = s$ the theorem follows in this case. □

3. Weighted estimates

Our next aim is to obtain weighted $L^p - L^q$ estimates for the operator T_α and certain classes of weights. The fundamental tool to get these results is the following theorem where we prove a Coifman type inequality.

Theorem 3.1. *Let $0 \leq \alpha < n$ and let T_α the integral operator defined by (1.2). We suppose that for $1 \leq i \leq m$, the matrices A_i and the functions Ω_i satisfy the hypothesis (H), (H_1) and (H_2). Let $s \geq 1$ be defined by $\frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{s} = 1$, $0 < p < \infty$ and let $w \in A_\infty$ satisfying (1.4). Then there exists $C > 0$ such that, for $f \in L_c^\infty(\mathbb{R}^n, dx)$,*

$$\int_{\mathbb{R}^n} |T_\alpha f(x)|^p w(x)\, dx \leq C \int_{\mathbb{R}^n} |M_{\alpha,s} f(x)|^p w(x)\, dx,$$

always holds if the left hand side is finite.

Proof. Let $w \in A_\infty$, then there exists $r > 1$ such that $w \in A_r$. For $0 < p < \infty$ we take $0 < \delta < 1$, such that $1 < r < p/\delta$, thus $w \in A_{p/\delta}$. If $||T_\alpha f||_{p,w} < \infty$ then also $||(T_\alpha f)^\delta||_{\frac{p}{\delta},w} < \infty$. Under these conditions we can apply Theorem 2.20 in [5], p. 410, and from theorem (2.2) we get

$$\begin{aligned}
\int_{\mathbb{R}^n} |T_\alpha f(x)|^p w(x)\, dx &\leq \int_{\mathbb{R}^n} (M(T_\alpha f)^\delta(x))^{p/\delta} w(x)\, dx \\
&\leq C \int_{\mathbb{R}^n} (M_\delta^\sharp(T_\alpha f)(x))^p w(x)\, dx \\
&\leq C \int_{\mathbb{R}^n} \left(\sum_{i=1}^m M_{\alpha,s} f(A_i^{-1}x) \right)^p w(x)\, dx \\
&\leq C \sum_{i=1}^m \int_{\mathbb{R}^n} (M_{\alpha,s} f)^p(x) w(A_i x)\, dx \\
&\leq C \int_{\mathbb{R}^n} (M_{\alpha,s} f(x))^p w(x)\, dx,
\end{aligned}$$

where the last inequality follows since w satisfies (1.4).

□

Lemma 3.2. *Let $0 \leq \alpha < n$ and let T_α the integral operator defined by (1.2). We suppose that for $1 \leq i \leq m$, the matrices A_i and the functions Ω_i satisfy the hypothesis (H), (H_1) and (H_2). Let $s \geq 1$ be defined by $\frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{s} = 1$, let $w^s \in A\left(\frac{p}{s}, \frac{q}{s}\right)$ with $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If $f \in L^\infty_c(\mathbb{R}^n, dx)$ then $T_\alpha(f) \in L^q(\mathbb{R}^n, w^q)$.*

Proof. The proof follows similar lines than the proof of Lemma 2.2 in [11]. Since $w^s \in A\left(\frac{p}{s}, \frac{q}{s}\right)$ then $w^q \in A_r$ with $r = 1 + \frac{q}{s}\frac{1}{\left(\frac{p}{s}\right)'} = \frac{q}{n}(\frac{n}{s} - \alpha)$.

Let $\mathcal{M}_j = \max\{|A_j y| : |y| = 1\}$ and let $\mathcal{M} = \max_{1\leq j\leq m}\{\mathcal{M}_j\}$. Suppose supp $f \subseteq B(0,R)$. If $|x| > 2\mathcal{M}R$ and $y \in$ supp f, then for $1 \leq i \leq m$,

$$|x - A_i y| \geq |x| - |A_i y| = |x| - |y|\left|A_i\frac{y}{|y|}\right| \geq |x| - R\mathcal{M} \geq \frac{|x|}{2},$$

so by Hölder's inequality,

$$|T_\alpha f(x)| = \left|\int k_1(x - A_1 y)\cdots k_m(x - A_m y) f(y)\, dy\right|$$

$$\leq ||k_1(x - A_1\cdot)\chi_{\{|x-A_1\cdot|\geq\frac{|x|}{2}\}}||_{p_1}...||k_m(x - A_m\cdot)\chi_{\{|x-A_m\cdot|\geq\frac{|x|}{2}\}}||_{p_m}\|f\|_s.$$

Now,

$$||k_i(x - A_i\cdot)\chi_{\{|x-A_i\cdot|\geq\frac{|x|}{2}\}}||_{p_i} = \sum_{k\in\mathbb{N}}||k_i(x - A_i\cdot)\chi_{\{|x-A_i\cdot|\sim 2^{k-2}|x|\}}||_{p_i}$$

$$\leq C\sum_{k\in\mathbb{N}}\left(2^k|x|\right)^{-\frac{n}{q_i}}||\Omega_i\chi_{\{|\cdot|\sim 2^{k-2}|x|\}}||_{p_i} \leq \sum_{k\in\mathbb{N}}\left(2^k|x|\right)^{-\frac{n}{q_i}+\frac{n}{p_i}}||\Omega_i||_{p_i}$$

$$= C|x|^{-\frac{n}{q_i}+\frac{n}{p_i}}\|\Omega_i\|_{p_i}.$$

So

$$|T_\alpha f(x)| \leq C|x|^{\sum_{i=1}^{m} -\frac{n}{q_i}+\frac{n}{p_i}}\|\Omega_1\|_{p_1}...\|\Omega_m\|_{p_m}\|f\|_s = C|x|^{\alpha-\frac{n}{s}}\|f\|_s.$$

Thus

$$\int_{|x|>2\mathcal{M}R}|T_\alpha f(x)|^q w^q(x)dx = \sum_{k\in\mathbb{N}}\int_{|x|\sim 2^k\mathcal{M}R}|T_\alpha f(x)|^q w^q(x)dx$$

$$\leq C\sum_{k\in\mathbb{N}}\int_{|x|\sim 2^k\mathcal{M}R}|x|^{\left(\alpha-\frac{n}{s}\right)q}w^q(x)dx \leq C\sum_{k\in\mathbb{N}}\left(2^k\mathcal{M}R\right)^{\left(\alpha-\frac{n}{s}\right)q}w^q(B(0,2^{k+1}\mathcal{M}R).$$

Since $w^q \in A_r$, there exists $\widetilde{r} < r = \frac{q}{n}(\frac{n}{s} - \alpha)$ such that $w^q \in A_{\widetilde{r}}$ so $w^q(B(0,2^{k+1}\mathcal{M}R) \leq C2^{kn\widetilde{r}}$ (see Lemma 2.2 in [5]) so the last sum is finite.

To study

$$\int_{|x|\leq 2\mathcal{M}R}|T_\alpha f(x)|^q w^q(x)dx,$$

we recall that in [6] the authors obtain the boundedness of T_α from $L^p(\mathbb{R}^n, dx)$ into $L^q(\mathbb{R}^n, dx)$ for $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, and so we continue the proof as in [11]. □

We are now ready to prove the weighted boundedness result.

Theorem 3.3. *Let $0 \le \alpha < n$ and let T_α the integral operator defined by (1.2). We suppose that for $1 \le i \le m$, the matrices A_i and the functions Ω_i satisfy the hypothesis (H), (H_1) and (H_2). Let $s \ge 1$ be defined by $\frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{s} = 1$. Suppose w satisfies (1.4) and $w^s \in A\left(\frac{p}{s}, \frac{q}{s}\right)$ with $s < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then there exits $C > 0$ such that for $f \in L^\infty_c(\mathbb{R}^n, dx)$,*

$$\left(\int_{\mathbb{R}^n} |T_\alpha f(x)|^q w^q(x)\, dx\right)^{\frac{1}{q}} \le C \left(\int_{\mathbb{R}^n} |f(x)|^p w^p(x)\, dx\right)^{\frac{1}{p}}. \tag{3.1}$$

Proof. Since $w^s \in A(\frac{p}{s}, \frac{q}{s})$ for $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ then $w^q \in A_r \subset A_\infty$, with $r = \frac{q}{n}(\frac{n}{s} - \alpha)$. By Lemma 3.2 we have that $T_\alpha f \in L^q(\mathbb{R}^n, w^q)$. Moreover we recall that $w^s \in A(\frac{p}{s}, \frac{q}{s})$ implies that $M_{\alpha s}$ is bounded from $L^{\frac{p}{s}}(\mathbb{R}^n, w^{\frac{p}{s}})$ into $L^{\frac{q}{s}}(\mathbb{R}^n, w^{\frac{q}{s}})$, so we apply Theorem 3.1 to obtain

$$\begin{aligned}
\left(\int |T_\alpha f(x)|^q w^q(x) dx\right)^{\frac{1}{q}} &\le C \left(\int (M_{\alpha,s} f(x))^q w^q(x) dx\right)^{\frac{1}{q}} \\
&= C \left(\int (M_{\alpha s} |f(x)|^s)^{\frac{q}{s}} w^q(x) dx\right)^{\frac{1}{q}} \\
&\le C \left(\int |f(x)|^p w^p(x) dx\right)^{\frac{1}{p}}.
\end{aligned}$$

□

By a standard duality argument we obtain the following Theorem.

Theorem 3.4. *Let $0 \le \alpha < n$ and let T_α the integral operator defined by (1.2). We suppose that for $1 \le i \le m$, the matrices A_i and the functions Ω_i satisfy the hypothesis (H), (H_1) and (H_2). Let $s \ge 1$ be defined by $\frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{s} = 1$. Suppose w satisfies $w^{-1}(A_i^{-1} x) \le C w^{-1}(x)$ for all $1 \le i \le m$ and $w^{-s} \in A\left(\frac{q'}{s}, \frac{p'}{s}\right)$ with $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $q < s'$. Then there exits $C > 0$ such that for $f \in L^\infty_c(\mathbb{R}^n, dx)$,*

$$\left(\int_{\mathbb{R}^n} |T_\alpha f(x)|^q w^q(x)\, dx\right)^{\frac{1}{q}} \le C \left(\int_{\mathbb{R}^n} |f(x)|^p w^p(x)\, dx\right)^{\frac{1}{p}}. \tag{3.2}$$

Proof. We observe that the adjoint T^*_α of the operator T_α is the integral operator with kernel

$$\tilde{K}(x, y) = \tilde{k_1}(x - A_1^{-1} y) \cdots \tilde{k_m}(x - A_m^{-1} y)$$

where for $1 \le i \le m$

$$\tilde{k_i}(x) = \frac{\tilde{\Omega}_i(x)}{|A_i x|^{\frac{n}{qi}}} = \frac{\bar{\Omega}_i(-A_i x)}{|A_i x|^{\frac{n}{qi}}}.$$

It is easy to check that $\tilde{\Omega}_i$ satisfy (H_1) and (H_2) and also that $\tilde{k_i} \in H_{\frac{n}{q'_i}, p_i}$ for all $1 \le i \le m$. We take g with $||g||_{q', w^{-q'}} \le 1$, thus

$$\int_{\mathbb{R}^n} T_\alpha f(x) g(x) dx = \int_{\mathbb{R}^n} f(x) T^*_\alpha g(x) dx.$$

Hence

$$||T_\alpha f||_{q,w^q} = \sup_g |\int_{\mathbb{R}^n} f(x)T^*_\alpha g(x)dx| \leq ||f||_{p,w^p} \sup_g ||T^*_\alpha g||_{p',w^{-p'}}.$$

Since $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $1 < p < q < s'$ then $\frac{1}{p'} = \frac{1}{q'} - \frac{\alpha}{n}$ and $s < q' < \frac{n}{\alpha}$, so as in Theorem 3.3 we obtain

$$||T^*_\alpha g||_{p',w^{-p'}} \leq C||g||_{q',w^{-q'}} \leq C,$$

thus

$$||T_\alpha f||_{q,w^q} \leq C||f||_{p,w^p}.$$

□

We now obtain an estimate of the type (1.6) for the operator T_α and for certain weights in the class $A(\frac{n}{\alpha}, \infty)$.

Theorem 3.5. *Let $0 \leq \alpha < n$ and let T_α the integral operator defined by (1.2). We suppose that for $1 \leq i \leq m$, the matrices A_i and the functions Ω_i satisfy the hypothesis (H), (H_1) and (H_2). If $s \geq 1$ is defined by $\frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{s} = 1$. If $w^s \in A(\frac{n}{\alpha s}, \infty)$ and satisfies (1.4), then there exits $C > 0$ such that for $f \in L^\infty_c(\mathbb{R}^n, dx)$,*

$$|||T_\alpha f|||_w \leq C \left(\int (|f(x)|\, w(x))^{\frac{n}{\alpha}}\, dx \right)^{\frac{\alpha}{n}}.$$

Proof. We observe that if

$$w^s \in A\left(\frac{n}{\alpha s}, \infty\right) \text{ then } ||wM_{\alpha,s}f||_\infty \leq C||fw||_{\frac{n}{\alpha}}. \tag{3.3}$$

Indeed by Hölder's inequality we get

$$\frac{1}{|B|^{1-\frac{\alpha s}{n}}} \int_B |f(x)|^s dx \leq \frac{1}{|B|^{1-\frac{\alpha s}{n}}} \left(\int_B |f(x)|^{\frac{n}{\alpha}} w^{\frac{n}{\alpha}}(x)dx \right)^{\frac{\alpha s}{n}} \left(\int_B w^{-s(\frac{n}{\alpha s})'}(x)dx \right)^{\frac{1}{(\frac{n}{\alpha s})'}}.$$

Then, for $x \in B$, since $w^s \in A\left(\frac{n}{\alpha s}, \infty\right)$ we get

$$\begin{aligned} w(x) &\left(\frac{1}{|B|^{1-\frac{\alpha s}{n}}} \int_B |f(x)|^s dx \right)^{\frac{1}{s}} \\ &\leq \left(\int_B |f(x)|^{\frac{n}{\alpha}} w^{\frac{n}{\alpha}}(x)dx \right)^{\frac{\alpha}{n}} ||w^s \chi_B||_\infty^{\frac{1}{s}} \left(\frac{1}{|B|} \int_B w^{-s(\frac{n}{\alpha s})'}(x)dx \right)^{\frac{1}{(\frac{n}{\alpha s})'s}} \\ &\leq C \left(\int_{\mathbb{R}^n} |f(x)|^{\frac{n}{\alpha}} w^{\frac{n}{\alpha}}(x)dx \right)^{\frac{\alpha}{n}}, \end{aligned}$$

thus

$$w(x)M_{\alpha,s}f(x) \leq C||fw||_{\frac{n}{\alpha}},$$

and (3.3) follows.

Now, using Theorem 2.2 and (3.3), we get

$$\begin{aligned}|||T_\alpha f|||_w \simeq ||wM^\sharp T_\alpha f||_\infty &\leq C\sum_{i=1}^m ||wM_{\alpha,s}f(A_i^{-1}\cdot)||_\infty \\ &\leq C\sum_{i=1}^m \left(\int |f(A_i^{-1}x)w(x)|^{\frac{n}{\alpha}}dx\right)^{\frac{\alpha}{n}} \\ &\leq C\sum_{i=1}^m \left(\int |f(x)w(A_ix)|^{\frac{n}{\alpha}}dx\right)^{\frac{\alpha}{n}} \\ &\leq C\left(\int |f(x)w(x)|^{\frac{n}{\alpha}}dx\right)^{\frac{\alpha}{n}},\end{aligned}$$

where the last inequality follows since w satisfies hypothesis (1.4). □

Finally we prove that T_α satisfies a weighted weak type $(1, \frac{n}{n-\alpha})$ estimate for certain weights in $A(1, \frac{n}{n-\alpha})$.

Theorem 3.6. *Let $0 \leq \alpha < n$ and let T_α the integral operator defined by (1.2). We suppose that for $1 \leq i \leq m$, the matrices A_i and the functions Ω_i satisfy the hypothesis (H), (H_1) and (H_2). If $s \geq 1$ is defined by $\frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{s} = 1$. If $w^s \in A(1, \frac{n}{n-\alpha s})$ and satisfies (1.4) then there exists $C > 0$ such that for $f \in L_c^\infty(\mathbb{R}^n, dx)$,*

$$\sup_{\lambda>0} \lambda(w^{\frac{sn}{n-\alpha s}}\{x : |T_\alpha f(x)| > \lambda\})^{\frac{n-\alpha s}{sn}} \leq C\left(\int |f(x)|^s w^s(x)dx\right)^{\frac{1}{s}}.$$

Proof. Given $w \in A_\infty$ there exists $\beta > 0$ and $C > 0$ such that

$$w\{x : Mf(x) > 2\lambda, M^\sharp f(x) \leq \gamma\lambda\} \leq C\gamma^\beta w\{x : Mf(x) > \lambda\},$$

for any $\gamma > 0$ (see [3] p.146).

For $q \geq 1$, as in Theorem 3.2 in [11], we obtain that

$$\sup_{\lambda>0} \lambda^q w\{x : Mf(x) > \lambda\} \leq C \sup_{\lambda>0} \lambda^q w\{x : M^\sharp f(x) > \gamma\lambda\},$$

for some $\gamma > 0$.

We consider first the case $s > 1$. If $w^s \in A(1, \frac{n}{n-\alpha s})$ then $w^{\frac{sn}{n-\alpha s}} \in A_\infty$. So for $q = \frac{sn}{n-\alpha s}$, we obtain

$$\begin{aligned}\sup_{\lambda>0} \lambda(w^{\frac{sn}{n-\alpha s}}\{x : |T_\alpha f|(x) > \lambda\})^{\frac{n-\alpha s}{sn}} &\leq C \sup_{\lambda>0} \lambda(w^{\frac{sn}{n-\alpha s}}\{x : MT_\alpha f(x) > \lambda\})^{\frac{n-\alpha s}{sn}} \\ &\leq C \sup_{\lambda>0} \lambda(w^{\frac{sn}{n-\alpha s}}\{x : M^\sharp T_\alpha f(x) > \gamma\lambda\})^{\frac{n-\alpha s}{sn}} \\ &\leq C \sup_{\lambda>0} \lambda(w^{\frac{sn}{n-\alpha s}}\{x : \sum_{i=1}^m M_{\alpha,s}f(A_i^{-1}x) > C\gamma\lambda\})^{\frac{n-\alpha s}{sn}},\end{aligned}$$

where the last inequality follows from Theorem 2.2, with $\delta = 1$. Since w satisfies (1.4), it is easy to check that

$$w^{\frac{sn}{n-\alpha s}}\{x : M_{\alpha,s}f(A_i^{-1}x) > \lambda\} \leq C_i w^{\frac{sn}{n-\alpha s}}\{x : M_{\alpha,s}f(x) > \lambda\},$$

so

$$\begin{aligned}\sup_{\lambda>0}\lambda(w^{\frac{sn}{n-\alpha s}}\{x:|T_\alpha f|(x)>\lambda\})^{\frac{n-\alpha s}{sn}} &\leq C\sup_{\lambda>0}\lambda(w^{\frac{sn}{n-\alpha s}}\{x:M_{\alpha,s}f(x)>\lambda\})^{\frac{n-\alpha s}{sn}}\\ &\leq C\sup_{\lambda>0}\lambda(w^{\frac{sn}{n-\alpha s}}\{x:M_{\alpha s}|f|^s(x)>\lambda^s\})^{\frac{n-\alpha s}{sn}}\\ &\leq C\left(\int|f(x)|^s w^s(x)dx\right)^{\frac{1}{s}},\end{aligned}$$

where the last inequality follows since $w^s \in A(1, \frac{n}{n-\alpha s})$, and since $M_{\alpha s}$ is of weak type $(1, \frac{n}{n-\alpha s})$.

If $s = 1$, T_α is bounded by the operator T defined in (2.17) so we proceed as in the proof of Theorem 3.2 in [11].

□

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