CROSSED EXTENSIONS OF THE COREPRESENTATION CATEGORY OF FINITE SUPERGROUP ALGEBRAS

ADRIANA MEJÍA CASTAÑO AND MARTÍN MOMBELLI

Abstract. We present explicit examples of finite tensor categories that are C_2-graded extensions of the corepresentation category of certain finite-dimensional non-semisimple Hopf algebras.

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1. Introduction

Throughout this paper we shall work over an algebraically closed field $\Bbbk$ of characteristic zero.

Given a finite group Γ, a (faithful) Γ-grading on a finite tensor category $\mathcal{D}$ is a decomposition $\mathcal{D} = \oplus_{g\in\Gamma}\mathcal{D}_g$, where $\mathcal{D}_g$ are full Abelian subcategories of $\mathcal{D}$ such that

- $\mathcal{D}_g \neq 0$;
- $\otimes : \mathcal{D}_g \times \mathcal{D}_h \to \mathcal{D}_{gh}$ for all $g, h \in \Gamma$.

In this case $\mathcal{C} = \mathcal{D}_e$ is a tensor subcategory of $\mathcal{D}$. The tensor category $\mathcal{D}$ is a Γ-*extension* of $\mathcal{C}$. The category $\mathcal{D}_g$ is an invertible $\mathcal{C}$-bimodule category for any $g \in \Gamma$. This gives rise to a group homomorphism $c : \Gamma \to \mathrm{BrPic}(\mathcal{C})$, where $\mathrm{BrPic}(\mathcal{C})$ is the so-called *Brauer-Picard group* of $\mathcal{C}$ introduced in [7]. The Brauer-Picard group of a finite tensor category $\mathcal{C}$ is the group of equivalence classes of invertible exact $\mathcal{C}$-bimodule categories.

Given a finite group Γ and a fusion category $\mathcal{C}$, Γ-extensions of $\mathcal{C}$ were classified in [7]. Any such extension depends on a group map $c : \Gamma \to \mathrm{BrPic}(\mathcal{C})$ and certain cohomological data. The problem of giving concrete examples of Γ-extensions of a given finite tensor category $\mathcal{C}$ is that, besides the cohomological obstructions, the explicit computation of the Brauer-Picard group is needed. The computation of Brauer-Picard group is in general complicated. Some computations of this group were done in [14], [15], [12].

A different version of Γ-extensions was studied in [10]. In *loc. cit.* the author studies and classifies Γ-gradings $\mathcal{D} = \oplus_{g\in\Gamma}\mathcal{D}_g$ such that there are equivalences $\mathcal{D}_g \simeq \mathcal{D}_e$ as $\mathcal{D}_e$-module categories for any $g \in \Gamma$. Such extensions are called Γ-*crossed products* and they are classified by equivalence

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classes of *crossed systems* of Γ over $\mathcal{C}$. A crossed system of Γ over $\mathcal{C}$ consists of a collection $\Sigma = ((a_*, \xi^a), (U_{a,b}, \sigma^{a,b}), \gamma_{abc})_{a,b,c\in\Gamma}$ where

- $(a_*, \xi^a) : \mathcal{C} \to \mathcal{C}$ are monoidal autoequivalences, with monoidal structure
$$\xi^a_{X,Y} : a_*(X\otimes Y) \to a_*(X)\otimes a_*(Y), \quad X, Y \in \mathcal{C};$$
- invertible objects $U_{a,b} \in \mathbb{C}$;
- pseudo-natural isomorphisms
$$\sigma^{a,b}_X : a_*b_*(X)\otimes U_{a,b} \to U_{a,b}\otimes (ab)_*X, \quad X \in \mathcal{C};$$
- isomorphisms $\gamma_{a,b,c} : a_*(U_{b,c})\otimes U_{a,bc} \to U_{a,b}\otimes U_{ab,c}$,

such that they satisfy certain conditions. If Σ is a crossed system of Γ over $\mathcal{C}$ we define a new category $\mathcal{C}(\Sigma) = \oplus_{a\in\Gamma}\mathcal{C}_a$ as Abelian categories and $\mathcal{C}_a = \mathcal{C}$ for all $a \in \Gamma$. Denote by $[V, a]$ the object $V \in \mathcal{C}_a$. In [10] the author introduces a new tensor product on the category $\mathcal{C}(\Sigma)$ given by
$$[V, a]\otimes[W, b] = [V\otimes a_*(W)\otimes U_{a,b}, ab],$$
for any $V, W \in \mathcal{C},\ a, b \in \Gamma$. The conditions of crossed system ensures that $\mathcal{C}(\Sigma)$ is indeed a monoidal category.

The present paper is devoted to give explicit examples of C_2-crossed products, where C_2 is the cyclic group of two elements, of the category $\mathrm{Comod}(H)$ of finite-dimensional H-comodules, where H is a supergroup algebra. Part of the information needed to compute crossed systems in this particular case is the computation of tensor autoequivalences $F : \mathrm{Comod}(H) \to \mathrm{Comod}(H)$, thus we need to compute the group $\mathrm{BiGal}(H)$ of equivalence classes of biGalois objects over H [16]. The group $\mathrm{BiGal}(H)$ is interesting from the Hopf algebraic point of view. It was computed only for few examples, see [3], [4], [17]. In this work we present a technique to compute the biGalois group for supergroup algebras. This technique is different to the one presented by Schauenburg in [17].

The paper is organized as follows. In Section 2 we give the required notations. In Section 3 we describe the Hopf algebra structure of the supergroup algebras introduced in [1]. For any supergroup algebra we describe the projective cover of its simple objects. This description will be useful when computing certain Frobenius-Perron dimensions. In Section 4 we classify biGalois objects for supergroup algebras. BiGalois objects are a fundamental piece of information to compute examples of crossed systems. In Section 5 we recall the definition of crossed product tensor category as introduced in [10] and how they are constructed from crossed systems. We also give a more concrete description of crossed systems in the case the tensor category is the category of corepresentations of a finite-dimensional Hopf algebra. In Section 6 we give explicit examples of crossed systems of C_2 over a supergroup algebra and we describe the monoidal structure. We obtain eight non-equivalent tensor categories and we compute their Frobenius-Perron dimensions.

2. Preliminaries and notation

If Γ is a finite group and $\psi \in Z^2(\Gamma, \Bbbk^\times)$ is a 2-cocycle, there is another 2-cocycle ψ' in the same cohomology class as ψ such that

$$\psi'(g,1) = \psi'(1,g) = 1, \quad \psi'(g,g^{-1}) = 1, \quad \psi'(g,h)^{-1} = \psi'(h^{-1},g^{-1}), \tag{2.1}$$

for all $g,h \in \Gamma$. From now on, all elements in $Z^2(\Gamma, \Bbbk^\times)$ representing some class in $H^2(\Gamma, \Bbbk^\times)$ will satisfy equation (2.1)

If H is a Hopf algebra and $g \in G(H)$ is a group-like element, we denote $\Bbbk_g$ the one dimensional vector space generated by w_g with left H-comodule given by

$$\lambda : \Bbbk_g \to H \otimes_{\Bbbk} \Bbbk_g, \quad \lambda(w_g) = g \otimes w_g.$$

A *coradically graded Hopf algebra* $H = \oplus_{i=0}^m H(i)$ is a Hopf algebra H that is a graded algebra and a graded coalgebra such that the coradical filtration is given by $H_n = \oplus_{i=0}^n H(i)$.

If H is a coradically graded Hopf algebra and (A, λ) is a left H-comodule algebra, the *Loewy series* on A is given by $A_n = \lambda^{-1}(H_n \otimes_{\Bbbk} A)$, $n = 1, \dots, m$, see [13]. The associated graded algebra $\operatorname{gr} A$ is again a left H-comodule algebra. If the coradical H_0 is a Hopf subalgebra then A_0 is a left H_0-comodule algebra. The comodule algebra A is H-*simple* if it has no non-trivial ideals $I \subseteq A$ such that $\lambda(I) \subseteq H \otimes_{\Bbbk} I$.

2.1. **Twisting Hopf algebras.**

Let H be a Hopf algebra . A Hopf 2-cocycle for H is a convolution invertible map $\sigma : H \otimes_{\Bbbk} H \to \Bbbk$, such that

$$\sigma(x_{(1)}, y_{(1)}) \sigma(x_{(2)} y_{(2)}, z) = \sigma(y_{(1)}, z_{(1)}) \sigma(x, y_{(2)} z_{(2)}), \tag{2.2}$$

$$\sigma(x,1) = \varepsilon(x) = \sigma(1,x), \tag{2.3}$$

for all $x,y,z \in H$. There is a new Hopf algebra structure constructed over the same coalgebra H with product described by

$$x \cdot_{[\sigma]} y = \sigma(x_{(1)}, y_{(1)}) \sigma^{-1}(x_{(3)}, y_{(3)})\, x_{(2)} y_{(2)}, \qquad x,y \in H. \tag{2.4}$$

This new Hopf algebra is denoted by $H^{[\sigma]}$. If (A, λ) is a left H-comodule algebra, then we can define a new product in A by

$$a \cdot_\sigma b = \sigma(a_{(-1)}, b_{(-1)})\, a_{(0)} . b_{(0)}, \quad a,b \in A. \tag{2.5}$$

We shall denote by A_σ this new algebra. With the same comodule structure, A_σ is a left $H^{[\sigma]}$-comodule algebra.

Let H be a pointed coradically graded Hopf algebra with coradical $\Bbbk G$, G a finite group. Let $\psi \in Z^2(G, \Bbbk^\times)$ be a 2-cocycle. There exists a Hopf 2-cocycle $\sigma_\psi : H \otimes_{\Bbbk} H \to \Bbbk$ such that for any homogeneous elements $x,y \in H$

$$\sigma_\psi(x,y) = \begin{cases} \psi(x,y), & \text{if } x,y \in H(0); \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

See [11, Lemma 4.1].

2.2. **Bicategories.** For a review on basic notions on bicategories we refer to [2]. Any monoidal category $\mathcal{C}$ gives rise to a bicategory $\underline{\mathcal{C}}$ with only one object. If $\mathcal{C}, \mathcal{D}$ are strict monoidal categories, a *pseudo-functor* $(F, \xi) : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a monoidal functor between the monoidal categories $\mathcal{C}$ and $\mathcal{D}$. If $(F, \xi), (G, \zeta) : \mathcal{C} \to \mathcal{D}$ are monoidal functors, a *pseudo-natural transformation* between them is a pair $(\eta_0, \eta) : (F, \xi) \to (G, \zeta)$ where $\eta_0 \in \mathcal{D}$ is an object and for any $X \in \mathcal{C}$ natural transformations

$$\eta_X : F(X) \otimes \eta_0 \to \eta_0 \otimes G(X),$$

such that for all $x, y \in \mathcal{C}$

$$(\mathrm{id}_{\eta_0} \otimes \zeta_{X,Y})\eta_{X\otimes Y} = (\eta_X \otimes \mathrm{id}_{G(Y)})(\mathrm{id}_{F(X)} \otimes \eta_Y)(\xi_{X,Y} \otimes \mathrm{id}_{\eta_0}). \tag{2.7}$$

Given two pseudo-natural transformations $(\eta_0, \eta) : (F, \xi) \to (G, \zeta)$ and $(\sigma_0, \sigma) : (G, \zeta) \to (H, \chi)$ their composition is given by

$$(\eta_0 \otimes \sigma_0, (\mathrm{id}_{\eta_0} \otimes \sigma)(\eta \otimes \mathrm{id}_{\sigma_0})) : (F, \xi) \to (H, \chi), \tag{2.8}$$

and their tensor product is given by

$$(F(\sigma_0) \otimes \eta_0, (\mathrm{id}_{F(\sigma_0)} \otimes \eta_{H(-)})(\xi^{-1}_{\sigma_0, H(-)} \otimes \mathrm{id}_{\eta_0})(F(\sigma)\xi_{G(-),\sigma_0} \otimes \mathrm{id}_{\eta_0})). \tag{2.9}$$

A pair (η_0, η) is a *pseudo-natural isomorphism* if there exists another pseudo-natural transformation (σ_0, σ) such that

$$(\eta_0, \eta)(\sigma, \sigma_0) = (\mathbf{1}_{\mathcal{D}}, \mathrm{id}_F), \quad (\sigma_0, \sigma)(\eta_0, \eta) = (\mathbf{1}_{\mathcal{D}}, \mathrm{id}_G).$$

Consequently, the object η_0 is invertible in $\mathcal{D}$, that is, there exists an object $\overline{\eta_0} \in \mathcal{D}$ such that $\eta_0 \otimes \overline{\eta_0} = \mathbf{1}_{\mathcal{D}} = \overline{\eta_0} \otimes \eta_0$.

If $(\eta_0, \eta), (\sigma_0, \sigma) : F \to G$ are pseudo-natural transformations, a *modification* $\gamma : (\eta_0, \eta) \rightrightarrows (\sigma_0, \sigma)$ is a morphism $\gamma \in \mathrm{Hom}_{\mathcal{C}}(\eta_0, \sigma_0)$ such that for all $V \in \mathcal{C}$

$$(\gamma \otimes \mathrm{id}_{G(V)})\eta_V = \sigma_V(\mathrm{id}_{F(V)} \otimes \gamma). \tag{2.10}$$

Given two modifications $\gamma : (\eta_0, \eta) \rightrightarrows (\sigma_0, \sigma)$ and $\overline{\gamma} : (\sigma_0, \sigma) \rightrightarrows (\tau_0, \tau)$ their composition is given by the composition of morphisms in $\mathcal{D}$.

γ is an *invertible modification* if there exist another modification $\overline{\gamma}$ such that $\gamma \circ \overline{\gamma} = \mathrm{id}_{\eta_0}$ and $\overline{\gamma} \circ \gamma = \mathrm{id}_{\sigma_0}$.

2.3. **Hopf biGalois objects.** Let H, L be finite-dimensional Hopf algebras. An (H, L)-*biGalois object* [16], is an algebra A that is a left H-Galois extension and a right L-Galois extension of the base field $\Bbbk$ such that the two comodule structures make it an (H, L)-bicomodule. Two biGalois objects are isomorphic if there exists a bijective bicomodule morphism that is also an algebra map. Any (H, L)-biGalois object A can be regarded as a left $H \otimes_{\Bbbk} L^{\mathrm{cop}}$-comodule algebra. It follows from [**?**, Corollary 8.3.10] that any biGalois object is $H \otimes_{\Bbbk} L^{\mathrm{cop}}$-simple as a left $H \otimes_{\Bbbk} L^{\mathrm{cop}}$-comodule algebra.

Denote by $\mathrm{BiGal}(H)$ the set of isomorphism classes of (H, H)-biGalois objects. It is a group with product given by the cotensor product $\square_H$.

If A is an (H, L)-biGalois object then the functor

$$\mathcal{F}_A : \mathrm{Comod}(L) \to \mathrm{Comod}(H), \quad \mathcal{F}_A(X) = A\Box_L X, \tag{2.11}$$

for all $X \in \mathrm{Comod}(L)$, has a tensor structure as follows. If $X, Y \in \mathrm{Comod}(L)$ then $\xi^A_{X,Y} : (A\Box_L X)\otimes_{\Bbbk}(A\Box_L Y) \to A\Box_L(X\otimes_{\Bbbk} Y)$ is defined by

$$\xi^A_{X,Y}(a_i\otimes x_i\otimes b_j\otimes y_j) = a_i b_j\otimes x_i\otimes y_j, \tag{2.12}$$

for any $a_i\otimes x_i \in A\Box_L X$, $b_j\otimes y_j \in A\Box_L Y$. If A, B are (H, L)-biGalois objects then there is a natural monoidal isomorphism between the tensor functors $\mathcal{F}_A, \mathcal{F}_B$ if and only if $A \simeq B$ as biGalois objects.

Assume that A is a H-biGalois object with left H-comodule structure $\lambda : A \to H\otimes_{\Bbbk} A$. If $g \in G(H)$ is a group-like element we can define a new H-biGalois object A^g on the same underlying algebra A with unchanged right comodule structure and a new left H-comodule structure given by $\lambda^g : A^g \to H\otimes_{\Bbbk} A^g$, $\lambda^g(a) = g^{-1}a_{(-1)}g\otimes a_{(0)}$ for all $a \in A$.

Recall [9] that two H-biGalois objects A, B are *equivalent*, and denote it by $A \sim B$ if there exists an element $g \in G(H)$ such that $A^g \simeq B$ as biGalois objects. The subgroup of $\mathrm{BiGal}(H)$ consisting of H-biGalois objects equivalent to H is denoted by $\mathrm{InnbiGal}(H)$. This group is a normal subgroup of $\mathrm{BiGal}(H)$. We denote $\mathrm{OutbiGal}(H) = \mathrm{BiGal}(H)/\, \mathrm{InnbiGal}(H)$.

Theorem 2.1. [9, Thm. 4.5] *Let* $A, B \in \mathrm{BiGal}(H)$. *The following statements are equivalent.*

1. $A \sim B$*;*
2. *there exists a pseudo-natural isomorphism* $(\eta_0, \eta) : \mathcal{F}_A \to \mathcal{F}_B$.

□

Remark 2.2. Given an isomorphism $f : A^g \to B$ of bicomodule algebras, there is an associated pseudo-natural isomorphism $(\eta_0, \eta^f) : \mathcal{F}_A \to \mathcal{F}_B$, given by

$$\eta_0 = \Bbbk_g, \quad \eta^f_V : A\Box_H V\otimes_{\Bbbk}\Bbbk_g \to \Bbbk_g\otimes_{\Bbbk} B\Box_H V,$$

$$\eta^f_V(a\otimes v\otimes r) = r\otimes f(a)\otimes v,$$

for all $a\otimes v\otimes r \in A\Box_H V\otimes_{\Bbbk}\Bbbk_g$. Moreover, any pseudo-natural isomorphism is of this form.

2.4. **Comodule algebras over graded Hopf algebras.** One of the goals of the paper is the classification of biGalois objects over a certain family of Hopf algebras. Since biGalois objects are in particular comodule algebras, we first recall some tools developed in [13] to study simple comodule algebras over coradically graded Hopf algebras.

Let $H = \oplus_{i=0}^m H(i)$ be a coradically graded finite-dimensional Hopf algebra. We shall also assume that H is pointed; the coradical is a group algebra $H_0 = \Bbbk G$ of a finite group G.

If A is right H-simple then A_0 is right $\Bbbk G$-simple, [13, Prop. 4.4], thus there exists a subgroup $F \subseteq G$ and a 2-cocycle $\psi \in Z^2(F, \Bbbk^\times)$ such that $A_0 = \Bbbk_\psi F$. The next result is [14, Lemma 5.4].

Lemma 2.3. *If A is right H-simple there exists a 2-cocycle $\widehat{\psi} \in Z^2(G, \Bbbk^\times)$ such that $\widehat{\psi}$ restricted to F equals ψ and $(\operatorname{gr} A)_{\sigma_{\widehat{\psi}}}$ is isomorphic to a homogeneous left coideal subalgebra of $H^{[\sigma_{\widehat{\psi}}]}$ as a left $H^{[\sigma_{\widehat{\psi}}]}$-comodule algebras.* □

Recall that the Hopf 2-cocycle $\sigma_{\widehat{\psi}}$ was defined in (2.6).

3. Finite supergroup algebras

Let G be a finite Abelian group, $u \in G$ be an element of order 2 and V a finite-dimensional G-module such that $u \cdot v = -v$ for all $v \in V$. The space V has a Yetter-Drinfeld module structure over $\Bbbk G$ as follows. The G-comodule structure $\delta : V \to \Bbbk G \otimes_{\Bbbk} V$ is given by $\delta(v) = u \otimes v$, for all $v \in V$. The Nichols algebra of V is the exterior algebra $\mathfrak{B}(V) = \wedge(V)$. The bosonization $\wedge(V)\#\Bbbk G$ is called in [1] a *finite supergroup algebra* and it is denoted by $\mathcal{A}(V, u, G)$. Hereafter we shall denote the element $v\#g$ simply by vg, for all $v \in V, g \in G$.

The algebra $\mathcal{A}(V, u, G)$ is generated by elements $v \in V, g \in G$ subject to relations

$$vw + wv = 0, \quad gv = (g \cdot v)g, \text{ for all } v, w \in V, g \in G.$$

The coproduct and antipode are determined for all $v \in V, g \in G$ by

$$\Delta(v) = v \otimes 1 + u \otimes v, \qquad \Delta(g) = g \otimes g, \qquad \mathcal{S}(v) = -uv, \qquad \mathcal{S}(g) = g^{-1}.$$

Lemma 3.1. *The algebra map $\phi : \mathcal{A}(V, u, G) \to \mathcal{A}(V, u, G)^{\mathrm{cop}}$ determined by*

$$\phi(v) = vu, \quad \phi(g) = g,$$

is a Hopf algebra isomorphism. □

Next, we shall compute the projective covers of simple $\mathcal{A}(V, u, G)$-comodules. For any $g \in G$, $\Bbbk_g$ is a simple $\mathcal{A}(V, u, G)$-comodule. Let $P_g = \wedge(V) \otimes_{\Bbbk} \Bbbk\langle g \rangle$ be the $\mathcal{A}(V, u, G)$-comodule with coaction determined by the restriction of the coproduct, that is

$$vg \mapsto vg \otimes g + ug \otimes vg, \quad g \mapsto g \otimes g, \quad v \in V.$$

Let us explain this coaction in a more explicit form. Let be $t \in \mathbb{N}$, and define

$$\mathcal{L}_t = \{(1, \ldots, t), (t, 1, \ldots, t-1), (t-1, t, 1, \ldots, t-2), \ldots, (2, 3, \ldots, t, 1)\} \subset \mathbb{N}^t,$$

then the coaction of P_g on the element $v_1 \cdots v_t g$ equals

$$v_1 \cdots v_t g \otimes g + u^t g \otimes v_1 \cdots v_t g + \sum_{(i_1,\ldots,i_t) \in \mathcal{L}_t} v_{i_1} \cdots v_{i_{t-1}} ug \otimes v_{i_t} g +$$
$$\sum_{(i_1,\ldots,i_t) \in \mathcal{L}_t} v_{i_1} \cdots v_{i_{t-2}} u^2 g \otimes v_{i_t} v_{i_t} g + \cdots + \sum_{(i_1,\ldots,i_t) \in \mathcal{L}_t} v_{i_1} u^{t-1} g \otimes v_{i_2} \cdots v_{i_t} g.$$

Theorem 3.2. *Let $\{v_1, \dots, v_k\}$ be a basis of V. The following assertions hold.*

1. *The family $\{\Bbbk_g : g \in G\}$ is a complete set of isomorphism classes of simple $\mathcal{A}(V, u, G)$-comodules.*
2. *The projective cover of the comodule $\Bbbk_{u^k g}$ is P_g.*
3. *For all $g, h \in G$, $\Bbbk_g \otimes \Bbbk_h \simeq \Bbbk_{gh}$ and $P_g \otimes \Bbbk_h \simeq P_{gh}$ as $\mathcal{A}(V, u, G)$-comodules.*

Proof. Since $\mathcal{A}(V, u, G)$ is pointed, every simple comodule is one-dimensional and they come from group-like elements of $\mathcal{A}(V, u, G)$. This proves (1).

Since $\mathcal{A}(V, u, G) = \oplus_{g \in G} P_g$, as left $\mathcal{A}(V, u, G)$-comodules, then P_g is a projective comodule for any $g \in G$.

Let $p_g : P_g \to \Bbbk_{u^k g}$ be the $\mathcal{A}(V, u, G)$-comodule epimorphism given by

$$p_g(x) = \begin{cases} w_{u^k g} & \text{if } x = v_1 \dots v_k g, \\ 0 & \text{elsewhere.} \end{cases} \tag{3.1}$$

Let us prove that this projection is essential. Let L be any $\mathcal{A}(V, u, G)$-comodule together with a comodule morphism $\psi : L \to P_g$ such that $p_g \circ \psi$ is an epimorphism. Let $y \in L$ such that $p_g \circ \psi(y) = w_{u^k g}$, then $\psi(y) = z + \alpha\, v_1 \dots v_k \otimes g$ for some $z \in \ker(p_g)$ and $0 \neq \alpha \in \Bbbk$.

Note that P_g is the subcomodule generated by $z + \alpha\, v_1 \dots v_k \otimes g$. Since the image of ψ is a subcomodule containing $z + \alpha\, v_1 \dots v_k \otimes g$, then it must be all P_g. Hence ψ is surjective and the map p_g is essential. We conclude that P_g is the projective cover of the comodule $\Bbbk_{u^k g}$.

Finally, for $g, h \in G$, let $\gamma : \Bbbk_g \otimes \Bbbk_h \to \Bbbk_{gh}$ and $\beta : P_g \otimes \Bbbk_h \to P_{gh}$ be the maps

$$\gamma(w_g \otimes w_h) = w_{gh}, \quad \beta(v \otimes g \otimes w_h) = v \otimes gh$$

for all $v \in V$. Clearly γ and β are comodule isomorphisms. □

The following result will be needed when computing the Frobenius-Perron dimension of certain tensor categories.

Corollary 3.3. *Assume* $\dim(V) = 2$. *For any $g \in G$ we have*

$$\langle P_g \rangle = 2\langle \Bbbk_g \rangle + 2\langle \Bbbk_{ug} \rangle.$$

Here $\langle P_g \rangle$ denotes the class of P_g in the Grothendieck group of the category of finite-dimensional left $\mathcal{A}(V, u, G)$-comodules.

Proof. Let $\{v, w\}$ be a basis of V. Recall the projection $p_g : P_g \to \Bbbk_g$ described in (3.1). Since in this case P_g is generated as a vector space by $\{vw \otimes g, v \otimes g, w \otimes g, 1 \otimes g\}$, the kernel of p_g is generated as a vector space by $\{v \otimes g, w \otimes g, 1 \otimes g\}$. Define $f : \ker(p_g) \to \Bbbk_{ug}$ the $\mathcal{A}(V, u, G)$-comodule epimorphism by

$$f(x) = \begin{cases} w_{ug} & \text{if } x = w \otimes g, \\ 0 & \text{elsewhere.} \end{cases}$$

Let $f_1 : \ker(f) \to \Bbbk_{ug}$ be the $\mathcal{A}(V,u,G)$-comodule epimorphism given by

$$f_1(x) = \begin{cases} w_{ug} & \text{if } x = v\otimes g, \\ 0 & \text{elsewhere.} \end{cases}$$

We have a composition series for P_g given by

$$P_g \supseteq \ker(p_g) \supseteq \ker(f) \supseteq \ker(f_1) \supseteq 0,$$

and satisfies

$$P_g/\ker(p_g) \simeq \Bbbk_g, \qquad \ker(p_g)/\ker(f) \simeq \Bbbk_{ug},$$
$$\ker(f)/\ker(f_1) \simeq \Bbbk_{ug}, \qquad \ker(f_1) \simeq \Bbbk_g.$$

□

3.1. **The tensor product $\mathcal{A}(V,u,G)\otimes_{\Bbbk}\mathcal{A}(V,u,G)^{\mathrm{cop}}$.** Let G_1, G_2 be finite Abelian groups and $u_i \in G_i$ be central elements of order 2. For $i = 1,2$ let V_i be finite-dimensional G_i-modules, such that u_i acts in V_i as -1.

Define $\mathcal{A}(V_1,V_2,u_1,u_2,G_1,G_2) = \mathcal{A}(V_1,u_1,G_1)\otimes_{\Bbbk}\mathcal{A}(V_2,u_2,G_2)$ with the tensor product Hopf algebra structure. For simplicity, we shall denote

$$\mathcal{B}(V,u,G) = \mathcal{A}(V,V,u,u,G,G).$$

Observe that $\mathcal{B}(V,u,G)$ is a coradically graded Hopf algebra.

If we denote $D = G_1 \times G_2$, then both vector spaces V_1, V_2 are D-modules by setting

$$(g,h)\cdot v_1 = g\cdot v_1, \quad (g,h)\cdot v_2 = h\cdot v_2, \quad (g,h) \in D, v_i \in V_i; i = 1,2.$$

The algebra $\mathcal{A}(V_1,V_2,u_1,u_2,G_1,G_2)$ is generated by elements V_1, V_2, D subject to relations

$$v_1w_1 + w_1v_1 = 0, \;\; v_2w_2 + w_2v_2 = 0, \;\; v_1v_2 = v_2v_1,$$
$$gv_1 = (g\cdot v_1)g, \quad gv_2 = (g\cdot v_2)g,$$

for all $g \in D$, $v_i, w_i \in V_i$, $i = 1,2$. The Hopf algebra structure is determined for all $(g_1,g_2) \in G$, $v_i \in V_i$, $i = 1,2$ by

$$\Delta(v_1) = v_1\otimes 1 + (u_1,1)\otimes v_1, \quad \Delta(v_2) = v_2\otimes 1 + (1,u_2)\otimes v_2,$$
$$\Delta(g_1,g_2) = (g_1,g_2)\otimes(g_1,g_2).$$

We shall define certain families of Hopf algebras that are cocycle deformations of $\mathcal{B}(V,u,G)$. Let $(V_1,V_2,u_1,u_2,G_1,G_2)$ be a data as above. Set $V = V_1 \oplus V_2$. Define $\mathcal{H}(V_1,V_2,u_1,u_2,G_1,G_2) = \wedge(V)\otimes_{\Bbbk}\Bbbk D$ with product determined by

$$vw + wv = 0, \quad gv = (g\cdot v)g, \text{ for any } v,w \in V_1 \oplus V_2, g \in D,$$

and coproduct determined by

$$\Delta(v_1) = v_1\otimes 1 + (u_1,1)\otimes v_1, \quad \Delta(v_2) = v_2\otimes 1 + (1,u_2)\otimes v_2,$$

for any $v_i \in V_i$, $i = 1,2$.

Lemma 3.4. [14, Prop. 6.2] *Let be* $H = \mathcal{A}(V_1, V_2, u_1, u_2, G_1, G_2)$*,* $\psi \in Z^2(D, \Bbbk^\times)$ *and* $\sigma_\psi : H \otimes_{\Bbbk} H \to \Bbbk$ *the Hopf 2-cocyle defined in* (2.6)*. Denote*

$$\xi = \psi((u_1, 1), (1, u_2))\psi((1, u_2), (u_1, 1))^{-1}.$$

Then

(i) *if* $\xi = 1$ *we have* $H^{[\sigma]} \simeq \mathcal{A}(V_1, V_2, u_1, u_2, G_1, G_2)$*;*
(ii) *if* $\xi = -1$ *then* $H^{[\sigma]} \simeq \mathcal{H}(V_1, V_2, u_1, u_2, G_1, G_2)$*.*

□

4. The classification of Hopf BiGalois objects over $\mathcal{A}(V, u, G)$

In this section we shall present a classification of BiGalois objects over the supergroup algebras. The idea to achieve this classification for an arbitrary Hopf algebra H is the following. Any biGalois object over H is an $H \otimes_{\Bbbk} H^{\text{cop}}$-simple left $H \otimes_{\Bbbk} H^{\text{cop}}$-comodule algebra with trivial coinvariants. Any such $H \otimes_{\Bbbk} H^{\text{cop}}$-comodule algebra is a *lifting* of a 2-cocycle deformation of a homogeneous left coideal subalgebra inside certain a twisting of the Hopf algebra $H \otimes_{\Bbbk} H^{\text{cop}}$. Since biGalois objects have dimension equal to the dimension of H, we can then detect the biGalois objects.

Let G be a finite Abelian group, $u \in G$ be an element of order 2 and V a finite-dimensional G-module such that $u \cdot v = -v$ for all $v \in V$.

First we classify all $\mathcal{A}(V, u, G) \otimes_{\Bbbk} \mathcal{A}(V, u, G)^{\text{cop}}$-simple left comodule algebras with trivial coinvariants. Hopf biGalois objects over $\mathcal{A}(V, u, G)$ are inside this family.

4.1. Simple comodule algebras over $\mathcal{B}(V, u, G)$.

We recall the description of all $\mathcal{B}(V, u, G)$-simple left comodule algebras presented in [14].

For a given finite-dimensional coradically graded Hopf algebra H, the idea to classify simple left H-comodule algebras, is roughly the following. If A is a H-simple left comodule algebra the graded algebra $\operatorname{gr} A$, with respect to the Loewy filtration, is also H-simple. A twisting of $\operatorname{gr} A$, by a certain Hopf 2-cocycle σ, is isomorphic to an homogeneous coideal subalgebra inside $H^{[\sigma]}$. Then, one has to classify homogeneous coideal subalgebras inside $H^{[\sigma]}$. At last, one has to compute all *liftings* of $\operatorname{gr} A$, that is H-comodule algebras A such that $\operatorname{gr} A$ is a twisting of a coideal subalgebra inside $H^{[\sigma]}$.

Definition 4.1. A collection $(W^1, W^2, W^3, \beta, F, \psi)$ is *compatible* with the triple (V, u, G) if

- $W^1, W^2 \subseteq V$, $W^3 \subseteq V \oplus V$ are subspaces, such that $W^3 \cap W^1 \oplus W^2 = 0$, $W^3 \cap V \oplus \{0\} = 0 = W^3 \cap \{0\} \oplus V$;
- $F \subseteq G \times G$ is a subgroup that leaves invariant all subspaces W^i, $i = 1, 2, 3$;
- if $W^3 \neq 0$ then $(u, u) \in F$;

- denote $W = W^1 \oplus W^2 \oplus W^3$. Then $\beta : W \times W \to \Bbbk$ is a bilinear form stable under the action of F, such that

$$\beta(w_1, w_2) = -\beta(w_2, w_1),\ \beta(w_1, w_3) = \beta(w_3, w_1),\ \beta(w_2, w_3) = -\beta(w_3, w_2),$$

  for all $w_i \in W^i$, $i = 1, 2, 3$, and β restricted to $W^i \times W^i$ is symmetric for any $i = 1, 2, 3$;
- if $(u, u) \notin F$ then β restricted to $W^1 \times W^2$ and $W^2 \times W^3$ is null;
- $\psi \in H^2(F, \Bbbk^\times)$.

If $(W^1, W^2, W^3, \beta, F, \psi)$ is compatible with (V, u, G) the left $\mathcal{B}(V, u, G)$-comodule algebra $\mathcal{K}(W, \beta, F, \psi)$ is defined as follows. The algebra $\mathcal{K}(W, \beta, F, \psi)$ is generated by W and $\{e_f : f \in F\}$, subject to relations

$$e_f e_h = \psi(f, h)\, e_{fh}, \quad e_f w = (f \cdot w) e_f,$$

$$w_i w_j + w_j w_i = \beta(w_i, w_j)1, \quad w_i \in W^i, w_j \in W^j,$$

for any $(i, j) \in \{(1, 1), (2, 2), (1, 3), (3, 3)\}$, and relations

$$w_2 w_3 - w_3 w_2 = \beta(w_2, w_3)\, e_{(u,u)}, \text{ for any } w_2 \in W^2, w_3 \in W^3,$$

$$w_1 w_2 - w_2 w_1 = \beta(w_1, w_2)\, e_{(u,u)}, \text{ for any } w_1 \in W^1, w_2 \in W^2.$$

The left coaction $\delta : \mathcal{K}(W, \beta, F, \psi) \to \mathcal{B}(V, u, G) \otimes_{\Bbbk} \mathcal{K}(W, \beta, F, \psi)$ is defined on the generators

$$\delta(e_f) = f \otimes e_f, \quad \delta(v, w) = v \otimes 1 + w(u, u) \otimes e_{(u,u)} + (u, 1) \otimes (v, w),$$

$$\delta(w_2) = w_2 \otimes 1 + (1, u) \otimes w_2, \quad \delta(w_1) = w_1 \otimes 1 + (u, 1) \otimes w_1,$$

for any $f \in F, w_1 \in W^1$, $w_2 \in W^2$, $(v, w) \in W^3$. This family of comodule algebras was introduced in [14] to classify certain module categories.

Definition 4.2. If $(W^1, W^2, W^3, \beta, F, \psi)$ is a compatible data with (V, u, G) such that $W^1 = W^2 = 0$ we shall denote $\mathcal{L}(W, \beta, F, \psi) = \mathcal{K}(W, \beta, F, \psi)$.

The following result is [14, Prop. 7.4, Thm. 7.10].

Theorem 4.3. *The following assertions hold.*

1. $\dim \mathcal{K}(W, \beta, F, \psi) = \dim W |F|$.
2. *The algebra* $\mathcal{K}(W, \beta, F, \psi)$ *is a* $\mathcal{B}(V, u, G)$*-simple left comodule algebra with trivial coinvariants.*

Moreover, any $\mathcal{B}(V, u, G)$*-simple left* $\mathcal{B}(V, u, G)$*-comodule algebra with trivial coinvariants is isomorphic to one* $\mathcal{K}(W, \beta, F, \psi)$ *for some compatible data* (W, β, F, ψ). □

For later use, we shall give explicitly the left and right coactions of the algebra $\mathcal{L}(W, \beta, \psi)$. Any left $\mathcal{B}(V, u, G)$-comodule is a $\mathcal{A}(V, u, G)$-bicomodule where the right coaction is obtained using the canonical projection

$$\epsilon \otimes \mathrm{id} : \mathcal{B}(V, u, G) = \mathcal{A}(V, u, G) \otimes_{\Bbbk} \mathcal{A}(V, u, G) \twoheadrightarrow \mathcal{A}(V, u, G),$$

composed with the isomorphism $\phi : \mathcal{A}(V, u, G) \to \mathcal{A}(V, u, G)^{\mathrm{cop}}$ given in Lemma 3.1.

The $\mathcal{A}(V,u,G)$-bicomodule structure on $\mathcal{L}(W,\beta,F,\psi)$ is given by the left and right actions $\lambda:\mathcal{L}(W,\beta,F,\psi)\to\mathcal{A}(V,u,G)\otimes_{\Bbbk}\mathcal{L}(W,\beta,F,\psi)$, $\rho:\mathcal{L}(W,\beta,F,\psi)\to\mathcal{L}(W,\beta,F,\psi)\otimes_{\Bbbk}\mathcal{A}(V,u,G)$ determined by

$$\lambda(v,w)=v\otimes 1+u\otimes(v,w),\quad \rho(v,w)=e_{(u,u)}\otimes w+(v,w)\otimes 1, \tag{4.1}$$
$$\lambda(e_{(g,f)})=g\otimes e_{(g,f)},\quad \rho(e_{(g,f)})=e_{(g,f)}\otimes f,$$

for all $(g,f)\in F$, $(v,w)\in W$.

The proof of the next result is straightforward.

Lemma 4.4. *If $F\subseteq G\times G$ is a subgroup such that $(u,u)\in F$, $|F|=|G|$, $F\cap G\times\{1\}=\{1\}=F\cap\{1\}\times G$ and $W\subseteq V\oplus V$ is a subspace stable under the action of F such that $\dim W=\dim V$, $W\cap V\oplus 0=0=W\cap 0\oplus V$; then the comodule algebras $\mathcal{L}(W,\beta,F,\psi)$ are $\mathcal{A}(V,u,G)$-biGalois objects.* □

4.2. **Hopf BiGalois objects over $\mathcal{A}(V,u,G)$.** We shall use the description of $\mathcal{B}(V,u,G)$-simple left comodule algebras given in the previous section to classify $\mathcal{A}(V,u,G)$-Hopf BiGalois objects.

Theorem 4.5. *Any $\mathcal{A}(V,u,G)$-biGalois object is isomorphic to an algebra of the form $\mathcal{L}(W,\beta,F,\psi)$, where*

- *$F\subseteq G\times G$ is a subgroup such that $F\cap G\times\{1\}=\{1\}=F\cap\{1\}\times G$, $|F|=|G|$, $(u,u)\in F$;*
- *$W\subseteq V\oplus V$ is a subspace stable under the action of F such that $\dim W=\dim V$, $W\cap V\oplus 0=0=W\cap 0\oplus V$;*
- *$\beta:W\times W\to\Bbbk$ is a F-invariant symmetric bilinear form;*
- *and $\psi\in H^2(F,\Bbbk^\times)$ is a 2-cocycle.*

Proof. Let A be a $\mathcal{A}(V,u,G)$-biGalois object. Then it is a $\mathcal{B}(V,u,G)$-simple left $\mathcal{B}(V,u,G)$-comodule algebra with trivial coinvariants. This implies that there exists a compatible data $(W^1,W^2,W^3,\beta,F,\psi)$ such that $A\simeq\mathcal{K}(W,\beta,F,\psi)$. Since the coinvariants of A are trivial, then $W^1=W^2=0$ and $W=W^3$. The conditions stated on F and W must be satisfied since the coinvariants of A are trivial and $\dim A=\dim H$. □

Now, we shall give an alternative description of compatible data (W,β,F,ψ) such that the comodule algebra $\mathcal{L}(W,\beta,F,\psi)$ is a biGalois object.

A collection (T,β,α,ψ) will be also called a *compatible data* if:

- $\alpha:G\to G$ is a group isomorphism such that $\alpha(u)=u$;
- $T:V\to V$ is a linear isomorphism such that
$$T(g\cdot v)=\alpha(g)\cdot T(v),\quad v\in V, g\in G;$$
- $\beta:V\times V\to\Bbbk$ is a symmetric G-invariant bilinear form;
- $\psi\in H^2(G,\Bbbk^\times)$ is a 2-cocycle.

Lemma 4.6. *There is a bijective correspondence between the set of compatible data (T,β,α,ψ) and collections (W,β,F,ψ) such that they satisfy conditions of Theorem 4.5.*

Proof. If (T,β,α,ψ) is a compatible data define $(W,\widehat{\beta},F,\widehat{\psi})$ as follows.

$$W=\{(T(v),v): v\in V\},\quad F=\{(\alpha(g),g): g\in G\}.$$

The bilinear form $\widehat{\beta}$ and the 2-cocycle $\widehat{\psi}$ are defined as

$$\widehat{\beta}((T(v),v),(T(w),w)=\beta(v,w),\quad \widehat{\psi}((\alpha(g),g),(\alpha(f),f)=\psi(g,f),$$

for all $v,w\in V$, $g,f\in G$. Let (W,β,F,ψ) be a compatible data such that it satisfy conditions of Theorem 4.5. If $(x,g)\in F$, since $F\cap G\times\{1\}=\{1\}$, then x is uniquely determined by the element g. So we can denote $x=\alpha(g)$. Since $F\cap\{1\}\times G=\{1\}$ the map α is injective. Since F is a group then α is a group homomorphism. Since $|F|=|G|$ the function α is defined for any $g\in G$, thus it is a group isomorphism. The definition of the linear isomorphism T is analogous. Both constructions are one the inverse of the other. □

Definition 4.7. If (T,β,α,ψ) is a compatible data denote $\mathcal{L}(T,\beta,\alpha,\psi)$ the algebra $\mathcal{L}(W,\beta,F,\psi)$ where the collection (W,β,F,ψ) is the associated data to (T,β,α,ψ) under the correspondence of Lemma 4.6. If (T,β,α,ψ), $(T',\beta',\alpha',\psi')$ are compatible data, define

$$(T,\beta,\alpha,\psi)\bullet(T',\beta',\alpha',\psi')=(T\circ T',\beta\circ T'+\beta',\alpha\circ\alpha',\psi\psi').$$

If $g\in G$ define $T_g: V\to V$ the isomorphism $T_g(v)=g\cdot v$ for all $v\in V$. Then $(T_g,0,\mathrm{id}\,,1)$ is a compatible data for all $g\in G$.

Lemma 4.8. *Let (T,β,α,ψ), $(T',\beta',\alpha',\psi')$ be compatible data.*

1. *The collection $(T\circ T',\beta\circ T'+\beta',\alpha\circ\alpha',\psi\psi')$ is a compatible data.*
2. *The set of compatible data with product*

$$(T,\beta,\alpha,\psi)\bullet(T',\beta',\alpha',\psi')=(T\circ T',\beta\circ T'+\beta',\alpha\circ\alpha',\psi\psi') \tag{4.2}$$

is a group with identity $(\mathrm{Id}\,,0,\mathrm{id}\,,1)$.

Proof. 1. Straightforward.

2. The proof of the associativity is straightforward. The inverse of the compatible data (T,β,α,ψ) is $(T^{-1},-\beta\circ T^{-1},\alpha^{-1},\psi^{-1})$. □

Definition 4.9. Define the group $\mathfrak{R}(V,u,G)$ as the quotient of the set of compatible data (T,β,α,ψ) with product described in (4.2) modulo the normal subgroup of order two generated by the element $(T_u,0,\mathrm{id}\,,1)$.

The set of compatible data $\{(T_g,0,\mathrm{id}\,,1): g\in G\}$ is a normal subgroup of $\mathfrak{R}(V,u,G)$. The quotient group $\mathfrak{R}(V,u,G)/\{(T_g,0,\mathrm{id}\,,1): g\in G\}$ is denoted by $\mathfrak{O}(V,u,G)$.

Proposition 4.10. *Let (T,β,α,ψ), $(T',\beta',\alpha',\psi')$ be compatible data. The following assertions hold.*

1. *There is an isomorphism $\mathcal{L}(T,\beta,\alpha,\psi)\simeq\mathcal{L}(T',\beta',\alpha',\psi')$ of biGalois objects if and only*

$$(T,\beta,\alpha,\psi)=(T',\beta',\alpha',\psi')\ \textit{or}\ (T_u\circ T,\beta,\alpha,\psi)=(T',\beta',\alpha',\psi').$$

2. $\mathcal{L}(T,\beta,\alpha,\psi) \in \text{InnbiGal}(\mathcal{A}(V,u,G))$ *if and only if* $(T,\beta,\alpha,\psi) = (T_g, 0, \text{id}, 1)$ *for some* $g \in G$.
3. *There is an isomorphism of* $\mathcal{B}(V,u,G)$*-comodule algebras*

$$\mathcal{L}(T,\beta,\alpha,\psi)\square_{\mathcal{A}(V,u,G)}\mathcal{L}(T',\beta',\alpha',\psi') \simeq \mathcal{L}(T\circ T', \beta\circ T'+\beta', \alpha\circ\alpha', \psi\psi').$$

Proof. 1. Let $f : \mathcal{L}(T,\beta,\alpha,\psi) \to \mathcal{L}(T',\beta',\alpha',\psi')$ be a $\mathcal{B}(V,u,G)$-comodule algebra isomorphism. This implies that for any $g \in G$ we have $f(e_{(g,\alpha(g))}) = \chi_g\, e_{(g,\alpha(g))}$ for some $\chi_g \in \Bbbk$. Whence $\psi = \psi'$ in $H^2(G,\Bbbk^\times)$. Since $e^2_{(u,u)} = 1$ we have $\chi_u = \pm 1$.

Denote by (W,β,ψ), (W',β',ψ') the collections associeted to the compatible data (T,β,α,ψ) and $(T',\beta',\alpha',\psi')$, respectively, under the correspondence of Lemma 4.6. It follows straightforward that $f(W) = W'$. If $f(x,y) = (x',y')$ for $(x,y) \in W$ then, since f is a $\mathcal{B}(V,u,G)$-comodule map, the element

$$x'\otimes 1 + y'(u,u)\otimes e_{(u,u)} + (u,1)\otimes(x',y')$$

is equals to

$$x\otimes 1 + \chi_u\, y(u,u)\otimes e_{(u,u)} + (u,1)\otimes(x',y').$$

Thus $f(x,y) = (x,\chi_u\, y)$. If $\chi_u = 1$ both collections (W,β,ψ), (W',β',ψ') are equal. If $\chi_u = -1$ then $(T_u\circ T,\beta,\alpha,\psi) = (T',\beta',\alpha',\psi')$.

2. Recall the definition of InnbiGal(H) given in Section 2.3. It follows directly from (1) and the definition of InnbiGal$(\mathcal{A}(V,u,G))$.

3. Define the algebra map

$$\vartheta : \mathcal{L}(T\circ T', \beta+\beta', \alpha\circ\alpha', \psi\psi') \to \mathcal{L}(T,\beta,\alpha,\psi)\square_{\mathcal{A}(V,u,G)}\mathcal{L}(T',\beta',\alpha',\psi')$$

as follows. If $g \in G, v \in V$ then

$$\vartheta(T\circ T'(v), v) = (T\circ T'(v), T'(v))\otimes 1 + e_{(u,u)}\otimes(T'(v), v),$$

$$\vartheta(e_{(\alpha\circ\alpha'(g),g)}) = e_{(\alpha\circ\alpha'(g),\alpha'(g))}\otimes e_{(\alpha'(g),g)}.$$

It follows by a straightforward calculation that the image of ϑ is inside $\mathcal{L}(T,\beta,\alpha,\psi)\square_{\mathcal{A}(V,u,G)}\mathcal{L}(T',\beta',\alpha',\psi')$. The map ϑ is an injective algebra map. Since both algebras have the same dimension, ϑ is an isomorphism. □

Remark 4.11. The proof of Part (1) of Proposition 4.10 gives a description of the possible bicomodule algebra isomorphisms between two biGalois objects. This fact will be used later.

Corollary 4.12. *There are group isomorphisms*

$$\mathfrak{R}(V,u,G) \simeq \text{BiGal}(\mathcal{A}(V,u,G)), \quad \mathfrak{O}(V,u,G) \simeq \text{OutbiGal}(\mathcal{A}(V,u,G)).$$

□

Remark 4.13. As a consequence of [9, Corollary 4.9] and Proposition 4.10 there is an exact sequence of groups

$$0 \to G/<u> \to \mathfrak{R}(V,u,G) \to \text{BrPic}\,(\text{Rep}(\mathcal{A}(V,u,G)).$$

Lemma 4.14. *Let (T, β, α, ψ) be a compatible data and $g \in G$. Then there is an ismorphism $\mathcal{L}(T, \beta, \alpha, \psi)\Box_{\mathcal{A}(V,u,G)}\Bbbk_g \simeq \Bbbk_{\alpha(g)}$ of left $\mathcal{A}(V, u, G)$-comodules.*

Proof. If $a\otimes r \in \mathcal{L}\Box_H\Bbbk_g$ then $\rho(a) = a\otimes g$, hence

$$\rho(ae_{(\alpha(g^{-1}),g^{-1})}) = (a\otimes g)(e_{(\alpha(g^{-1}),g^{-1})}\otimes g^{-1}) = ae_{(\alpha(g^{-1}),g^{-1})}\otimes 1,$$

therefore $ae_{(\alpha(g^{-1}),g^{-1})} \in \Bbbk 1 = \mathcal{L}(T, \beta, \alpha, \psi)^{\mathrm{co}\,\mathcal{A}(V,u,G)}$, and $a = \zeta e_{(\alpha(g),g)}$ for some $\zeta \in \Bbbk$. □

4.3. **A concrete example of biGalois extensions.** Assume V is the 2-dimensional vector space generated by $\{v_1, v_2\}$ and $G = C_2 =< u >$ the cyclic group with two elements. Then, V is a C_2-module with action determined by declaring $u \cdot v_i = -v_i$ for $i = 1, 2$.

For any $\xi \in \Bbbk$ define $T_\xi : V \to V$ the linear map

$$T_\xi(v_1) = v_1, \quad T_\xi(v_2) = \xi v_1 - v_2.$$

By Lemma 4.6, the compatible data $(T_\xi, 0, \mathrm{id}\,, 1)$ gives rise to a $\mathcal{A}(V, u, C_2)$-biGalois extension that we denote by $\mathbf{U}_\xi$. From Proposition 4.10 (3) it follows that $\mathbf{U}_\xi$ has order two, that is, there is a bicomodule algebra isomorphism $\mathbf{U}_\xi\Box_H\mathbf{U}_\xi \simeq H$.

5. Crossed product tensor categories

In this section $\mathcal{C}$ will denote a strict finite tensor category. We recall the definition of crossed system of a finite group Γ on the tensor category $\mathcal{C}$ introduced in [10] and the associated Γ-graded extension of $\mathcal{C}$.

Definition 5.1. [10] Let Γ be a finite group. *A crossed system of* Γ *over* $\mathcal{C}$ is a collection $\Sigma = ((a_*, \xi^a), (U_{a,b}, \sigma^{a,b}), \gamma_{a,b,c})_{a,b,c\in\Gamma}$ consisting of

- Monoidal autoequivalences $(a_*, \xi^a) : \mathcal{C} \to \mathcal{C}$ where $\xi^a_{X,Y} : a_*(X\otimes Y) \to a_*(X)\otimes a_*(Y)$ is the monoidal structure for $X, Y \in \mathcal{C}$. We also require that $a_*(\mathbf{1}) = \mathbf{1}$;
- Objects $U_{a,b} \in \mathcal{C}$ and for any $X \in \mathcal{C}$ natural isomorphisms
$$\sigma^{a,b}_X : a_*b_*(X)\otimes U_{a,b} \to U_{a,b}\otimes(ab)_*X, \quad X \in \mathcal{C};$$
- isomorphisms $\gamma_{a,b,c} : a_*(U_{b,c})\otimes U_{a,bc} \to U_{a,b}\otimes U_{ab,c}$;

such that for all $a, b, c \in \Gamma$, $X, Y \in \mathcal{C}$:

$$\sigma^{a,b}_{\mathbf{1}} = \mathrm{id}\,_{U_{a,b}}, \quad 1_* = \mathrm{Id}\,_{\mathcal{C}}, \quad (U_{1,a}, \sigma^{1,a}) = (\mathbf{1}, \mathrm{id}\,_{a_*}) = (U_{a,1}, \sigma^{a,1}), \tag{5.1}$$

$$\gamma_{a,1,b} = \gamma_{1,a,b} = \gamma_{a,b,1} = \mathrm{id}\,_{U_{a,b}}, \tag{5.2}$$

$$(\mathrm{id}\,_{U_{a,b}}\otimes\xi^{ab}_{X,Y})\sigma^{a,b}_{X\otimes Y} = \tag{5.3}$$
$$(\sigma^{a,b}_X\otimes\mathrm{id}\,_{(ab)_*(Y)})(\mathrm{id}\,_{a_*b_*(X)}\otimes\sigma^{a,b}_Y)(\xi^a_{b_*X,b_*Y}a_*(\xi^b_{X,Y})\otimes\mathrm{id}\,_{U_{a,b}}),$$

$$(\gamma_{a,b,c}\otimes\mathrm{id}\,_{(abc)_*(X)})(\mathrm{id}\,_{a_*(U_{b,c})}\otimes\sigma^{a,bc}_X)(\xi^a_{U_{b,c},(bc)_*(X)}\circ a_*(\sigma^{b,c}_X)\otimes\mathrm{id}\,_{U_{a,bc}}) = \tag{5.4}$$

$$= (\mathrm{id}_{U_{a,b}} \otimes \sigma^{ab,c}_X)(\sigma^{a,b}_{c_*X} \otimes \mathrm{id}_{U_{ab,c}})(\mathrm{id}_{a_*b_*c_*(X)} \otimes \gamma_{a,b,c})(\xi^a_{b_*c_*(X),U_{bc}} \otimes \mathrm{id}_{U_{a,bc}}).$$

Remark 5.2. 1. Condition (5.3) of Definition (5.1) implies that $(U_{a,b}, \sigma^{a,b})$ is a pseudo-natural isomorphism in the bicategory $\underline{\mathcal{C}}$ with only one object. In particular the object $U_{a,b}$ is invertible in $\mathcal{C}$ with inverse $\overline{U_{a,b}}$.

2. Condition (5.4) implies that $\gamma_{a,b,c}$ is an invertible modification in the same bicategory.

Definition 5.3. A crossed system $\Sigma = ((a_*, \xi^a), (U_{a,b}, \sigma^{a,b}), \gamma_{a,b,c})_{a,b,c\in\Gamma}$ is a *coherent outer* Γ*-action* on $\mathcal{C}$ if for all $a, b, c, d \in \Gamma$

$$(5.5) \qquad (\gamma_{a,b,c} \otimes \mathrm{id}_{U_{abc,d}})(\mathrm{id}_{a_*(U_{b,c})} \otimes \gamma_{a,bc,d})(\xi^a_{U_{bc},U_{bc,d}} a_*(\gamma_{b,c,d}) \otimes \mathrm{id}_{U_{a,bcd}}) =$$

$$(\mathrm{id}_{U_{a,b}} \otimes \gamma_{ab,c,d})(\sigma^{a,b}_{U_{cd}} \otimes \mathrm{id}_{U_{ab,cd}})(\mathrm{id}_{a_*b_*(U_{cd})} \otimes \gamma_{a,b,cd})(\xi^a_{b_*(U_{c,d}),U_{b,cd}} \otimes \mathrm{id}_{U_{a,bcd}}).$$

In this case, we say that Γ *acts* on the category $\mathcal{C}$.

If Γ acts on $\mathcal{C}$ via a crossed system Σ, then the Γ*-crossed product tensor category*, introduced in [10], associated to this action is $\mathcal{C}(\Sigma)$, where $\mathcal{C}(\Sigma) = \oplus_{a\in\Gamma}\mathcal{C}_a$ as Abelian categories and $\mathcal{C}_a = \mathcal{C}$ for all $a \in \Gamma$. Denote by $[V, a]$ the object $V \in \mathcal{C}_a$. Morphisms from $\oplus_{a\in\Gamma}[V_a, a]$ to $\oplus_{a\in\Gamma}[W_a, a]$ are given by $\oplus_{a\in\Gamma}[f_a, a]$ where $f_a : V_a \to W_a$ is a morphism in $\mathcal{C}$ for all $a \in \Gamma$.

Theorem 5.4. [10, Sec. 3.3] *$\mathcal{C}(\Sigma)$ is a tensor category with tensor product $\otimes : \mathcal{C}(\Sigma) \times \mathcal{C}(\Sigma) \to \mathcal{C}(\Sigma)$ defined by*

$$(5.6) \qquad [V, a] \otimes [W, b] = [V \otimes a_*(W) \otimes U_{a,b}, ab] \text{ in objects},$$

$$(5.7) \qquad [f, a] \otimes [g, b] = [f \otimes a_*(g) \otimes \mathrm{id}_{U_{a,b}}, ab] \text{ in morphisms},$$

with unit object $[\mathbf{1}_{\mathcal{C}}, 1]$, and associativity constraints given by

$$(5.8) \qquad \alpha_{[V,a][W,b][Z,c]} = (\mathrm{id}_{V\otimes a_*W} \otimes \sigma^{a,b}_Z \otimes \mathrm{id}_{U_{ab,c}})(\mathrm{id}_{V\otimes a_*W\otimes a_*b_*Z} \otimes \gamma_{a,b,c}) \circ$$
$$(\mathrm{id}_{V\otimes a_*W} \otimes \xi^a_{b_*Z,U_{b,c}} \otimes \mathrm{id}_{U_{a,bc}})(\mathrm{id}_V \otimes \xi^a_{W,b_*Z\otimes U_{b,c}} \otimes \mathrm{id}_{U_{a,bc}}).$$

The dual objects are given by

$$([V, 1])^* = [V^*, 1] \text{ and } ([\mathbf{1}, a])^* = [\overline{U_{a,a^{-1}}}, a^{-1}].$$

□

The next result explains when, for two coherent outer actions Σ, Σ', the tensor categories $\mathcal{C}(\Sigma)$, $\mathcal{C}(\Sigma')$ are monoidally equivalent.

Theorem 5.5. [10, Th. 4.1] *Let $\Sigma = ((a_*, \varrho^a), (U_{a,b}, \sigma^{a,b}), \gamma_{a,b,c})_{a,b,c\in\Gamma}$, $\Sigma' = ((a', \zeta^a), (U'_{a,b}, \tau^{a,b}), \gamma'_{a,b,c})_{a,b,c\in\Gamma}$ be two coherent outer Γ-actions over $\mathcal{C}$. Any graded monoidal equivalence $F : \mathcal{C}(\Sigma) \to \mathcal{C}(\Sigma')$ comes from a collection $((H, \xi), f, (\theta_a, \beta^a), \chi_{a,b})_{a,b\in\Gamma}$ where:*

- *$(H, \xi) : \mathcal{C} \to \mathcal{C}$ is a monoidal equivalence;*
- *$f : \Gamma \to \Gamma$ is a group isomorphism;*
- *for any $a \in \Gamma$ the pair $(\theta_a, \beta^a) : H \circ a_* \to f(a)' \circ H$ is a pseudo-natural isomorphism such that $(\theta_1, \beta^1) = (\mathbf{1}, \mathrm{id})$;*

- $\chi_{a,b} : H(U_{a,b}) \otimes \theta_{ab} \to \theta_a \otimes f(a)'(\theta_b) \otimes U'_{f(a),f(b)}$ *is an invertible morphism in* $\mathcal{C}$ *such that* $\chi_{a,1} = \chi_{1,a} = \mathrm{id}_{\theta_a}$ *and*

$$p_V(\mathrm{id}_{H(a_*b_*(V))} \otimes \chi_{a,b}) = (\chi_{a,b} \otimes \mathrm{id}_{f(ab)'(H(V))}) q_V, \quad V \in \mathcal{C}, \tag{5.9}$$

where

$$p_V = (\mathrm{id}_{\theta_a} \otimes (\mathrm{id}_{f(a)'(\theta_b)} \otimes \tau^{f(a),f(b)}_{H(V)})(s_V \otimes \mathrm{id}_{U'_{f(a),f(b)}})) \circ (\beta^a_{b_*(V)} \otimes \mathrm{id}_{f(a)'(\theta_b) \otimes U'_{f(a),f(b)}}),$$

$$s_V = \zeta^{f(a)}_{\theta_b, f(b)'(H(V))} \circ f(a)'(\beta^b) \circ (\zeta^{f(a)}_{H(b_*(V)),\theta_b})^{-1},$$

$$q_V = (\mathrm{id}_{H(U_{a,b})} \otimes \beta^{ab}_V)(\xi_{U_{a,b},(ab)_*(V)} \circ H(\sigma^{ab}_V) \circ (\xi_{a_*b_*(V),U_{ab}})^{-1} \otimes \mathrm{id}_{\theta_{ab}}).$$

□

Given the collection $((H,\xi), f, (\theta_a, \beta^a), \chi_{a,b})_{a,b \in \Gamma}$ as in the previous Theorem, the monoidal equivalence $F : \mathcal{C}(\Sigma) \to \mathcal{C}(\Sigma')$ is defined by

$$F([V,a]) = [H(V) \otimes \theta_a, f(a)], \quad [V,a] \in \mathcal{C}(\Sigma),$$

for any $[V,a] \in \mathcal{C}(\Sigma)$.

Remark 5.6. In [10] the author defines crossed systems in terms of equivalence classes of monoidal functors, up to monoidal isomorphisms, and equivalence classes of pseudo-natural isomorphisms, up to invertible modifications. This is done this way since it is shown that equivalence classes of crossed product extensions of the tensor category $\mathcal{C}$ by the group Γ are classified by crossed systems. Since we are only interested in giving examples, our definition of crossed systems is a *representative* of a crossed systems according to [10].

5.1. **Coherent outer actions for the corepresentation category of a Hopf algebra.** Let H be a finite-dimensional Hopf algebra. We shall give an explicit description for coherent outer actions on the tensor category $\mathrm{Comod}(H)$ of finite-dimensional left H-comodules in terms of Hopf algebraic data. Let Γ be a finite group.

Let us fix the following notation. If $g \in G(H)$ and L is a (H,H)-biGalois object then the cotensor product $L \Box_H \Bbbk_g$ is one-dimensional. Let $\phi(L,g) \in \Gamma$ be the group-like element such that $L \Box_H \Bbbk_g \simeq \Bbbk_{\phi(L,g)}$ as left H-comodules.

Lemma 5.7. *From a collection* $\Upsilon = (L_a, (g(a,b), f^{a,b}), \gamma_{a,b,c})_{a,b,c \in \Gamma}$*, where*

- *for any* $a \in \Gamma$, L_a *is a* (H,H)*-biGalois object;*
- $g(a,b) \in G(H)$ *is a group-like element and bicomodule algebra isomorphisms* $f^{a,b} : (L_a \Box_H L_b)^{g(a,b)} \to L_{ab}$*;*
- $\gamma_{a,b,c} \in \Bbbk^\times$*,*

such that for all $a,b,c \in \Gamma$*:*

$$L_1 = H, \quad (g(1,a), f^{1,a}) = (1, \mathrm{id}_{L_a}) = (g(a,1), f^{a,1}); \tag{5.10}$$

$$\phi(L_a, g(b,c)) g(a,bc) = g(a,b) g(ab,c); \tag{5.11}$$

$$\gamma_{a,1,b} = \gamma_{1,a,b} = \gamma_{a,b,1} = 1; \tag{5.12}$$

$$(f^{a,b} \otimes \mathrm{id}_{L_c}) f^{ab,c} = (\mathrm{id}_{L_a} \otimes f^{b,c}) f^{a,bc}, \tag{5.13}$$

there is associated a crossed system $\overline{\Upsilon}$ *of* Γ *over* Comod(H). *Moreover, the crossed system* $\overline{\Upsilon}$ *is a coherent outer action on* Comod(H) *if and only if* γ *is a 3-cocycle, that is, for all* $a, b, c, d \in \Gamma$

$$\gamma_{a,b,c}\gamma_{a,bc,d}\gamma_{b,c,d} = \gamma_{ab,c,d}\gamma_{a,b,cd}. \tag{5.14}$$

Proof. For any $a, b \in \Gamma$ define the monoidal functor $a_* : \mathrm{Comod}(H) \to \mathrm{Comod}(H)$, $a_* = L_a \square_H -$ and $U_{a,b} = \Bbbk_{g(a,b)}$.

Define the pseudo-natural isomorphism $(\Bbbk_{g(a,b)}, \sigma^{a,b}) : a_* \circ b_* \to (ab)_*$ which comes from the bicomodule algebra isomorphism

$$f^{a,b} : (L_a \square_H L_b)^{g(a,b)} \to L_{ab}$$

as explained in Remark 2.2.

The existence of the isomorphisms $\gamma_{a,b,c} : L_a \square_H \Bbbk_{g(b,c)} \to \Bbbk_{g(a,b)g(ab,c)}$ is equivalent to $\phi(L_a, g(b,c))g(a,bc) = g(a,b)g(ab,c)$. Since both vector spaces $L_a \square_H \Bbbk_{g(b,c)} \otimes \Bbbk_{g(a,bc)}$ and $\Bbbk_{g(a,b)} \otimes \Bbbk_{g(ab,c)}$ are one-dimensional, the map $\gamma_{a,b,c} : \Bbbk \to \Bbbk$ is given by multiplication of a scalar $\gamma_{a,b,c} \in \Bbbk^\times$.

Equation (5.1) is equivalent to (5.10), (5.2) is equivalent to (5.12), and Equation (5.4) is equivalent to (5.13). Since $f^{a,b}$ is an algebra morphism then Equation (5.3) is satisfied. Equation (5.14) follows from (5.5). □

Definition 5.8. Given a collection Υ as in the previous Lemma, define $\mathrm{Comod}(H)(\Upsilon) := \mathrm{Comod}(H)(\overline{\Upsilon})$ the Γ-crossed product tensor category associated to the coherent outer action $\overline{\Upsilon}$.

The next Lemma is a direct consequence of the Theorem 5.5 applied to $\mathcal{C} = \mathrm{Comod}(H)$.

Lemma 5.9. *Let* $\Upsilon = (L_a, (g(a,b), f^{a,b}), \gamma_{a,b,c})_{a,b,c\in\Gamma}$ *and* $\Upsilon' = (L'_a, (g'(a,b), z^{a,b}), \gamma'_{a,b,c})_{a,b,c\in\Gamma}$ *be collections that satisfy conditions in Lemma 5.7 (so that* $\overline{\Upsilon}, \overline{\Upsilon'}$ *are coherent outer* Γ*-actions).*

Any graded monoidal equivalence $F : Comod(H)(\Upsilon) \to Comod(H)(\Upsilon')$ *comes from a collection* $(L, \lambda, (h(a), h^a), \tau_{a,b})_{a,b\in\Gamma}$ *where*

- L *is a* (H,H)*-biGalois object,*
- $\lambda : \Gamma \to \Gamma$ *is a group isomorphism,*
- $h(a) \in G(H)$ *is a group-like and* $h^a : (L \square_H L_a)^{h(a)} \to L'_{\lambda(a)} \square_H L$ *is a biGalois object isomorphism, where* $(h(1), h^1) = (1, \mathrm{id})$*,*
- $\tau_{a,b} \in \Bbbk^\times$ *such that* $\tau_{a,1} = \tau_{1,a} = 1$*,*

whose satisfies that

$$\phi(L, g(a,b))h(ab) = h(a)\phi(L'_{\lambda(a)}, h(b))g'(\lambda(a), \lambda(b)), \tag{5.15}$$

$$h^{ab}(\mathrm{id}_L \otimes f^{a,b}) = (z^{\lambda(a),\lambda(b)} \otimes \mathrm{id}_L)(\mathrm{id}_{L'_{\lambda(a)}} \otimes h^b)(h^a \otimes \mathrm{id}_{L_b}). \tag{5.16}$$

□

6. Examples of C_2-extensions of $\mathrm{Comod}(\mathcal{A}(V,u,C_2))$

Let C_2 be the cyclic group of 2 elements. In this section we shall give explicit examples of tensor categories that are C_2-extensions of the tensor category $\mathrm{Comod}(\mathcal{A}(V,u,C_2))$ with V a 2-dimensional vector space.

6.1. **C_2-extensions of $\mathrm{Comod}(H)$.** Let H be a finite-dimensional Hopf algebra. First, we explicitly describe data giving rise to C_2-extensions of $\mathrm{Comod}(H)$ in the particular case the group-like elements of the Hopf algebra H is a cyclic group of order 2 generated by u.

Assume that (L,g,f,γ) is a collection where

- L is a (H,H)-biGalois object;
- $g\in G(H)$ is a group-like element such that $\varpi : L\Box_H \Bbbk_g \simeq \Bbbk_g$ as left H-comodules;
- $f:(L\Box_H L)^g\to H$ is a bicomodule algebra isomorphism and
- $\gamma\in\Bbbk^\times$, $\gamma^2=1$.

According to Lemma 5.7 from data (L,g,f,γ) we obtain a crossed system of C_2 over $\mathrm{Comod}(H)$. Just take $L_u=L$, $L_1=H$, $g(u,u)=g$, $1=g(1,u)=g(u,1)=g(1,1)$, $f^{u,u}=f$, $f^{1,u}=f^{u,1}=f^{1,1}=\mathrm{id}$ and $\gamma_{a,b,c}=1\in\Bbbk$ for any $a,b,c\in C_2$ except $\gamma_{u,u,u}:=\gamma$. Let us denote this crossed system Υ.

The monoidal structure of the category $\mathrm{Comod}(H)(\Upsilon)$, given by Theorem 5.4 explicitly writes as follows. For any $V,W,Z\in\mathrm{Comod}(H)$ and $b\in C_2$:

$$\begin{aligned}
[V,1]\otimes[W,b] &= [V\otimes_{\Bbbk}W,b],\\
[V,u]\otimes[W,1] &= [V\otimes_{\Bbbk}(L\Box_H W),u],\\
[V,u]\otimes[W,u] &= [V\otimes_{\Bbbk}(L\Box_H W)\otimes_{\Bbbk}\Bbbk_g,1],
\end{aligned}$$

The unit object is $[\Bbbk,1]$ and dual objects are given by

$$([V,1])^*=[V^*,1],\quad ([\Bbbk,1])^*=[\Bbbk,1]\quad\text{and}\quad ([\Bbbk,u])^*=[\Bbbk_{g^{-1}},u].$$

Finally the associativity is given by

$$\begin{aligned}
\alpha_{[V,u][W,u][Z,u]} = \gamma(&\mathrm{id}_{V\otimes L\Box_H W}\otimes f\Box_H \mathrm{id}_Z\otimes \mathrm{id}_{\Bbbk_g})(\mathrm{id}_{V\otimes L\Box_H W\otimes L\Box_H L\Box_H Z}\otimes\varpi)\circ\\
&(\mathrm{id}_{V\otimes L\Box_H W}\otimes\xi_{L\Box_H Z,\Bbbk_g})(\mathrm{id}_V\otimes\xi_{W,L\Box_H Z\otimes\Bbbk_g}),
\end{aligned}$$

and the others are trivials. Here $\xi=(\xi^L)^{-1}$ is the morphism defined in the Equation (2.12).

6.2. **Explicit examples of C_2-extensions of $\mathrm{Comod}(\mathcal{A}(V,u,C_2))$.** In this section $H=\mathcal{A}(V,u,C_2)$ where V is a 2-dimensional vector space. Using the results of previous sections, we describe families of crossed systems of C_2 over $\mathrm{Comod}(\mathcal{A}(V,u,C_2))$. These crossed systems come from a collection (L,g,f,γ) as presented in Section 6.1. Below, we present two such families depending on the biGalois object L. For the first family the biGalois object L is the one presented in Section 4.3 and for the second family the biGalois object L is trivial.

Lemma 6.1. *Let be $\xi, \gamma \in \Bbbk, g \in C_2$, and let $f \in Hom(H^g, H)$ be a comodule algebra isomorphism. Assume $\gamma^2 = 1$.*

1. *The collection (ξ, g, f, γ) has associated a coherent outer C_2-action over* $\operatorname{Comod}(\mathcal{A}(V, u, C_2))$ *and the corresponding C_2-crossed product tensor category will be denoted by $\mathcal{C}_\xi(g, f, \gamma)$.*
2. *The collection (g, f, γ) has associated a coherent outer C_2-action over* $\operatorname{Comod}(\mathcal{A}(V, u, C_2))$ *and the corresponding C_2-crossed product tensor category will be denoted by $\mathcal{D}(g, f, \gamma)$.*

Proof. 1. Let $L = \mathbf{U}_\xi$ be the (H, H)-biGalois object defined in the Section 4.3. It follows from Lemma 4.14 that $\mathbf{U}_\xi \Box_H \Bbbk_g \simeq \Bbbk_g$.

2. Following the same idea, take $L = H$. Then $H \Box_H \Bbbk_g \simeq \Bbbk_g$. □

We want to be more explicit in the determination of the comodule algebra isomorphism $f : H^g \to H$ that appears in Lemma 6.1. We make use of the proof Proposition 4.10 (1), were such comodule algebra maps are explicitly determined. Let (ξ, g, f, γ) be a collection as in Lemma 6.1. There are two options:

- If $g = 1$, then $f : H \to H$. Let $\delta : H \to \mathcal{L}(\operatorname{Id}, 0, \operatorname{id}, 1)$ be the canonical isomorphism $h \mapsto (h, h)$ and define $\overline{f} := \delta \circ f \circ \delta^{-1}$. By (the proof of) Proposition 4.10
$$\overline{f} : \mathcal{L}(\operatorname{Id}_V, 0, \operatorname{id}, 1) \to \mathcal{L}(\operatorname{Id}_V, 0, \operatorname{id}, 1),$$
satisfies that $\overline{f}(x, y) = (x, y)$ if $(x, y) \in \{(v, v) : v \in V\}$ which implies that $f(x) = x$ if $x \in V$. Moreover $\overline{f}(e_{1,1}) = \chi_1 e_{1,1} = e_{1,1}$ and $\overline{f}(e_{u,u}) = \chi_u e_{u,u} = e_{u,u}$. Then $f = \operatorname{id}_H$.
- If $g = u$, then $f : H^u \to H$. By (the proof of) Proposition 4.10 (1)
$$\overline{f} : \mathcal{L}(\operatorname{Id}_V^u, 0, \operatorname{id}, 1) \to \mathcal{L}(\operatorname{Id}_V, 0, \operatorname{id}, 1),$$
satisfies that $\overline{f}(x, y) = (x, -y)$ if $(x, y) \in \{(u \cdot v, v) | v \in V\}$ which implies that $f(x) = u \cdot x = -x$ if $x \in V$. Moreover $\overline{f}(e_{1,1}) = e_{1,1}$ and $\overline{f}(e_{u,u}) = \chi_u e_{u,u} = -e_{u,u}$, so $f(u) = -u$. We shall denote by $\iota : H^u \to H$ this unique bicomodule algebra isomorphism.

Hence, we obtain four families of C_2-crossed product tensor categories

$$\mathcal{C}_\xi(1, \operatorname{id}, \gamma), \mathcal{C}_\xi(u, \iota, \gamma), \mathcal{D}(1, \operatorname{id}, \gamma), \mathcal{D}(u, \iota, \gamma). \tag{6.1}$$

Some of these tensor categories are equivalent. We shall use Lemma 5.9 to distinguish them.

Theorem 6.2. *Let be $\xi, \xi', \gamma, \gamma' \in \Bbbk$ with $\gamma^2 = 1 = (\gamma')^2$. As tensor categories*

$$\mathcal{C}_\xi(1, \operatorname{id}, \gamma) \not\cong \mathcal{C}_{\xi'}(u, \iota, \gamma'), \quad \mathcal{C}_\xi(1, \operatorname{id}, \gamma) \cong \mathcal{C}_0(1, \operatorname{id}, \gamma'),$$
$$\mathcal{C}_\xi(u, \iota, \gamma) \cong \mathcal{C}_0(u, \iota, \gamma'), \quad \mathcal{D}(1, \operatorname{id}, \gamma) \not\cong \mathcal{D}(u, \iota, \gamma'),$$
$$\mathcal{D}(1, \operatorname{id}, \gamma) \not\cong \mathcal{C}_0(1, \operatorname{id}, \gamma'), \quad \mathcal{D}(u, \iota, \gamma) \not\cong \mathcal{C}_0(u, \iota, \gamma').$$

Proof. Using Lemma 5.9, there exists a monoidal equivalence

$$\mathcal{C}_\xi(g,f,\gamma)\simeq\mathcal{C}_{\xi'}(g',f',\gamma')$$

if there exists

(1) $L=\mathcal{L}(T,0,\alpha,1)$ a biGalois object over H,
(2) $h:=h(u)\in C_2$ and $h^u:\mathcal{L}(T_hTT_\xi,0,\mathrm{id}\,,1)\to\mathcal{L}(T_{\xi'}T,0,\mathrm{id}\,,1)$ a biGalois isomorphism,
(3) $\tau:=\tau_{u,u}\in\Bbbk^\times$,

satisfying

$$\alpha(g)=g',\quad \Phi(\mathrm{id}\,_L\otimes f)=(f'\otimes\mathrm{id}\,_L)(\mathrm{id}\,_{\mathbf{U}_{\xi'}}\otimes h^u)(h^u\otimes\mathrm{id}\,_{\mathbf{U}_\xi}), \tag{6.2}$$

where $\Phi:L\square_H H\to H\square_H L$ is the isomorphism given by $l\otimes h\mapsto l_{-1}\otimes l_0\varepsilon(h)$.

The second condition of (6.2) comes from Equation (5.16), and the first condition from Equation (5.15):

For all $a,b\in C_2$, $L\square_h\Bbbk_{g(a,b)}\simeq\Bbbk_{\alpha g(a,b)}$ and $L'_a\square_H\Bbbk_{h(b)}\simeq\Bbbk_{h(b)}$, then Equation (5.15) implies that $\alpha(g(a,b))h(ab)=h(a)h(b)g'(a,b)$. For $a=1$ or $b=1$ this equation is valid. For $a=u=b$, we obtain $\alpha(g)=h^2g'=g'$.

Since $\alpha=\mathrm{id}\,$, we obtain that $\mathcal{C}_\xi(1,\mathrm{id}\,,\gamma)\not\simeq\mathcal{C}_{\xi'}(u,\iota,\gamma')$.

By Lemma 4.10(1), h^u is an isomorphism if and only if

$$T_hTT_\xi=T_{\xi'}T,\quad\text{or}\quad T_uT_hTT_\xi=T_{\xi'}T.$$

To prove that there is a monoidal equivalence $\mathcal{C}_\xi(1,\mathrm{id}\,,\gamma)\simeq\mathcal{C}_0(1,\mathrm{id}\,,\gamma')$ choose $h=1$ and $h^u=\mathrm{id}$ then $TT_\xi=T_0T$ if T is given by the matrix

$$\begin{pmatrix}1&\xi/2\\0&1\end{pmatrix}.$$

We only need to check that

$$\Phi(\mathrm{id}\,_L\otimes\varphi_2)=(\varphi_3\otimes\mathrm{id}\,_L)(\mathrm{id}\,_{U_0}\otimes\varphi_1)(\varphi_1\otimes\mathrm{id}\,_{\mathbf{U}_\xi}),$$

where

- $\varphi_1:L\square_H\mathbf{U}_\xi\to\mathbf{U}_0\square_H L$, coming from $h^u=\mathrm{id}\,:\mathcal{L}(TT_\xi,0,\mathrm{id}\,,1)\to\mathcal{L}(T_0T,0,\mathrm{id}\,,1)$ up to isomorphism,
- $\varphi_2:\mathbf{U}_\xi\square_H\mathbf{U}_\xi\to H$, coming from $\mathrm{id}\,:\mathcal{L}(T_\xi^2,0,\mathrm{id}\,,1)\to\mathcal{L}(\mathrm{id}\,,0,\mathrm{id}\,,1)$, which satisfies $(\varphi_2)^{-1}(v)=(T_\xi T_\xi(v),T_\xi(v))\otimes1+e_{u,u}\otimes(T_\xi(v),v)$ and $(\varphi_2)^{-1}(e_{g,g})=e_{g,g}\otimes e_{g,g}$ for $v\in V$ and $g\in C_2$,
- $\varphi_3:\mathbf{U}_0\square_H\mathbf{U}_0\to H$ coming from $\mathrm{id}\,:\mathcal{L}(T_0^2,0,\mathrm{id}\,,1)\to\mathcal{L}(\mathrm{id}\,,0,\mathrm{id}\,,1)$ up to isomorphism.

Let $v\in V$. If $a=(TT_\xi(v),T_\xi(v))\otimes1+e_{u,u}\otimes(T_\xi(v),v)\in L\square_H\mathbf{U}_\xi$ then $\varphi_1(a)=(T_0T(v),T(v))\otimes1+e_{u,u}\otimes(T(v),v)$:

Let $\zeta_1:\mathcal{L}(TT_\xi,0,\mathrm{id}\,,1)\to L\square_H\mathbf{U}_\xi$ and $\zeta_2:\mathcal{L}(T_0T,0,\mathrm{id}\,,1)\to\mathbf{U}_0\square_H L$ be the isomorphisms give in the Lemma 4.10(3), whose satisfy

$$\zeta_1(TT_\xi(v),v)=TT_\xi(v),T_\xi(v))\otimes1+e_{u,u}\otimes(T_\xi(v),v),$$

$$\zeta_2(T_0T(v),v)=(T_0T(v),T(v))\otimes1+e_{u,u}\otimes(T(v),v).$$

By definition of φ_1, we have that $\varphi_1 \circ \zeta_1 = \zeta_2 \circ \mathrm{id}_{\mathcal{L}(TT_\xi,0,\mathrm{id},1)}$, and this implies the claim.

By the same argument, if $b = (T_0T_0(v), T_0(v))\otimes 1 + e_{u,u}\otimes(T_0(v), v) \in \mathbf{U}_0\Box_H\mathbf{U}_0$ then $\varphi_3(b) = v$.

Moreover $\Phi = \alpha_1 \circ \alpha_2$ where $\alpha_1 : L \to H\Box_H L$, $\alpha_2 : L\Box_H L \to L$ and $(\alpha_1)^{-1}(h\otimes l) = \varepsilon(h)l$ and $(\alpha_2)^{-1}(l) = l_0\otimes l_1$.

Let $x = (T(w), w) \in L$, then

$$(\alpha_1)^{-1}(\varphi_3\otimes\mathrm{id}_L)(\mathrm{id}_{\mathbf{U}_0}\otimes\varphi_1)(\varphi_1\otimes\mathrm{id}_{\mathbf{U}_\xi})(\mathrm{id}_L\otimes(\varphi_2)^{-1})(\alpha_2)^{-1}(x) = x,$$

since

$$\begin{aligned}
x &\mapsto e_{u,u}\otimes w + (T(w), w)\otimes 1\\
&\mapsto e_{u,u}\otimes e_{u,u}\otimes(T_\xi(w), w) + e_{u,u}\otimes(T_\xi T_\xi(w), T_k(w))\otimes 1 + (T(w), w)\otimes 1\otimes 1\\
&\mapsto e_{u,u}\otimes e_{u,u}\otimes(T_\xi(w), w) + (T_0TT_\xi(w), TT_\xi(w))\otimes 1\otimes 1\\
&+ e_{u,u}\otimes(TT_\xi(w), T_\xi(w))\otimes 1\\
&\mapsto (T_0TT_\xi(w), TT_\xi(w))\otimes 1\otimes 1 + e_{u,u}\otimes(T_0T(w), T(w))\otimes 1\\
&+ e_{u,u}\otimes e_{u,u}\otimes(T(w), w)\\
&\mapsto u\otimes(T(w), w) + T(w)\otimes 1\\
&\mapsto x.
\end{aligned}$$

In the same way, $(g, g) \mapsto (g, g)$ for all $g \in C_2$. Which implies that $\mathcal{C}_\xi(1, \mathrm{id}, \gamma) \simeq \mathcal{C}_0(1, \mathrm{id}, \gamma')$.

To prove $\mathcal{C}_\xi(u, \iota, \gamma) \simeq \mathcal{C}_0(u, \iota, \gamma')$, is enough to take $h = u$ and $h^u : \mathcal{L}(T_uTT_k, 0, \mathrm{id}, 1) \to \mathcal{L}(T_0T, 0\mathrm{id}, 1)$ given for $x, y \in V$ by

$$h^u(x, y) = (x, -y), \quad h^u(e_{u,u}) = -e_{u,u}.$$

It follows from Lemma 5.9, that there is a monoidal equivalence

$$\mathcal{D}(1, \mathrm{id}, \gamma) \simeq \mathcal{D}(u, \iota, \gamma')$$

if and only if there exist $M = \mathcal{L}(R, 0, \alpha, 1)$ a biGalois object over H, $h \in C_2$, $h^u : \mathcal{L}(T_hR, 0, \mathrm{id}, 1) \to \mathcal{L}(R, 0, \mathrm{id}, 1)$ a biGalois object isomorphism and $\tau \in \Bbbk^\times$. As before, they have to satisfy that $\alpha(1) = u$, but $\alpha = \mathrm{id}$. This proves that $\mathcal{D}(1, \mathrm{id}, \gamma) \not\simeq \mathcal{D}(u, \iota, \gamma')$.

Again, using Lemma 5.9, $\mathcal{D}(1, \mathrm{id}, \gamma) \simeq \mathcal{C}_0(1, \mathrm{id}, \gamma')$ as monoidal categories if and only if there exist $M = \mathcal{L}(R, 0, \alpha, 1)$ a biGalois object over H, $h \in C_2$, $h^u : \mathcal{L}(T_hR, 0, \mathrm{id}, 1) \to \mathcal{L}(T_0R, 0, \mathrm{id}, 1)$ a biGalois object isomorphism and $\tau \in \Bbbk^\times$.

By Lemma 4.10(3), h^u is an isomorphism if and only if $T_hR = T_0R$ or $T_uT_hR = T_0R$, but the last two equations do not have a solution for T invertible. So $\mathcal{D}(1, \mathrm{id}, \gamma) \not\simeq \mathcal{C}_0(1, \mathrm{id}, \gamma')$ and $\mathcal{D}(u, \iota, \gamma) \not\simeq \mathcal{C}_0(u, \iota, \gamma')$. □

In conclusion, we obtain eight pairwise non-equivalent tensor categories

$$\begin{aligned} &\mathcal{C}_0(1,\mathrm{id}\,,1), \mathcal{C}_0(1,\mathrm{id}\,,-1), \mathcal{C}_0(u,\iota,1), \mathcal{C}_0(u,\iota,-1),\\ &\mathcal{D}(1,\mathrm{id}\,,1), \mathcal{D}(1,\mathrm{id}\,,-1), \mathcal{D}(u,\iota,1), \mathcal{D}(u,\iota,-1). \end{aligned} \tag{6.3}$$

6.3. **Explicit description of the monoidal structure.** Using Theorem 5.4, we can explicitly describe the tensor product and the associativity constraint for the eight tensor categories presented above. Recall that all those categories have the same underlying Abelian category $\mathrm{Comod}(\mathcal{A}(V,u,C_2))\oplus \mathrm{Comod}(\mathcal{A}(V,u,C_2))$ where V is a 2-dimensional vector space. The associativity constraints that we describe are the non-trivial ones.

Let $V, W, Z \in \mathrm{Comod}(\mathcal{A}(V,u,C_2))$ and $g \in C_2$.

• The tensor product, dual objects and associativity in the category $\mathcal{C}_0(1,\mathrm{id}\,,\pm 1)$ are given by

$$\begin{aligned} [V,1][W,g] &= [V\otimes W, g], & [V,u][W,g] &= [V\otimes \mathbf{U}_0\square_H W, ug],\\ [V,1]^* &= [V^*,1], & [\mathbf{1},u]^* &= [\Bbbk, u], \end{aligned}$$

$\alpha_{[V,u],[W,u],[Z,u]} = [\pm(\mathrm{id}\,_{V\otimes\mathbf{U}_0\square W}\otimes\epsilon\varphi_2\otimes\mathrm{id}\,_Z)(\mathrm{id}\,_V\otimes\xi_{W,\mathbf{U}_0\square Z}), u]$.

Here $\xi = (\xi^{\mathbf{U}_0})^{-1}$ is the morphism defined in the Equation (2.12).

• The tensor product, dual objects and associativity in $\mathcal{C}_0(u,\iota,\pm 1)$ are given by

$$\begin{aligned} [V,1][W,1] &= [V\otimes W, 1], & [V,u][W,u] &= [V\otimes\mathbf{U}_0\square_H W\otimes\Bbbk_u, 1],\\ [V,1][W,u] &= [V\otimes W, u], & [V,u][W,1] &= [V\otimes\mathbf{U}_0\square_H W, u],\\ [V,1]^* &= [V^*,1], & [\mathbf{1},u]^* &= [\Bbbk_u, u], \end{aligned}$$

The associativity constraint $\alpha_{[V,u],[W,u],[Z,u]}$ is equal to

$$[\pm(\mathrm{id}\,_{V\otimes\mathbf{U}_0\square W}\otimes(\epsilon\iota\varphi_2\otimes\mathrm{id}\,_{Z\otimes\mathbf{U}_0\square\Bbbk_u})(\xi_{\mathbf{U}_0\square Z,\Bbbk_u}))(\mathrm{id}\,_V\otimes\xi_{W,\mathbf{U}_0\square Z\otimes\Bbbk_u}), u].$$

• The tensor product, dual objects and associativity in $\mathcal{D}(1,\mathrm{id}\,,\pm 1)$ are given by

$$\begin{aligned} [V,1][W,g] &= [V\otimes W, g], & [V,u][W,g] &= [V\otimes W, ug],\\ [V,1]^* &= [V^*,1], & [\mathbf{1},u]^* &= [\Bbbk, u], \end{aligned}$$

$\alpha_{[V,u],[W,u],[Z,u]} = [\pm(\mathrm{id}\,_{V\otimes W\otimes Z}, u]$.

• The tensor product, dual objects and associativity in $\mathcal{D}(u,\iota,\pm 1)$ are given by

$$\begin{aligned} [V,1][W,1] &= [V\otimes W, 1], & [V,u][W,u] &= [V\otimes W\otimes\Bbbk_u, 1],\\ [V,1][W,u] &= [V\otimes W, u], & [V,u][W,1] &= [V\otimes W, u],\\ [V,1]^* &= [V^*,1], & [\mathbf{1},u]^* &= [\Bbbk_u, u], \end{aligned}$$

$\alpha_{[V,u],[W,u],[Z,u]} = [\pm(\mathrm{id}\,_{V\otimes W}\otimes\varepsilon\iota\otimes\mathrm{id}\,_{Z\otimes\Bbbk_u}, u]$.

6.4. **Frobenius-Perron dimension of the C_2-crossed extensions.** For a review on Frobenius-Perron dimension we refer to [6]. For any object X in a category $\mathcal{C}$ we denote by $\langle X\rangle$ the class of X in the Grothendieck group of $\mathcal{C}$.

For the categories presented in (6.3), the isomorphism classes of the simple objects are

$$\langle[\Bbbk_1, 1]\rangle, \quad \langle[\Bbbk_1, u]\rangle, \quad \langle[\Bbbk_u, 1]\rangle, \quad \langle[\Bbbk_u, u]\rangle.$$

Using Theorem 3.2, the projective covers of these simple objects are respectively

$$\langle[P_1, 1]\rangle, \quad \langle[P_1, u]\rangle, \quad \langle[P_u, 1]\rangle, \quad \langle[P_u, u]\rangle.$$

Using Corollary 3.3 it follows from a straightforward computation that in any of the categories listed in (6.3)

$$\operatorname{FPdim}\langle[\Bbbk_g, h]\rangle = 1, \quad \operatorname{FPdim}\langle[P_g, h]\rangle = 4,$$

for any $g, h \in C_2$. This implies the next result.

Theorem 6.3. *If $\mathcal{C}$ is any of the tensor categories listed in* (6.3) *then* $\operatorname{FPdim}\mathcal{C} = 16$. □

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E-mail address: sighana25@gmail.com
E-mail address: martin10090@gmail.com, mombelli@mate.uncor.edu
URL: http://www.mate.uncor.edu/∼mombelli