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Normal holonomy of orbits and Veronese submanifolds

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Abstract. It was conjectured, twenty years ago, the following result that would generalize the so-called rank rigidity theorem for homogeneous Euclidean submanifolds: let M^n, $n \geq 2$, be a full and irreducible homogeneous submanifold of the sphere $S^{N-1} \subset \mathbb{R}^N$ such that the normal holonomy group is not transitive (on the unit sphere of the normal space to the sphere). Then M^n must be an orbit of an irreducible s-representation (i.e. the isotropy representation of a semisimple Riemannian symmetric space).

If $n = 2$, then the normal holonomy is always transitive, unless M is a homogeneous isoparametric hypersurface of the sphere (and so the conjecture is true in this case). We prove the conjecture when $n = 3$. In this case M^3 must be either isoparametric or a Veronese submanifold. The proof combines geometric arguments with (delicate) topological arguments that use information from two different fibrations with the same total space (the holonomy tube and the caustic fibrations).

We also prove the conjecture for $n \geq 3$ when the normal holonomy acts irreducibly and the codimension is the maximal possible $n(n+1)/2$. This gives a characterization of Veronese submanifolds in terms of normal holonomy. We also extend this last result by replacing the homogeneity assumption by the assumption of minimality (in the sphere).

Another result of the paper, used for the case $n = 3$, is that the number of irreducible factors of the local normal holonomy group, for any Euclidean submanifold M^n, is less or equal than $[n/2]$ (which is the rank of the orthogonal group $SO(n)$). This bound is sharp and improves the known bound $n(n-1)/2$.

1. Introduction.

The holonomy of the normal connection turns out to be a useful tool in Euclidean submanifold geometry [**BCO**]. The most important applications of this tool were the alternative proof of Thorbergsson theorem [**Th**], given in [**O2**], and the rank rigidity theorems for submanifolds [**O3**], [**CO**], [**DO**] (see Section 2.1). Moreover, the extension of Thorbergsson's result to infinite dimensional geometry, given by [**HL**], makes also use of normal holonomy.

It is interesting to remark that normal holonomy is related, in a very subtle way, to Riemannian holonomy. Namely, by using submanifold geometry, with normal holonomy ingredients, one can give short and geometric proofs of both Berger holonomy theorem [**B**] and Simons holonomy (systems) theorem [**S**] (see [**O5**], [**O6**]). Moreover, by applying this methods, it was proved in [**OR**] the so-called skew-torsion holonomy theorem with applications to naturally reductive spaces.

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The starting point for this theory was the normal holonomy theorem [**O1**] which asserts that the (restricted) normal holonomy group representation, of a submanifold of a space form, is, up to a trivial factor, an s-representation (equivalently, the normal holonomy is a Riemannian non-exceptional holonomy). This implies that the so-called principal holonomy tubes have flat normal bundle (holonomy tubes are the image, under the normal exponential map, of the holonomy subbundles of the normal bundle). Such tubes, despite to the classical spherical tubes, behave nicely with respect to products of submanifolds.

But the normal holonomy, which is invariant under conformal transformations of the ambient space, gives much weaker information in submanifold geometry than the Riemannian holonomy in Riemannian geometry. For instance, the reducibility of the normal holonomy representation does not imply that the manifold splits. So, interesting applications of the normal holonomy can be expected only within a restrictive class of submanifolds. For instance:

(1) submanifolds with constant principal curvatures,
(2) complex submanifolds of the complex projective space,
(3) homogeneous submanifolds.

For the first two classes of submanifolds there are "Berger-type" theorems.

For (1) one has the following reformulation of the Thorbergsson theorem [**Th**]: *a full and irreducible submanifold with constant principal curvatures, such that the normal holonomy, as a submanifold of the sphere, is non-transitive must be either a inhomogeneous isoparametric hypersurface or an orbit of an s-representation.*

For (2) we have the following result [**CDO**]: *a complete full and irreducible complex submanifold M of the complex projective space with non-transitive normal holonomy is the complex orbit (in the projectivized tangent space) of the isotropy representation of a Hermitian symmetric space or, equivalently, M is extrinsically symmetric.* This result is not true without the completeness assumption.

For the class (3) we have the rank rigidity theorem for submanifolds [**O3**], [**DO**]: *if the normal holonomy of a full and irreducible Euclidean homogeneous submanifold $M^n = K.v$, $n \geq 2$ has a fixed non-null vector, then M is contained in a sphere. If the dimension of the fixed set of the normal holonomy has dimension at least 2, then M is an orbit of an s-representation (perhaps by enlarging the group K).*

But this last result would be only a particular case of a Berger-type result that it was conjectured twenty years ago in [**O3**]: *if the normal holonomy of a full and irreducible homogeneous submanifold M^n of the sphere, $n \geq 2$, is non-transitive then M is an orbit of an s-representation.*

For $n = 2$ the normal holonomy must be always transitive or trivial (see [**BCO**, Section 4.5 (c)]).

The goal of this article is twofold. On the one hand, to give some progress on this conjecture. On the other hand, to characterize the classical (Riemannian) Veronese submanifolds in terms of normal holonomy.

If a submanifold M^n of the sphere has irreducible and non-transitive normal holonomy, then the first normal space, as a Euclidean submanifold, coincides with the normal space (see Remark 2.11). This imposes the restriction that the codimension is at most

$n(n+1)/2$. We will prove the above mentioned conjecture in the case that the normal holonomy acts irreducibly and the (Euclidean) codimension is the maximal one $n(n+1)/2$. The proof uses most of the techniques developed in the theory of submanifolds and holonomy [**BCO**]. Moreover, the most difficult case is in dimension $n = 3$ for which we have to use also delicate topological arguments involving two different fibrations on a partial holonomy tube: *the holonomy tube fibration* and *the caustic fibration*.

We extend these results by replacing the homogeneity by the property that the submanifold is minimal in a sphere. But the proof of this result is simpler than the homogeneous case and a general proof works also for $n = 3$.

We also prove the sharp bound $n/2$ on the number of irreducible factors of the normal holonomy, which implies, from the above mentioned result, the conjecture for $n = 3$ (see Proposition 6.1).

Let us explain our main results which are related to the so-called Veronese submanifolds.

The isotropy representation of the symmetric space $SL(n+1)/SO(n+1)$ is naturally identified with the action of $SO(n+1)$, by conjugation, on the traceless symmetric matrices. A Veronese (Riemannian) submanifold M^n, which has parallel second fundamental form, is the orbit of a matrix with exactly two eigenvalues, one of which has multiplicity 1. Being M a submanifold with constant principal curvatures, the first normal space $\nu^1(M)$ coincides with the normal space $\nu(M)$. Moreover, $\nu^1(M)$ has maximal dimension. Namely, the codimension of M is $n(n+1)/2$.

The restricted normal holonomy of M, as a submanifold of the sphere, is the image, under the slice representation, of the (connected) isotropy. Then the normal holonomy representation of M is irreducible and it is equivalent to the isotropy representation of $SL(n)/SO(n)$. So, the normal holonomy of M is non-transitive if and only if $n \geq 3$. We have the following geometric characterization of Veronese submanifolds in terms of normal holonomy, which proves a special case of the conjecture on normal holonomy of orbits, when the normal holonomy, of a submanifold of the sphere, acts irreducibly, not transitively and the codimension is maximal.

THEOREM A. *Let $M^n \subset S^{n-1+n(n+1)/2}$, $n \geq 3$, be a homogeneous submanifold of the sphere. Then M is a (full) Veronese submanifold if and only if the restricted normal holonomy group of M acts irreducibly and not transitively.*

For dimension 3 the conjecture on normal holonomy is true. Namely,

THEOREM B. *Let $M^3 \subset S^{N-1}$ be a full irreducible homogeneous 3-dimensional submanifold of the sphere. Assume that the restricted normal holonomy group of M is non-transitive. Then M is an orbit of an s-representation. Moreover, M is either a principal orbit of the isotropy representation of $SL(3)/SO(3)$ or a Veronese submanifold.*

The irreducibility and fullness conditions on M are always with respect to the Euclidean ambient space.

We can replace, in Theorem A, the homogeneity condition by the assumption of minimality in the sphere.

THEOREM C. *Let M^n, $n \geq 3$, be a complete (immersed) submanifold of the sphere $S^{n-1+n(n+1)/2}$. Then M^n is, up to a cover, a (full) Veronese submanifold if and only if M is a minimal submanifold and the restricted normal holonomy group acts irreducibly and not transitively.*

The assumptions of homogeneity or minimality, in our main results, cannot be dropped, since a conformal (arbitrary) diffeomorphism of the sphere transforms M into a submanifold with the same normal holonomy but in general not any more minimal. Last theorem admits a local version.

We will explain the main ideas in the proof of Theorem A, when $n \geq 4$.

Let $\tilde{A}$ be the traceless shape operator of $M = H.v$, i.e. $\tilde{A}_\xi = A_\xi - (1/n)\langle \vec{H}, \xi \rangle Id$, where $\vec{H}$ is the mean curvature vector. Let us consider the map $\tilde{A}$, from the normal space $\bar{\nu}_q(M)$ to sphere into the traceless symmetric endomorphisms $Sim_0(T_qM)$. Then $\tilde{A}$ maps normal spaces to the $\Phi(q)$-orbits into normal spaces to the $SO(n)$-orbits, by conjugation, in $Sim_0(T_qM)$. By using the results in Section 2, which are related to Simons theorem, we obtain that $\tilde{A}$ is a homothecy which maps the normal holonomy group $\Phi(q)$ into $SO(n)$. This implies that the eigenvalues of $\tilde{A}_\xi$ do not change if ξ is parallel transported along a loop. From the homogeneity, since the group H is always inside the $\nabla^\perp$-transvections, we obtain that the eigenvalues of $\tilde{A}_{\xi(t)}$ are constant, if $\xi(t)$ is a parallel normal field along a curve. Now we pass to an appropriate, singular, holonomy tube, M_ξ, where A_ξ has exactly two eigenvalues one of them of multiplicity 2. Let $\hat{\xi}$ be the parallel normal field of M_ξ such that M coincides with the parallel focal manifold $(M_\xi)_{-\hat{\xi}}$ to M_ξ. One obtains that the three eigenvalue functions, $\hat{\lambda}_1, \hat{\lambda}_2$ and $\hat{\lambda}_3 = -1$, of the shape operator $\hat{A}_{\hat{\xi}}$ of M_ξ have constant multiplicities. The two horizontal eigendistributions of $\hat{A}_{\hat{\xi}}$, let us say E_1 and E_2, have multiplicities 2 and $(n-2)$ respectively. The vertical distribution is the eigendistribution associated to the constant eigenvalue -1. From the above mentioned properties of $\tilde{A}$ and the tube formulas one obtains that $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are functionally related (so if one eigenvalue is constant along a curve, the other is also constant). From the Dupin condition, since $\dim(E_1) \geq 2$, $\hat{\lambda}_1$, and so $\hat{\lambda}_2$, as previously remarked, are constant along the integral manifolds of E_1. If $n \geq 4$, the same is true for the distribution E_2. So, the eigenvalues of $\hat{A}_{\hat{\xi}}$ are constant along horizontal curves. But any two points in a holonomy tube can be joined by a horizontal curve. Then $\hat{A}_{\hat{\xi}}$ has constant eigenvalues and so $\hat{\xi}$ is an isoparametric non-umbilical parallel normal field. Then, by the isoparametric rank rigidity theorem, the holonomy tube M_ξ, and therefore M, is an orbit of an s-representation. From this we prove, without using classification results, that M must be a Veronese submanifold.

If $n = 3$, the proof is much harder, since the Dupin condition does not apply for E_2, and requires topological arguments, not valid for $n > 3$, as pointed out before.

2. Preliminaries and basic facts.

In this section, as well as in the appendix, for the reader convenience, we recall the basic notions and results that are needed in this article. We also include in this part some new results that are auxiliary for our purposes. Some of them have a small interest

in its own right, or the proofs are different from the standard ones.

The general reference for this section is [**PT**], [**Te**], [**BCO**].

2.1. Orbits of *s*-representations and Veronese submanifolds.

A submanifold $M \subset \mathbb{R}^N$ has *constant principal curvatures* if the shape operator $A_{\xi(t)}$ has constant eigenvalues, for any $\nabla^\perp$-parallel normal vector field $\xi(t)$ along any arbitrary (piece-wise differentiable) curve $c(t)$ in M. If, in addition, the normal bundle $\nu(M)$ is flat, then M is called isoparametric.

A submanifold M with constant principal curvatures (extrinsically) splits as $M = \mathbb{R}^k \times M'$, where M' is compact and contained in a sphere.

The (extrinsic) homogeneous isoparametric submanifolds are exactly the principal orbits of polar representations [**PT**]. The other orbits have constant principal curvatures (and, in particular, this family of orbits contains the submanifolds with parallel second fundamental form). But it is not true that all homogeneous submanifolds with constant principal curvatures are orbits of polar representations (there exists a homogeneous focal parallel manifold to an inhomogeneous isoparametric hypersurface of the sphere [**FKM**]). It turns out, from Dadok's classification [**Da**], that polar representations are orbit-like equivalent to the so-called *s*-representations, i.e. the isotropy representations of semisimple simply connected Riemannian symmetric spaces. So, a full and homogeneous (not contained in a proper affine subspace) Euclidean submanifold M is isoparametric if and only if it is a principal orbit of an *s*-representation. It is interesting to remark that there is a classification free proof [**EH**], for cohomogeneity different from 2, of the fact that any polar representation is orbit-like to an *s*-representation.

One has the following remarkable result.

THEOREM 2.1 (Thorbergsson, [**Th**], [**O3**]). *A compact full irreducible isoparametric Euclidean submanifold of codimension at least* 3 *is homogeneous* (and so the orbit of an irreducible *s*-representation).

The *rank* at p, of a Euclidean submanifold M, $\operatorname{rank}_p(M)$, is the maximal number of linearly independent parallel normal fields, locally defined around p. The rank of M, $\operatorname{rank}(M)$, is the minimum, over $p \in M$, of $\operatorname{rank}_p(M)$. If M is homogeneous then $\operatorname{rank}_p(M) = \operatorname{rank}(M)$, independent of $p \in M$. The submanifold M is said to be of *higher rank* if its rank is at least 2.

One has the following important result.

THEOREM 2.2 (Rank Rigidity for Submanifolds, [**O3**], [**O4**], [**DO**], [**BCO**]). *Let M^n, $n \geq 2$, be a Euclidean homogeneous submanifold which is full and irreducible. Then,*

(a) $\operatorname{rank}(M) \geq 1$, *if and only if M is contained in a sphere.*
(b) *If* $\operatorname{rank}(M) \geq 2$, *then M is an orbit of an s-representation.*

A parallel normal field ξ of M is called *isoparametric* if the shape operator A_ξ has constant eigenvalues. If the shape operator A_ξ, of a parallel isoparametric normal field, is umbilical, i.e. a multiple λ of the identity, then M is contained in a sphere, if $\lambda \neq 0$, or M is not full, if $\lambda = 0$.

One has the following result (see, [**BCO**, Theorem 5.5.2 and Corollary 5.5.3]).

THEOREM 2.3 (isoparametric local rank rigidity, [**CO**]). *Let M^n be a full (local) and locally irreducible submanifold of $S^{N-1} \subset \mathbb{R}^N$ which admits a non-umbilical parallel isoparametric normal field. Then M is an inhomogeneous isoparametric hypersurface or M is (an open subset of) an orbit of an s-representation.*

One has also a global version of the above result (see [**DO**, Theorem 1.2] and [**BCO**, Section 5.5 (b)]).

THEOREM 2.4 (isoparametric rank rigidity, [**DO**]). *Let M^n be a connected, simply connected and complete Riemannian manifold and let $f : M \to \mathbb{R}^{\mathbb{N}}$ be an irreducible isometric immersion. If there exists a non-umbilical isoparametric parallel normal section, then $f : M \to \mathbb{R}^{\mathbb{N}}$ has constant principal curvatures* (and so, if $f(M)$ is not an isoparametric hypersurface of a sphere, then it is an orbit of an s-representation).

Let K act (by linear isometries) on $\mathbb{R}^N$ as an s-representation. Let (G, K) be the associated simple (simply connected) symmetric pair with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p} \simeq \mathbb{R}^N$. Let $M = K.v$ be an orbit where $v \in \mathfrak{p}$.

One has that the normal space to M at v is given by [**BCO**]

$$\nu_v(M) = \mathcal{C}(v) := \{x \in \mathfrak{p} : [x, v] = 0\} \qquad (*)$$

where $[\,,\,]$ is the bracket of $\mathfrak{g}$.

An s-representation is always the product of irreducible ones. Then the orbit $M = K.v$ is a full submanifold if and only if the component of v, in any K-irreducible subspace of $\mathbb{R}^N$, is not zero.

Let M be a full orbit of an s-representation and let $p \in M$. Then the map $\xi \mapsto A_\xi$, from $\nu_p(M)$ into the symmetric endomorphisms of T_pM, is injective. In other words, the first normal space of M at p coincides with the normal space (see [**BCO**]).

One has the following result from [**HO**]; see also [**BCO**, Theorem 4.1.7].

THEOREM 2.5 ([**HO**]). *Let K act on $\mathbb{R}^N$ as an s-representation and let $M = K.v$ be a full orbit. Then the normal holonomy group $\Phi(v)$ of M at v coincides with the image of the representation of the isotropy K_v on $\nu_v(M)$* (the so-called *slice representation*).

For a Euclidean vector space $(\mathbb{V}, \langle\,,\,\rangle)$, let $Sim(\mathbb{V})$ denote the vector space of (real) symmetric endomorphisms of $\mathbb{V}$. The inner product on $Sim(\mathbb{V})$ is the usual one, $\langle A, B\rangle = \operatorname{trace}(A.B)$.

We denote by $Sim_0(\mathbb{V})$ the vector space of traceless symmetric endomorphisms.

COROLLARY 2.6. *Let K act (by linear isometries) on $\mathbb{R}^N$ as an s-representation and let $M = K.v$, where $|v| = 1$. Assume that the normal holonomy group $\Phi(v)$ acts irreducibly on $\bar{\nu}_v(M) := \{v\}^\perp \cap \nu_v(M)$. Then M is a minimal submanifold of the sphere $S^{N-1} \subset \mathbb{R}^N$. Moreover, the map $\xi \mapsto A_\xi$ is a homothecy, from $\bar{\nu}_v(M)$ onto its image in $Sim_0(T_vM)$.*

PROOF. The mean curvature vector $\vec{H}(v)$ must be fixed by the isotropy, represented on the normal space. Then, from Theorem 2.5, $\vec{H}(v)$ must be fixed by $\Phi(v)$.

Then, from the assumptions, $\vec{H}(v)$ must be proportional to v (which is fixed by the normal holonomy group). Then M is a minimal submanifold of the sphere.

Let us consider the following inner product $(\ ,\ )$ of $\bar{\nu}_v(M)$: $(\xi,\eta)=\langle A_\xi, A_\eta\rangle$. Then, $(\ ,\ )$ is $\Phi(v)$-invariant. In fact, if $\phi\in\Phi(v)$, there exists, from Theorem 2.5, $g\in K_v$ such that $g_{|\bar{\nu}_v(M)}=\phi$. Then

$$(\phi(\xi),\phi(\eta))=(g.\xi,g.\eta)=\langle A_{g.\xi},A_{g.\eta}\rangle=\langle gA_\xi g^{-1},gA_\eta g^{-1}\rangle=\langle A_\xi,A_\eta\rangle=(\xi,\eta).$$

Since $\Phi(v)$ acts irreducibly, then $(\ ,\ )$ is proportional to $\langle\ ,\ \rangle$. Then $\xi\mapsto A_\xi$ is a homothecy. □

Recall that the normal holonomy (group) representation, of a submanifold of a space form, on the normal space, is, up to the fixed set, an s-representation [**O1**], [**BCO**].

The proof of the above mentioned result depends on the construction of the so-called *adapted normal curvature tensor* $\mathcal{R}^\perp$ (see [**O1**] and [**BCO**, Section 4.3 c]). In fact, if M is an arbitrary submanifold of a space of constant curvature, then $\mathcal{R}^\perp$ is an algebraic curvature tensor on the normal space $\nu(M)$. Namely, if $p\in M$ and $R^\perp$ is the normal curvature tensor at p, regarded as a linear map for $\Lambda^2(T_pM)\to\Lambda^2(\nu_p(M))$, the adapted normal curvature tensor is defined by

$$\mathcal{R}^\perp=R^\perp\circ(R^\perp)^t$$

where $(\ )^t$ is the transpose endomorphism. This implies that $\mathcal{R}^\perp$ has the same image as $R^\perp$.

From the Ricci identity one has the nice formula, if $\xi_1,\xi_2,\xi_3,\xi_4\in\nu_p(M)$,

$$\begin{aligned}\langle\mathcal{R}^\perp_{\xi_1,\xi_2}\xi_3,\xi_4\rangle&=\operatorname{trace}([A_{\xi_1},A_{\xi_2}]\circ[A_{\xi_3},A_{\xi_4}])\\&=-\langle[A_{\xi_1},A_{\xi_2}],[A_{\xi_3},A_{\xi_4}]\rangle=-\langle[[A_{\xi_1},A_{\xi_2}],A_{\xi_3}],A_{\xi_4}\rangle\qquad(**)\end{aligned}$$

where A is the shape operator of M.

Since $\mathcal{R}^\perp(\Lambda^2(\nu_p(M)))=R^\perp(\Lambda^2(T_pM))$, one has that $\mathcal{R}^\perp_{\xi_1,\xi_2}$ belongs to the normal holonomy algebra at p (since curvature tensors, take values in the holonomy algebra).

Since the isotropy representation of a semisimple symmetric space coincides with that of the dual symmetric space, we may always assume that the symmetric space is compact. Let then (G,K) be a compact simply connected symmetric pair and let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition associated to such a pair. The isotropy representation of K is naturally identified with the Ad-representation of K on $\mathfrak{p}$. The Euclidean metric on $\mathfrak{p}$ is $-B$, where B is the Killing form of $\mathfrak{g}$. We denote by a dot the Ad-action of K on $\mathfrak{p}$. Let $0\neq v\in\mathfrak{p}$ and let us consider the orbit $M=K.v\simeq K/K_v$ which is a Euclidean submanifold with constant principal curvatures (and rank at least 2 if and only if it is not most singular, i.e. the isotropy type of M is not maximal).

Let us consider the restriction $\langle\ ,\ \rangle$ of $-B$ to $\mathfrak{k}$. This is an Ad-K invariant positive definite inner product on $\mathfrak{k}$. Let us consider the (normally) reductive decomposition

$$\mathfrak{k} = \mathfrak{k}_v \oplus \mathfrak{m}$$

where $\mathfrak{k}_v$ is the Lie algebra of the isotropy group K_v and $\mathfrak{m}$ is the orthogonal complement, with respect to $\langle\,,\,\rangle$, of $\mathfrak{k}$. The restriction of $\langle\,,\,\rangle$ to $\mathfrak{m} \simeq T_{[e]}K/K_v \simeq T_vM$ induced a so-called normal homogeneous metric on M, which is in particular naturally reductive, that we also denote by $\langle\,,\,\rangle$. Such a Riemannian metric on M will be called the canonical normal homogeneous metric. In general this metric is different from the induced metric as a Euclidean submanifold. Namely,

PROPOSITION 2.7. *Let K act on $\mathbb{R}^N$ as an irreducible s-representation and let $M = K.v$, $v \neq 0$. If the (canonical) normal homogeneous metric on M coincides with the induced metric, then M has parallel second fundamental form* (or equivalently, M is extrinsically symmetric [**Fe**]).

PROOF. We keep the notation previous to this proposition. Let ∇^c be the canonical connection on M associated to the reductive decomposition $\mathfrak{k} = \mathfrak{k}_v \oplus \mathfrak{m}$. Then the second fundamental form α of M is parallel with respect to the connection $\bar{\nabla}^c = \nabla^c \oplus \nabla^\perp$, i.e. $\bar{\nabla}^c\alpha = 0$ [**OSa**], [**BCO**]. Let $\bar{\nabla} = \nabla \oplus \nabla^\perp$, where ∇ is the Levi-Civita connection on M associated to the induced metric which coincides, by assumption, with the normal homogeneous metric. Then

$$(\bar{\nabla}_x\alpha)(y,z) = \alpha(D_xy, z) + \alpha(y, D_xz)$$

where $D = \nabla - \nabla^c$. We have that $D_xy = -D_yx$. This is a general fact, for naturally reductive spaces, since the canonical geodesics coincide with the Riemannian geodesics (see, for instance, [**OR**]).

Then

$$(\bar{\nabla}_x\alpha)(x,x) = 2\alpha(D_xx, x) = 0.$$

But, from the Codazzi identity, $(\bar{\nabla}_x\alpha)(y,z)$ is symmetric in all of its three variables. Then $\bar{\nabla}\alpha = 0$ and so M has parallel second fundamental form. □

COROLLARY 2.8. *Let K act on $\mathbb{R}^N$ as an s-representation and let $M = K.v$, $v \neq 0$. Assume that K_v acts irreducibly on T_vM. Then M has parallel second fundamental form* (or, equivalently, M is extrinsically symmetric [**Fe**]).

REMARK 2.9. A submanifold of the Euclidean space with parallel second fundamental form is, up to a Euclidean factor, an orbit of an s-representation [**Fe**] (see also [**BCO**]).

LEMMA 2.10. *Let $M^n, \bar{M}^n \subset S^{N-1}$ be submanifolds of the sphere with parallel second fundamental forms (or, equivalently, extrinsically symmetric spaces). Assume also that M is a full submanifold of the Euclidean space $\mathbb{R}^N$ and that there exists $p \in M \cap \bar{M}$ with $T_pM = T_p\bar{M}$. Assume, furthermore, that the associated fundamental forms at p, $\alpha, \bar{\alpha}$ of M and $\bar{M}$, respectively, as submanifolds of the sphere, are proportional (i.e.*

$\bar{\alpha} = \lambda\alpha$, $\lambda \neq 0$). *Then* $M = \bar{M}$ (*and so* $\lambda = 1$) *or* $M = \sigma(\bar{M})$, *where* σ *is the orthogonal transformation of* $\mathbb{R}^N$ *which is the identity on* $\mathbb{R}p \oplus T_pM$ *and minus the identity on* $\bar{\nu}_p(\bar{M}) = (\mathbb{R}p \oplus T_pM)^{\perp}$ (*and so* $\lambda = -1$).

Proof. Observe, in our assumptions, that the second fundamenal forms of M and $\bar{M}$, as Euclidean submanifolds, are not proportional, unless they coincide (since the shapes operators of M and $\bar{M}$, coincide in the direction of the position vector p).

Let us write $M = K.p$ where K acts as an irreducible s-representation. One has that the restricted holonomy at p, of the bundle $TM \oplus \bar{\nu}(M)$, is the representation, of the connected isotropy $(K_p)_0$, on $T_pM \oplus \bar{\nu}_p(M)$. This is a well-known fact that follows form the following property: if X belongs to the Cartan subalgebra associated to the symmetric pair (K, K_p), then $\mathrm{d}l_{\mathrm{Exp}(tX)}$ gives the Levi-Civita parallel transport, when restricted to T_pM, along the geodesic $\gamma(t) = \mathrm{Exp}(tX).p$, and at the same time, when restricted to $\bar{\nu}_p(M)$, the normal parallel transport along $\gamma(t)$.

Since curvature endomorphisms take values in the holonomy algebra, one has that $(R_{x,y}, R^{\perp}_{x,y}) \in \mathfrak{t}_p$, where $\mathfrak{t}_p = \mathrm{Lie}(K_p) = \mathrm{Lie}((K_p)_0) \subset \mathfrak{so}(T_pM) \oplus \mathfrak{so}(\bar{\nu}_p(M))$ and R, $R^{\perp}$ are the tangent and normal curvature tensors of M at p, respectively.

Let R^S be the curvature tensor of the sphere S^{N-1} at p, restricted to T_pM. Then, from the Gauss equation,

$$R_{x,y} = T_{x,y} + R^S_{x,y}$$

where $\langle T_{x,y}z, w\rangle = \langle \alpha(x,w), \alpha(y,z)\rangle - \langle \alpha(x,z), \alpha(y,w)\rangle$.

For $\bar{M} = \bar{K}.p$ we have similar objects $\bar{R}$, $\bar{R}^{\perp}$, $\bar{\mathfrak{t}}_p$, $\bar{T}$. From the assumptions one has that $\bar{T} = \lambda^2 T$. So,

$$\bar{R}_{x,y} = \lambda^2 T_{x,y} + R^S_{x,y}. \tag{a}$$

From the assumptions, and Ricci equation, one has that

$$\bar{R}^{\perp}_{x,y} = \lambda^2 R^{\perp}_{x,y}. \tag{b}$$

Now observe that, for any $X \in \mathfrak{t}_p \subset \mathfrak{so}(T_pM) \oplus \mathfrak{so}(\bar{\nu}_p(M))$,

$$X.\alpha = 0 = X.(\lambda.\alpha) = X.\bar{\alpha} \tag{c}$$

and the same is true for any $\bar{X} \in \bar{\mathfrak{t}}_p$ (the actions of X and $\bar{X}$ are derivations).

As we observed, $(R_{x,y}, R^{\perp}_{x,y}) \in \mathfrak{t}_p$, and $(\bar{R}_{x,y}, \bar{R}^{\perp}_{x,y}) \in \bar{\mathfrak{t}}_p$. Then, from (a), (b) and (c) one obtains, if $\lambda \neq \pm 1$ that

$$(R^S_{x,y}, 0).\alpha = 0 = (R^S_{x,y}, 0).\bar{\alpha}.$$

Since the linear span of $\{R^S_{x,y} : x, y \in T_pM\}$ is $\mathfrak{so}(T_pM)$, one has that

$$\alpha(g.x, g.y) = \alpha(x, y)$$

for all $g \in SO(T_pM)$. Then, from the Gauss equation $\langle A_\xi x, y\rangle = \langle \alpha(x,y), \xi\rangle$, one obtains that all the shape operators of M at p commute with any element of $SO(T_pM)$. Then M is umbilical at p and hence, since it is homogeneous, at any point. Then M is an extrinsic sphere. Since M is full we conclude that $M = S^{N-1}$. Then, since $n = N-1$, $M = \bar{M}$.

Observe that the fullness condition is essential. In fact, if M and $\bar{M}$ are umbilical submanifolds of the sphere of different radios, the second fundamental forms at p are proportional.

If $\lambda = 1$, then M and $\bar{M}$ have both the same second fundamental form at p. Since both submanifolds have parallel second fundamental forms, it is well-known and standard to prove that $M = \bar{M}$.

If $\lambda = -1$, then we replace $\bar{M}$ by $\sigma(\bar{M})$ and the second fundamental forms of M and $\bar{M}$ must coincide. Therefore, $M = \sigma(\bar{M})$. □

Remark 2.11. Let us enounce Theorem 4.1 in [**O6**]: *let M^n be a locally full submanifold either of the Euclidean space or the sphere, such that the local normal holonomy group at p acts without fixed non- zero vectors. Assume, furthermore, that no factor of the normal holonomy is transitive on the sphere. Then there are points in M, arbitrary close to p, where the first normal space coincides with the normal space. In particular, codim$(M) \leq n(n+1)/2$.*

This bound on the codimension is correct. But the better and sharp estimate is $\operatorname{codim}(M) \leq n(n+1)/2 - 1$. In fact, from the proof one has that if the shape operator, at a generic $q \in M$, A_ξ is a multiple of the identity (it needs not to be zero, as in that proof), then ξ is in the nullity of the adapted normal curvature tensor $\mathcal{R}^\perp$. But this last tensor is not degenerate. This implies that the injective map $A : \nu_q(M) \to Sim(T_qM)$ cannot be onto. Then $\dim(\nu_q(M)) = \operatorname{codim}(M) \leq \dim(Sim(T_qM)) - 1 = n(n+1)/2 - 1$.

If M, in the above assumptions, is a submanifold of the sphere, then the codimension of M, as a Euclidean submanifold, is bounded by $n(n+1)/2$.

2.2. Holonomy systems.

We recall here some facts about holonomy systems that are useful in submanifold geometry.

A *holonomy system* is a triple $[\mathbb{V}, R, H]$, where $\mathbb{V}$ is a Euclidean vector space, H is a connected compact Lie subgroup of $SO(\mathbb{V})$ and $R \neq 0$ is an algebraic Riemannian curvature tensor on $\mathbb{V}$ that takes values $R_{x,y} \in \mathfrak{h} = \operatorname{Lie}(H)$. The holonomy system is called:

- *irreducible*, if H acts irreducibly on $\mathbb{V}$.
- *transitive*, if H acts transitively on the unit sphere of $\mathbb{V}$.
- *symmetric*, if $h(R) = R$, for all $h \in H$.

Observe that a Lie subgroup $H \subset SO(\mathbb{V})$ that acts irreducibly on $\mathbb{V}$ must be compact, as it is well-known (since the center of H has dimension at most 1).

A holonomy system $[\mathbb{V}, R, H]$ is the product (eventually, after enlarging H) of irreducible holonomy systems (up to a Euclidean factor).

One has the following remarkable result.

THEOREM 2.12 (Simons holonomy theorem, [**S**], [**O6**]). *An irreducible and non-transitive holonomy system $[\mathbb{V}, R, H]$ is symmetric. Moreover, R is, up to a scalar multiple, unique.*

REMARK 2.13. If $[\mathbb{V}, R, H]$ is an irreducible symmetric holonomy system, then $\mathfrak{h}$ coincides with the linear span of $R_{x,y}$, $x, y \in \mathbb{V}$. In this case, since $\langle R_{x,y}v, \xi\rangle = \langle R_{v,\xi}x, y\rangle$, one has that the normal space at v to the orbit $H.v$ is given by

$$\nu_v(H.v) = \{\xi \in \mathbb{V} : R_{v,\xi} = 0\}.$$

From a symmetric holonomy system one can build an involutive algebraic Riemannian symmetric pair $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{V}$. The bracket $[\,,\,]$ is given by:

a) $[\,,\,]_{|\mathfrak{h}\times\mathfrak{h}}$ coincides with the bracket of $\mathfrak{h}$.
b) $[X, v] = -[v, X] = X.v$, if $X \in \mathfrak{h} \subset \mathfrak{so}(\mathbb{V})$ and $v \in \mathbb{V}$.
c) $[v, w] = R_{v,w}$, if $v, w \in \mathbb{V}$.

This implies the following: *if $[\mathbb{V}, R, H]$ is an irreducible and symmetric holonomy system, then H acts on $\mathbb{V}$ as an irreducible s-representation.*

Observe that, in this case, the scalar curvature $sc(R)$ of R is different from 0 (since this is true for the curvature tensor of an irreducible symmetric space).

LEMMA 2.14. *Let $[\mathbb{V}, R, K]$ be an irreducible and non-transitive holonomy system. Let $T \in SO(\mathbb{V})$ be such that $R_{x,y} = 0$ if and only if $R_{T(x),T(y)} = 0$. Then $T(R) = R$.*

PROOF. Let $R' = T(R)$. If $\xi \in \nu_v(K.v) = \{\xi \in \mathbb{V} : R_{v,\xi} = 0\}$, then, from the assumptions, $R'_{v,\xi} = T.R_{T(v),T(\xi)}.T^{-1} = 0$. So, $0 = \langle R'_{v,\xi}x, y\rangle = \langle R'_{x,y}v, \xi\rangle$, for all $x, y \in \mathbb{V}$. Then the Killing field $R'_{x,y} \in \mathfrak{so}(\mathbb{V})$ of $\mathbb{V}$ is tangent to any orbit $K.v$. This implies that $R'_{x,y} \in \tilde{\mathfrak{h}} = \mathrm{Lie}(\tilde{K})$, where $\tilde{K} = \{g \in SO(\mathbb{V}) : g \text{ preserves any } K\text{-orbit}\}$. Observe that $\tilde{K}$ is a (compact) Lie subgroup of $SO(\mathbb{V})$ which is non-transitive (on the unit sphere of $\mathbb{V}$). Since $K \subset \tilde{K}$ we have that $[\mathbb{V}, R, \tilde{K}]$ is also an irreducible and non-transitive holonomy system. From the Simons holonomy theorem we have that $[\mathbb{V}, R, K]$ and $[\mathbb{V}, R, \tilde{K}]$ are both symmetric. Then $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ are (linearly) spanned by $R_{x,y}$, $x, y \in \mathbb{V}$. Then $\mathfrak{h} = \tilde{\mathfrak{h}}$ and therefore, $K = \tilde{K}$.

Since R' takes values in $\tilde{\mathfrak{h}} = \mathfrak{h}$, then $[\mathbb{V}, R', K]$ is also an irreducible and non-transitive holonomy system. Then, from the uniqueness part of Simons theorem, $R' = \lambda R$, for some scalar $\lambda \neq 0$. Since T is an isometry, it induces an isometry on the space of tensors. Then $\lambda = \pm 1$. But $0 \neq sc(R) = sc(R')$. Then $\lambda = 1$ and hence $R' = R$. □

REMARK 2.15. Let $M^n = K.v$, where K acts (by linear isometries) on $\mathbb{R}^{n+n(n+1)/2}$ as an s-representation ($|v| = 1$). Assume that the restricted normal holonomy group $\Phi(v)$ acts irreducibly on $\bar{\nu}_v(M) = \{v\}^\perp \cap \nu_v(M)$. In this case M is a minimal submanifold of the sphere $S^{n-1+n(n+1)/2}$ (see Corollary 2.6).

Let A be the shape operator of M and let $Sim_0(T_pM)$ be the space of traceless symmetric endomorphisms of T_pM. Then the map $A : \bar{\nu}_v(M) \mapsto Sim_0(T_vM)$ is a linear isomorphism. In fact, it is injective, since the first normal space of M coincides with the normal space, and $\dim(\bar{\nu}_v(M)) = \dim(Sim_0(T_vM))$. Moreover, by the second part of

Corollary 2.6, A is a homothecy from $\bar{\nu}_v(M)$ onto $Sim_0(T_vM)$, let us say, of constant $\beta > 0$.

Let us consider the following two irreducible and symmetric holonomy systems:

$$[Sim_0(T_pM), R, SO(T_pM)] \quad \text{and} \quad [\bar{\nu}_v(M), \mathcal{R}^\perp, \Phi(v)],$$

where $\mathcal{R}^\perp$ is the adapted normal curvature tensor of M at v and R is the curvature tensor of $SL(n)/SO(n)$ (which is explicitly given by $(***)$ of Section 2.3).

Observe that $[\bar{\nu}_v(M), \mathcal{R}^\perp, \Phi(v)]$ is symmetric since, by Theorem 2.5, the restricted normal holonomy group is given by

$$\Phi(v) = \{k_{|\bar{\nu}_v(M)} : k \in (K_v)_0\}$$

and $\mathcal{R}^\perp$ is left fixed by K_v.

Both algebraic curvature tensors are related by the formula $(**)$ of Section 2.1. This implies that the homothecy A maps $\mathcal{R}^\perp$ into R. Then the isometry $\beta^{-1}A$ maps $\mathcal{R}^\perp$ into $\beta^4 R$.

Since in a symmetric irreducible holonomy system the Lie algebra of the group is (linearly) generated by the curvature endomorphisms, we conclude that A maps $\Phi(v)$ onto $SO(T_pM) \simeq SO(n)$. In particular, the two holonomy systems are equivalent and $\Phi(v) \simeq SO(n)$.

2.3. Veronese submanifolds.

Let us consider the isotropy representation of the symmetric space of the non-compact type $X = SL(n+1)/SO(n+1)$ (which coincides with the isotropy representation of its compact dual $SU(n+1)/SO(n+1)$). The Cartan decomposition of such a space is

$$\mathfrak{sl}(n+1) = \mathfrak{so}(n+1) \oplus Sim_0(n+1)$$

where $Sim_0(n+1)$ denotes the vector space of the traceless symmetric (real) $(n+1) \times (n+1)$-matrices. The Ad-representation of $SO(n+1)$ on $Sim_0(n+1)$ coincides with the action, by conjugation, of $SO(n+1)$ on $Sim_0(n+1)$.

The curvature tensor of X at $[e]$ is given (up to a positive multiple) by

$$R_{A,B}C = -[[A, B], C]$$

and

$$\langle R_{A,B}C, D\rangle = -\langle [[A, B], C], D\rangle = \langle [A, B], [C, D]\rangle \qquad (***)$$

where $A, B, C, D \in Sim_0(n+1) \simeq T_{[e]}X$.

Let $S \in Sim_0(n+1)$ with exactly two eigenvalues, one of multiplicity 1 (whose associated eigenspace we denote by E_1) and the other of multiplicity n (whose associated eigenspace we denote by E_2).

The orbit $V^n = SO(n+1).S = \{kSk^{-1} : k \in SO(n+1)\}$ is called a *Veronese-type* orbit (see Appendix).

The following assertions are easy to verify or well-known.

FACTS 2.16.

(i) The Veronese-type orbit $V^n = SO(n+1).S$ is a full and irreducible submanifold of $Sim_0(n+1)$ which has dimension n and codimension $n(n+1)/2$. Moreover, V^n is a minimal submanifold of the sphere of radius $\|S\|$.

(ii) An orbit of $SO(n+1)$ in $Sim_0(n+1)$ has minimal dimension if and only if it is of Veronese-type; see Lemma 8.1.

(iii) The normal holonomy group at S, of the Veronese-type orbit V^n, coincides with the image of the slice representation of the isotropy group $(SO(n+1))_S = S(O(E_1) \times O(E_2)) \simeq S(O(1) \times O(n))$. So, from $(*)$, the restricted normal holonomy representation, on $\bar{\nu}_S(V^n) = \{S\}^\perp \cap \nu_S(V^n)$, is equivalent to the isotropy representation of the symmetric space $SL(n)/SO(n)$ of rank $n-1$. Then, this normal holonomy representation is irreducible. Moreover, it is non-transitive (on the unit sphere of $\bar{\nu}_S(V^n)$) if and only if $n \geq 3$.

(iv) A Veronese-type orbit $V^n = SO(n+1).S = SO(n+1)/(SO(n+1))_S$ is intrinsically a real projective space $\mathbb{R}P^n$. Moreover, $(SO(n+1), (SO(n+1))_S)$ is a symmetric pair and so $(SO(n+1))_S$ acts irreducibly on $T_S V^n$. Then, from Corollary 2.8, V^n has parallel second fundamental form (as it is well known).

A submanifold $M \subset \mathbb{R}^N$ is called a *Veronese submanifold* if it is extrinsically isometric to a Veronese-type orbit.

PROPOSITION 2.17. *Let $M^n = K.v \subset \mathbb{R}^{n+n(n+1)/2}$, where K acts on $\mathbb{R}^{n+n(n+1)/2}$ as an s-representation $(n \geq 2)$. Assume that the restricted normal holonomy group $\Phi(v)$ of M at v, restricted to $\bar{\nu}_v(M) = \{v\}^\perp \cap \nu_v(M)$, acts irreducibly (eventually, in a transitive way). Then,*

(i) *The normal holonomy representation of $\Phi(v)$ on $\bar{\nu}_v(M)$ is equivalent to the isotropy representation of the symmetric space $Sl(n)/SO(n)$.*

(ii) *M^n is a Veronese submanifold.*

PROOF. Part (i) is a consequence of Remark 2.15.

Since K acts as an s-representation, then the image under the slice representation, of the (connected) isotropy group $(K_v)_0$, coincides with the restricted normal holonomy group $\Phi(v)$. But, from part (i), $\dim(\Phi(v)) = \dim(SO(n))$. Then the isotropy group K_v has dimension at least $\dim(SO(n)) = \dim(SO(T_vM))$.

Observe that the isotropy representation of K_v on T_vM is faithful. Otherwise, M would be contained in the proper subspace which consists of the fixed vector of K_v in $\mathbb{R}^N$.

Then, $(K_v)_0 = SO(T_vM)$. So, K_v acts irreducibly on T_vM. Then, from Corollary 2.8, M has parallel second fundamental form.

Let V^n be a Veronese submanifold of $\mathbb{R}^{n+n(n+1)/2}$. We may assume that $v \in V^n$ and that $T_vM = T_vV^n = \mathbb{R}^n \subset \mathbb{R}^{n+n(n+1)/2}$. For V^n we have, from Corollary 2.6 and Remark 2.15, that its shape operator $\bar{A} : \{v\}^\perp \cap \nu_v(V^n) = \{v\}^\perp \cap \nu_v(M) \to Sim_0(T_vV^n) = Sim_0(T_vM)$ is a homothecy which induces an isomorphism from the normal holonomy

group $\bar{\Phi}(v)$ of V^n onto $SO(n)$.

The same is true, again from Corollary 2.6 and Remark 2.15, for the shape operator A of M. Namely, $A : \{v\}^\perp \cap \nu_v(M) \to Sim_0(T_vV^n) = Sim_0(T_vM) = Sim_0(\mathbb{R}^n)$ is a homothecy which induces an isomorphism from the (restricted) normal holonomy group $\Phi(v)$ of M onto $SO(n)$. Then the map $A^{-1} \circ \bar{A}$ is a homothecy with constant, let us say, $\beta > 0$, of the space $\{v\}^\perp \cap \nu_p(M)$. Let $h = \beta^{-1}A^{-1} \circ \bar{A}$. Then h is a linear isometry of $\{v\}^\perp \cap \nu_v(M)$.

Let now g be the linear isometry of $\mathbb{R}^{n+n(n+1)/2}$ defined by the following properties:

(i) $g(v) = v$.
(ii) $g_{|\{v\}^\perp \cap \nu_v(M)} = h^{-1}$.
(iii) $g_{|T_vM} = \mathrm{Id}$.

Then V^n and $g(M)$ have proportional second fundamental forms and satisfy all the other assumptions of Lemma 2.10. Then, by this lemma, $g(M)$, and hence M, is a Veronese submanifold. □

2.4. Coxeter groups and holonomy systems.

The goal of this section is to prove Proposition 2.21 that will be important for proving our main theorems. In order to prove this proposition we need some basic results, related to Coxeter groups, that we have not found through the mathematical literature. So, and also for the sake of self-completeness, we include the proofs.

LEMMA 2.18. *Let C be a Coxeter group acting irreducibly, by linear isometries, on the Euclidean n-dimensional vector space $(\mathbb{V}, \langle\,,\,\rangle)$. Let $H_1, \dots, H_r$ be the family of (different) reflection hyperplanes, associated to the symmetries of C (that generates C). Let us define the group $G = \{g \in End(\mathbb{V}) : g$ permutes $H_1, \dots, H_r$ and $\det(g) = \pm 1\}$. Then G is finite.*

PROOF. Let P_r be the (finite) group of bijections of the set $\{1, \dots, r\}$. Let $\rho : G \to P_r$ be the group morphism defined by $\rho(g)(i) = j$, if $g(H_i) = H_j$. The group G is finite if and only if $\ker(\rho)$ is finite. Let us prove that $\ker(\rho)$ is finite. If $g \in \ker(\rho)$ then it induces the trivial permutation on the family $H_1, \dots, H_r$. Then, its transpose g^t, with respect to $\langle\,,\,\rangle$, induces the trivial permutation on the set of lines $L_1, \dots, L_r$, where L_i is the line which is perpendicular to H_i, $i = 1, \dots, r$ (and hence, any vector in any line $L_1, \dots, L_r$ is an eigenvector of g^t). Let us define, for $i \neq j$, the 2-dimensional subspace $\mathbb{V}_{i,j} :=$ the linear span of $(L_i \cup L_j)$. This subspace is called *generic* if there exists $k \in \{1, \dots, r\}$, $i \neq k \neq j$ such that $L_k \subset \mathbb{V}_{i,j}$. In other words, $\mathbb{V}_{i,j}$ is generic if there are at least three different lines of $\{L_1, \dots, L_r\}$ which are contained in $\mathbb{V}_{i,j}$. We have, if $\mathbb{V}_{i,j}$ is generic, that $g^t : \mathbb{V}_{i,j} \to \mathbb{V}_{i,j}$ is a scalar multiple of the identity $\mathrm{Id}_{i,j}$ of $V_{i,j}$. In fact, any vector in $L_i \cup L_j \cup L_k$ is an eigenvector of $(g^t)_{|V_{i,j}}$. Then, since $\dim(V_{i,j}) = 2$, $(g^t)_{|V_{i,j}} = \lambda \mathrm{Id}_{i,j}$, for some $\lambda \in \mathbb{R}$. Let us define the following equivalence relation $\sim$ on the set $\{1, \dots, r\}$: $i \sim i'$ if there exist $i_1, \dots, i_l \in \{1, \dots, r\}$ with $i_1 = i$, $i_l = i'$ and such that $\mathbb{V}_{i_s, i_{s+1}}$ is generic, for $s = 1, \dots, l-1$. Let $i \in \{1, \dots, r\}$ be fixed. By the previous observations one has that there must exist $\lambda \in \mathbb{R}$ such that for any $j \in [i]$ (the equivalence class of i) and for any $v_j \in L_j$, $g^t(v_j) = \lambda v_j$. In order to prove this lemma, it suffices to show that there is only one equivalence class on $\{1, \dots, r\}$. In

fact, if $[i] = \{1, \dots, r\}$, then $g^t = \lambda Id$, since $L_1, \dots, L_r$ span $\mathbb{V}$ (because of its othogonal complement is point-wise fixed by g). So $g = \lambda Id$. But $\det(g) = \pm 1$. Then $\lambda^n = \pm 1$ and hence $\lambda = \pm 1$. So, $g = \pm Id$ and therefore there are at most two elements in $\ker(\rho)$.

Let $i \in \{1, \dots, r\}$ be fixed. Let us show that $[i] = \{1, \dots, r\}$. If $j \notin [i]$ then L_j is perpendicular to any L_k, for all $k \in [i]$. In fact, assume that this is not true for some $k \in [i]$. Let $s_j \in C$ be the symmetry across the hyperplane H_j. Then $s_j(L_k)$ is a line, which belongs to $\{L_1, \dots, L_r\}$, that is contained in $\mathbb{V}_{k,j}$ and it is different from both L_k and L_j. Then $j \sim k$ and therefore $j \sim i$. A contradiction. Then, if $j \notin [i]$, $L_k \subset H_j$, for all $k \in [i]$. So, s_j acts trivially on $\mathbb{V}_{[i]}$, the subspace spanned by $\bigcup_{k \in [i]} L_k$. Observe that s_j commutes with s_k, for all $k \in [i]$. Let now $\mathbb{V}_0$ be the maximal subspace of $\mathbb{V}$ such that it is point-wise fixed by all the symmetries s_j with $j \notin [i]$. Observe that this space is not the null subspace, since $\mathbb{V}_{[i]} \subset \mathbb{V}_0$. If there exists $j \notin [i]$, then $\mathbb{V}_0$ must be a proper subspace of $\mathbb{V}$, since $s_j \neq \mathrm{Id}$. On the other hand, if $k \in [i]$, then $s_k(\mathbb{V}_0) \subset \mathbb{V}_0$, since s_k commutes with all the symmetries s_j, $j \notin [i]$. Then $\mathbb{V}_0$ is a proper and non-trivial subspace of $\mathbb{V}$ which is invariant under the irreducible Coxeter group C. A contradiction. So, $[i] = \{1, \dots, r\}$. □

Lemma 2.19. *We are under the assumptions and notation of the above lemma. Then G acts by isometries.*

Proof. By the above lemma, G is finite. By averaging the inner product $\langle\, ,\, \rangle$ over the elements of G, we obtain a G-invariant inner product $(\, ,\,)$ on $\mathbb{V}$. Since $C \subset G$, then $(\, ,\,)$ is C-invariant. Since C acts irreducible, $\langle\, ,\, \rangle$ must be proportional to $(\, ,\,)$. Then G acts by isometries on $(\mathbb{V}, \langle\, ,\, \rangle)$. □

Corollary 2.20. *Let $(\mathbb{V}_i, \langle\, ,\, \rangle_i)$ be a Euclidean vector spaces and let C_i be a Coxeter group acting irreducibly, by linear isometries, on $(\mathbb{V}_i, \langle\, ,\, \rangle_i)$, $i = 1, 2$. Let $h : \mathbb{V}_1 \to \mathbb{V}_2$ be a linear map such that it induces a bijection from the family of reflection hyperplanes of C_1 into the family of reflection hyperplanes of C_2. Then h is a homothetical map.*

Proof. Let $(\, ,\,) = h^*(\langle\, ,\, \rangle_2)$ and let $C^* = h^*(C_2) = h^{-1}C_2h$. Observe that the determinant of any element of C_2 is ± 1, since it is an isometry of $(\mathbb{V}_2, \langle\, ,\, \rangle_2)$. So, any element in C^* has determinant ± 1. From the assumptions, we obtain that the family of reflection hyperplanes of the irreducible Coxeter group C^* of $(\mathbb{V}_1, (\, ,\,))$ coincides with the family $H_1, \dots, H_r$ of reflection hyperplanes of C_1. Then any element of C^* induces a permutation in this family of hyperplanes. Then, by Lemma 2.19, C^* acts by isometries on $(\mathbb{V}_1, \langle\, ,\, \rangle_1)$. Since C^* acts irreducibly, one has that $\langle\, ,\, \rangle_1$ is proportional to $(\, ,\,)$. This implies that h is a homothecy. □

Proposition 2.21. *Let $(\mathbb{V}, R, K)$ and $(\mathbb{V}', R', K')$ be irreducible, non-transitive (and hence symmetric) holonomy systems. Let $h : \mathbb{V} \to \mathbb{V}'$ be a linear isomorphism such that, for any K-orbit $K.v$ in $\mathbb{V}$, $h(\nu_v(K.v)) = \nu_{h(v)}(K'.h(v))$, where ν denotes the normal space. Then h is a homothecy and $h_*^{-1}(K') = K$.*

Proof. Observe that the groups K and K' act as irreducible s-representations.

We have that $K.v$ is a maximal dimensional orbit if and only if $K'.h(v)$ is so.

Recall that, for s-representations, an orbit is maximal dimensional if and only if it is principal.

Let $K.v$ be a principal K-orbit. This orbit is an irreducible (homogeneous) isoparametric submanifold of $\mathbb{V}$. There is an irreducible Coxeter group C, associated to this isoparametric submanifold, that acts on the normal space $\nu_v(K.v)$ [**Te**], [**PT**], [**BCO**]. If $H_1, \ldots, H_r$ are the reflection hyperplanes of the symmetries of C, then

$$\bigcup_{i=1}^{r} H_i = \{z \in \nu_v(K.v) : K.z \text{ is a singular orbit}\}. \tag{a}$$

If $v' = h(v)$ one has the similar objects $K'.v'$, $\nu_{v'}(K'v')$, C' and $H'_1, \ldots, H'_s$ and

$$\bigcup_{i=1}^{s} H'_i = \{z' \in \nu_{v'}(K'.v') : K'.z' \text{ is a singular orbit}\}. \tag{b}$$

Moreover, from (a) and (b), one has that h maps, bijectively, the family $H_1, \ldots, H_r$ onto the family $H'_1, \ldots, H'_s$. Then, $s = r$ and so we may assume that $h(H_i) = H'_i$, $i = 1, \ldots, s$.

Then, from Corollary 2.20, one has that

$$h : \nu_w(K.w) \to \nu_{w'}(K'.w')$$

is a homothecy, for any principal K-vector w, where $w' = h(w)$. Denote by $\lambda(w) > 0$ the homothecy constant of this map.

Observe, since $w \in \nu_w(K.w)$ and $w' \in \nu_{w'}(K'.w')$, that

$$\langle h(w), h(w)\rangle' = \lambda(w)\langle w, w\rangle$$

where $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle'$ are the inner products on $\mathbb{V}$ and $\mathbb{V}'$, respectively.

Let v_0 be a fixed K-principal vector and let $M = K.v_0$.

Let $TM = E_1 \oplus \ldots \oplus E_r$, where $E_1, \ldots, E_r$ are the (autoparallel) eigendistributions of TM associated to the commuting family of shape operators A_ξ of the isoparametric submanifold $M \subset \mathbb{V}$. Associated to any E_i there is a parallel normal field η_i, a so-called curvature normal, such that, for any normal field ξ,

$$A_{\xi|E_i} = \langle \xi, \eta_i\rangle \mathrm{Id}_{E_i}.$$

Let, for $q \in M$, $S_i(q)$ denote the integral manifold of E_i by q. Such integral manifold is a so-called curvature sphere. If $x \in S_i(q)$ then

$$\nu_x(M) \cap \nu_q(M) = (\eta_i(q))^\perp$$

where the orthogonal complement is inside $\nu_q(M)$. Observe that this intersection is non-trivial, since the codimension of M in $\mathbb{V}$ is at least 2. This implies $\lambda(x) = \lambda(q)$. Since the eigendistributions span TM, one has that moving along different curvature sphere

one can reach, from v_0, any other point of M. Then $\lambda(x) = \lambda(v_0)$, for all $x \in M$.

Observe now that, for any $y \in \mathbb{V}$, there exists $\bar{x} \in M$ such that $y \in \nu_{\bar{x}}(M)$. In fact, such an $\bar{x}$ can be chosen as a point where the function, from M into $\mathbb{R}$, $x \to \langle x, y\rangle$ attains a maximum.

Then

$$\frac{\langle h(y), h(y)\rangle'}{\langle y, y\rangle} = \frac{\langle h(\bar{x}), h(\bar{x})\rangle'}{\langle \bar{x}, \bar{x}\rangle} = \lambda(\bar{x}) = \lambda(v_0),$$

for all $0 \neq y \in \mathbb{V}$. Then h is a homothecy of constant $\lambda := \lambda(v_0)$. This proves the first assertion.

Let $g \in K'$ and let $T = h^{-1} \circ g \circ h$. Since h is a homothecy, $T \in SO(\mathbb{V})$. Then, from the assumptions and Remark 2.13 one has that T satisfies hypothesis of Lemma 2.14. Then, by this lemma, $T(R) = R$. This implies, since the Lie algebra of K is generated by $\{R_{x,y}\}$, that T belongs to $N(K)$, the normalizer of K in $O(\mathbb{V})$. Moreover, T must belong to the connected component $N_0(K)$ (because of T can be deformed to the identity, since K' is connected). But $N_0(K) = K$, since K acts as an s-representation (see [**BCO**, Lemma 6.2.2]). Then $T \in K$, thus $h_*^{-1}(K') = K$. □

Remark 2.22. The above proposition is not true if the holonomy systems are transitive. In fact, let $(\mathbb{V}, R, K)$ and $(\mathbb{V}', R', K')$ be the (symmetric) holonomy systems associated to the rank 1 symmetric spaces $S^{2n} = SO(2n+1)/SO(2n)$ and $\mathbb{C}P^n = SU(n+1)/S(U(1) \times \mathrm{U}(n))$, respectively. In this case $\dim(\mathbb{V}) = \dim(\mathbb{V}') = 2n$. Then any linear isomorphism from $\mathbb{V}$ into $\mathbb{V}'$, satisfies the assumption of Proposition 2.21, since the normal spaces of non-trivial K or K'-orbits are lines.

3. Non-transitive normal holonomy.

Let $M^n = H.v \subset S^{n-1+n(n+1)/2}$ be a homogeneous submanifold of the sphere. Assume that the (restricted) normal holonomy group, as a submanifold of the sphere, acts irreducibly and it is not transitive (on the unit normal sphere).

From now on, we will regard M^n as a submanifold of the Euclidean space $\mathbb{R}^{n+n(n+1)/2}$. Let $\nu(M)$ be the normal bundle and let $\Phi(v)$ be the restricted normal holonomy group at v (regarding M as a Euclidean submanifold). Observe that $\Phi(v)$ acts trivially on $\mathbb{R}.v$ and that $\Phi(v)$, restricted to $\bar{\nu}_v(M) := \{v\}^\perp \cap \nu_v(M)$, is naturally identified with the (restricted) normal holonomy group of M at v, as a submanifold of the sphere.

Observe that the irreducibility of the normal holonomy group representation on $\{v\}^\perp \cap \nu_v(M)$ implies that $\mathrm{rank}(M) = 1$. Namely, v is the only vector of $\nu_v(M)$ which is fixed by $\Phi(v)$. This implies that *M is a full and irreducible submanifold of the Euclidean space*. In fact, if M is not full then any non-zero constant normal vector is a parallel normal field which is not a multiple of the position vector. Then $\mathrm{rank}(M) \geq 2$. A contradiction. If M is reducible it must be a product of submanifolds contained in spheres. Then $\mathrm{rank}(M) \geq 2$. Also a contradiction.

One has, from Remark 2.11, that the first normal space $\nu^1(M)$ coincides with the normal space $\nu(M)$, regarding M as a Euclidean submanifold. This means, that the linear

map, from $\nu_v(M)$ into $Sim(T_vM)$, $\xi \mapsto A_\xi$ is injective, where A is the shape operator of M. Since $\dim(\nu_v(M)) = n(n+1)/2 = \dim(Sim(T_vM))$, then $A : \nu_v(M) \to Sim(T_vM)$ *is a linear isomorphism.*

Let $\mathcal{R}^\perp_{\xi_1,\xi_2}$ be the adapted normal curvature tensor (see Section 2). This tensor is given by

$$\begin{aligned}\langle \mathcal{R}^\perp_{\xi_1,\xi_2}\xi_3,\xi_4\rangle &= \operatorname{trace}([A_{\xi_1},A_{\xi_2}]\circ[A_{\xi_3},A_{\xi_4}])\\ &= -\langle [A_{\xi_1},A_{\xi_2}],[A_{\xi_3},A_{\xi_4}]\rangle = -\langle [[A_{\xi_1},A_{\xi_2}],A_{\xi_3}],A_{\xi_4}\rangle.\end{aligned}$$

Observe that the right hand side of the above equality is, with the usual identifications, the Riemannian curvature tensor $\langle \tilde{R}_{A_{\xi_1},A_{\xi_2}}A_{\xi_3},A_{\xi_4}\rangle$ of the symmetric space $GL(n)/SO(n)$.

Observe that such a symmetric space is isometric to the following product:

$$GL(n)/SO(n) = \mathbb{R}\times SL(n)/SO(n).$$

The tangent space of the second factor is canonically identified with the traceless symmetric matrices $Sim_0(n)$.

Let us consider the so-called *traceless shape operator* $\tilde{A}$ of M. Namely,

$$\tilde{A}_\xi := A_\xi - \frac{1}{n}\operatorname{trace}(A_\xi)\mathrm{Id} = A_\xi - \frac{1}{n}\langle \xi,\vec{H}\rangle \mathrm{Id}$$

where $\vec{H}$ is the mean curvature vector.

Observe that

$$\begin{aligned}\langle \mathcal{R}^\perp_{\xi_1,\xi_2}\xi_3,\xi_4\rangle &= -\langle [\tilde{A}_{\xi_1},\tilde{A}_{\xi_2}],[\tilde{A}_{\xi_3},\tilde{A}_{\xi_4}]\rangle\\ &= \langle \tilde{R}_{A_{\xi_1},A_{\xi_2}}A_{\xi_3},A_{\xi_4}\rangle = \langle R_{\tilde{A}_{\xi_1},\tilde{A}_{\xi_2}}\tilde{A}_{\xi_3},\tilde{A}_{\xi_4}\rangle \qquad (****)\end{aligned}$$

where R is the curvature tensor at $[e]$ of the symmetric space $SL(T_vM)/SO(T_vM)$ (see formula $(***)$ of Section 2.3).

If $\bar{\nu}_v(M) = \{v\}^\perp \cap \nu_v(M)$, we have the following two symmetric non-transitive irreducible holonomy systems: $[\bar{\nu}_v,\mathcal{R}^\perp,\Phi(v)]$ and $[Sim_0(T_vM),R,SO(T_vM)]$.

Recall that for a symmetric irreducible holonomy system $[\mathbb{V},\bar{R},K]$, from Remark 2.13, the normal space to an orbit $K.v$ is given by $\nu_v(K.v) = \{\xi\in\mathbb{V} : \bar{R}_{v,\xi} = 0\}$.

Then, from $(****)$, we have that the map $\tilde{A}$ is a liner isomorphism that maps normal spaces to $\Phi(v)$-orbits into normal spaces to $SO(T_vM)$-orbits. Then, by Proposition 2.21, $\tilde{A}$ is a homothecy and $\tilde{A} : \bar{\nu}_v(M) \to Sim_0(T_vM)$ transforms $\Phi(v)$ into $SO(T_vM)$. Then $\Phi(v)$ is isomorphic to $SO(T_vM)$. Therefore, we have the following result:

LEMMA 3.1. *Let $M^n = K.v \subset S^{n-1+n(n+1)/2}$ be a homogeneous submanifold. Assume that the restricted normal holonomy group of M acts irreducibly and it is non-transitive. Then the representation of the normal holonomy group $\Phi(v)$ on $\bar{\nu}_v(M)$ is (orthogonally) equivalent to the isotropy representation of the symmetric*

space $SL(n)/SO(n) \simeq SL(T_vM)/SO(T_vM)$. *Moreover, the traceless shape operator* $\tilde{A} : \bar{\nu}_v(M) \to Sim_0(T_vM)$ *is a homothecy that transforms, equivariantly,* $\Phi(v)$ *into* $SO(T_vM)$. (In particular, $\dim(\Phi(v)) = n(n-1)/2 = \dim(SO(n))$).

PROPOSITION 3.2. *Let* $M^n = K.v \subset S^{n-1+n(n+1)/2}$ *be a homogeneous submanifold. Assume that the restricted normal holonomy group of* M *acts irreducibly and it is non-transitive. Then, for any parallel normal section* $\xi(t)$ *along a curve, the traceless shape operator* $\tilde{A}_{\xi(t)}$ *has constant eigenvalues.*

PROOF. Note that M must be full and irreducible as a Euclidean submanifold (see the beginning of this section). Let $p \in M$ be arbitrary and let K_p be the isotropy subgroup of K at p. Let us decompose

$$\mathrm{Lie}(K) = \mathfrak{m} \oplus \mathrm{Lie}(K_p)$$

where $\mathfrak{m}$ is a complementary subspace of $\mathrm{Lie}(K_p)$. Let $B_r(0)$ be an open ball, centered at the origin, of radius r of $\mathfrak{m}$ such that $\mathrm{Exp} : B_r(0) \to M$ is a diffeomorphism onto its image $U = \mathrm{Exp}(B_r(0))$, which is a neighbourhood of p (the inner product on $\mathrm{Lie}(K)$ is irrelevant).

Let $\beta : [0,1] \to U$ be an arbitrary piece-wise differentiable curve with $\beta(0) = p$. Since $\beta(1) \in U$, there exits $X \in \mathfrak{m}$ such that $\beta(1) = \mathrm{Exp}(X).p$. Let $\gamma : [0,1] \to M$ be defined by $\gamma(t) = \mathrm{Exp}(tX).p$. Let us denote, for $k \in K$, by l_k the linear isometry $v \mapsto k.v$ of $\mathbb{V}$. Let $\tau_t^\perp$ denote the $\nabla^\perp$-parallel transport along $\gamma_{|[0,t]}$. Then, from remarks 6.2.8 and 6.2.9 of [**BCO**],

$$\tau_t^\perp = (\mathrm{d}l_{\mathrm{Exp}(tX)})_{|\nu_p(M)} \circ \mathrm{e}^{-t\mathcal{A}_X} \tag{A}$$

where $\mathcal{A}_X$ belongs to the normal holonomy algebra $\mathrm{Lie}(\Phi(p))$ and it is defined by

$$\mathcal{A}_X = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \tau_{-t}^\perp \circ (\mathrm{d}l_{\mathrm{Exp}(tX)})_{|\nu_p(M)}.$$

Let $\tau_\beta^\perp$ be the $\nabla^\perp$-parallel transport along β and $\phi = (\tau_1^\perp)^{-1} \circ \tau_\beta^\perp$. Then ϕ belongs to $\Phi(p)$, the restricted normal holonomy group at p. In fact, ϕ coincides with the $\nabla^\perp$-parallel transport along the null-homotopic, since it is contained in U, loop $\beta * \tilde{\gamma}$, obtained from gluing the curve β together with the curve $\tilde{\gamma}$, where $\tilde{\gamma}(t) = \gamma(1-t)$.

We have that $\tau_\beta^\perp = \tau_1^\perp \circ \phi$ and so, by (A),

$$\tau_\beta^\perp = ((\mathrm{d}l_{\mathrm{Exp}(X)})_{|\nu_p(M)} \circ \mathrm{e}^{-\mathcal{A}_X}) \circ \phi = (dl_{\mathrm{Exp}(X)})_{|\nu_p(M)} \circ \bar{\phi}$$

where $\bar{\phi} = \mathrm{e}^{-\mathcal{A}_X} \circ \phi$ belongs to $\Phi(p)$. Then, for any $\xi \in \nu_p(M)$,

$$\begin{aligned} \tilde{A}_{\tau_\beta^\perp(\xi)} &= \tilde{A}_{\mathrm{d}l_{\mathrm{Exp}(X)}(\bar{\phi}(\xi))} = \mathrm{d}l_{\mathrm{Exp}(X)} \circ \tilde{A}_{\bar{\phi}(\xi)} \circ (\mathrm{d}l_{\mathrm{Exp}(X)})^{-1} \\ &= \mathrm{Exp}(X).\tilde{A}_{\bar{\phi}(\xi)}.(\mathrm{Exp}(X))^{-1}. \end{aligned}$$

Then, from the paragraph just before Lemma 3.1, we have that there exists $g \in SO(T_p(M))$ such that $\tilde{A}_{\bar{\phi}(\xi)} = g.\tilde{A}_\xi.g^{-1}$. Then

$$\tilde{A}_{\tau_\beta^\perp(\xi)} = (\mathrm{Exp}(X).g).\tilde{A}_\xi.(\mathrm{Exp}(X).g)^{-1}.$$

This shows that the eigenvalues of $\tilde{A}_{\tau_\beta^\perp(\xi)}$ are the same as the eigenvalues of $\tilde{A}_\xi$.

The curve β was assumed to be contained in U. Since p is arbitrary, one obtains that the eigenvalue of $\tilde{A}_{\xi(t)}$ are locally constant for any parallel normal section $\xi(t)$ along a curve $c(t)$. This implies that the eigenvalues of $\tilde{A}_{\xi(t)}$ are constant. □

The following lemma is well known and the proof is similar to the case of hypersurfaces of a space form.

Lemma 3.3 (Dupin Condition). *Let M be a submanifold of a space of constant curvature and let ξ be a parallel normal field such that the eigenvalues of the shape operator A_ξ have constant multiplicities. Let $\lambda : M \to \mathbb{R}$ be an eigenvalue function of A_ξ such that its associated (and integrable from Codazzi identity) eigendistribution E has dimension at least 2. Then λ is constant along any integral manifold of E (or equivalently, $d\lambda(E) = 0$).*

Theorem 3.4. *Let $M^n \subset S^{n-1+n(n+1)/2}$ be a homogeneous submanifold, where $n > 3$. Assume that the restricted normal holonomy group acts irreducibly and not transitively. Then M is a Veronese submanifold.*

Proof. Note that M must be full and irreducible as a Euclidean submanifold (see the beginning of this section).

We will regard M as a submanifold of the Euclidean space $\mathbb{R}^{n+n(n+1)/2}$. Then, as we have observed at the beginning of this section, $A : \nu_p(M) \to Sim(T_pM)$ is an isomorphism ($p \in M$ is arbitrary). Now choose $\xi \in \nu_p(M)$ such that A_ξ has exactly two eigenvalues $\lambda_1(p)$, $\lambda_2(p)$ with multiplicities $m_1, m_2 \geq 2$ (this is not possible if $n \leq 3$). In particular, we assume that $m_1 = 2$ and $m_2 = n-2$. We may assume that ξ is small enough such that the holonomy tube [**BCO**] M_ξ is an immersed Euclidean submanifold (see Remark 3.5). We may also assume that ξ is perpendicular to the position (normal) vector p, since $A_p = -Id$.

There is a natural projection $\pi : M_\xi \to M$, $\pi(c(1)+\bar{\xi}(1)) = c(1)$. Moreover, $\hat{\xi}$ defines a parallel normal field to M_ξ, where $\hat{\xi}(q) = q - \pi(q)$. In this way M is a parallel focal manifold to M_ξ. Namely, $M = (M_\xi)_{-\hat{\xi}}$. Observe that the holonomy tube M_ξ is not a maximal one and so it has not a flat normal bundle (this would have been the case, in our situation, where all of the eigenvalues of A_ξ have multiplicity one). Let $\bar{\xi}(t)$ be a parallel normal field along an arbitrary curve $c(t)$ with $c(0) = p$, $\bar{\xi}(0) = \xi$. Then, from Proposition 3.2, the eigenvalues of the traceless shape operator $\tilde{A}_{\bar{\xi}(t)}$ are constant and hence the same as the eigenvalues of $\tilde{A}_\xi$ which are $\tilde{\lambda}_1 = \lambda_1(p) - (1/n)(2\lambda_1(p) + (n-2)\lambda_2(p))$, with multiplicity 2 and $\tilde{\lambda}_2 = \lambda_2(p) - (1/n)(2\lambda_1(p) + (n-2)\lambda_2(p))$, with multiplicity $n-2$.

Let $\vec{H}$ be the mean curvature vector field on M. Then the eigenvalues of the shape operator $A_{\bar{\xi}(1)}$ can be written as

$$\lambda_i(c(1)) = \tilde{\lambda}_i + \frac{1}{n}\langle \bar{\xi}(1), \vec{H}(c(1))\rangle \qquad i = 1, 2$$

with multiplicities 2 and $n-2$, respectively (independent of $c(1) \in M$).

From the tube formula [**BCO**], one has that the eigenvalues functions $\hat{\lambda}_1$ and $\hat{\lambda}_2$ of the shape operator $\hat{A}_{\hat{\xi}}$ of the holonomy tube, restricted to the horizontal subspace $\mathcal{H}_q$ of the holonomy tube M_ξ, at a point $q = c(1) + \bar{\xi}(1)$ are:

$$\hat{\lambda}_1(q) = \frac{\tilde{\lambda}_1 + (1/n)\langle \bar{\xi}(1), \vec{H}(c(1))\rangle}{1 - \tilde{\lambda}_1 - (1/n)\langle \bar{\xi}(1), \vec{H}(c(1))\rangle}$$

and

$$\hat{\lambda}_2(q) = \frac{\tilde{\lambda}_2 + (1/n)\langle \bar{\xi}(1), \vec{H}(c(1))\rangle}{1 - \tilde{\lambda}_2 - (1/n)\langle \bar{\xi}(1), \vec{H}(c(1))\rangle}$$

or, equivalently,

$$\hat{\lambda}_1(q) = \frac{\tilde{\lambda}_1 + (1/n)\langle \hat{\xi}(q), \vec{H}(\pi(q))\rangle}{1 - \tilde{\lambda}_1 - (1/n)\langle \hat{\xi}(q), \vec{H}(\pi(q))\rangle}$$

and

$$\hat{\lambda}_2(q) = \frac{\tilde{\lambda}_2 + (1/n)\langle \hat{\xi}(q), \vec{H}(\pi(q))\rangle}{1 - \tilde{\lambda}_2 - (1/n)\langle \hat{\xi}(q), \vec{H}(\pi(q))\rangle}$$

with (constant) multiplicities 2 and $n-2$, respectively. Observe that $\hat{A}_{\hat{\xi}(q)}$, restricted to the vertical distribution (tangent to the orbits in M_ξ of the normal holonomy group of M at projected points) is minus the identity. So, $\hat{A}_{\hat{\xi}(q)}$ has a third eigenvalue $\hat{\lambda}_3(q) = -1$ with constant multiplicity $m_3 = \dim(M_\xi) - \dim(M)$.

The real injective function $s \overset{f}{\mapsto} s/(1+s)$ transforms $\hat{\lambda}_i(q)$ into $\tilde{\lambda}_i + (1/n)\langle \hat{\xi}(q), \vec{H}(\pi(q))\rangle$ $(i = 1, 2)$. Then,

$$\hat{\lambda}_1(q) = \hat{\lambda}_1(q') \iff \hat{\lambda}_2(q) = \hat{\lambda}_2(q'). \tag{I}$$

In fact, any of both equalities implies $(1/n)\langle \hat{\xi}(q), \vec{H}(\pi(q))\rangle = (1/n)\langle \hat{\xi}(q'), \vec{H}(\pi(q'))\rangle$. This, by the above equalities, implies (I).

Let now E_1 and E_2 be the (horizontal) eigendistributions associated to eigenvalue functions $\hat{\lambda}_1$ and $\hat{\lambda}_2$ of the shape operator $\hat{A}_{\hat{\xi}}$. Observe that $\dim(E_1) = 2$ and $\dim(E_2) = n - 2 \geq 2$.

Up to here everything is valid, except the last inequality, also for $n = 3$. (II)
(This will be used in next section where we deal with the case $n = 3$).

If $\gamma(t)$ is a curve that lies in E_1 then, from the Dupin Condition (see Lemma 3.3)

we have that $\hat{\lambda}_1$ is constant along γ. So, by (I), $\hat{\lambda}_2$ is also constant along γ. The same is true if γ lies in E_2. This implies that $0 = v(\hat{\lambda}_1) = v(\hat{\lambda}_2) = v(\hat{\lambda}_3)$ for any vector v that lies in $\mathcal{H}$. Then the eigenvalues of the shape operator $\hat{A}_{\hat{\xi}}$ are constant along any horizontal curve. Since any two points, in a holonomy tube, can be joined by a horizontal curve we conclude that the (three) eigenvalues of $\hat{A}_{\hat{\xi}}$ are constant on M_ξ.

Then $\hat{\xi}$ is a parallel isoparametric (non-umbilical) normal section. Observe that M_ξ is a full irreducible Euclidean submanifold, since M is so. Moreover, M_ξ is complete with the induced metric (see Remark 3.5). Then, by [**BCO**], [**DO**], M_ξ must be a submanifold with constant principal curvatures. Since $M = (M_\xi)_{-\hat{\xi}}$, we have that M is also a submanifold with constant principal curvatures. Any principal holonomy tube of M has codimension at least 3 in the Euclidean space, since the normal holonomy of M, as a submanifold of the sphere, is non-transitive. Then, by the theorem of Thorbergsson [**Th**], [**O2**], [**BCO**], M is an orbit of an (irreducible) s-representation.

The fact that M is a Veronese submanifold follows from Proposition 2.17. □

Remark 3.5. Let $M^n = H.v$ be a full irreducible homogeneous submanifold of $\mathbb{R}^N$ which is (properly) contained in the sphere S^{N-1}. We are not assuming that M is compact (in which case the assertions of this remark are trivial).

By making use of the homogeneity of M one obtains that there exists $\varepsilon > 0$ such that: if $\xi \in \nu(M)$ with $0 < \|\xi\| < \varepsilon$ then any of the eigenvalues λ of the shape operator A_ξ satisfies $|\lambda| < 1 - a$, for some $0 < a < 1$.

Let us assume that $\operatorname{rank}(M) = 1$, i.e., M is not a submanifold of higher rank (otherwise, M would be an orbit of an s-representation and hence compact).

Let $\xi \in \nu_v(M)$ with $0 < \|\xi\| < \varepsilon$ and let us consider the normal holonomy subbundle by ξ [**BCO**] of the normal bundle $\pi : \nu(M) \to M$.

$$\operatorname{Hol}_\xi(M) = \left\{\eta \in \nu(M) : \eta \overset{\mathcal{H}}{\sim} \xi\right\}$$

where $\mathcal{H}$ is the horizontal distribution of $\nu(M)$ and $\eta \overset{\mathcal{H}}{\sim} \xi$ if η and ξ can be joined by a horizontal curve. Equivalently, $\eta \overset{\mathcal{H}}{\sim} \xi$ implies η is the $\nabla^\perp$-parallel transport of ξ along some curve.

One has that the fibres of $\pi : \operatorname{Hol}_\xi(M) \to M$ are compact. In fact, $\pi^{-1}(\{\pi(\eta)\}) = \Phi(\pi(\eta)).\eta$, where Φ denotes the normal holonomy group. Observe that such a group is compact, since its connected component acts as an s-representation (see the discussion inside the proof of Theorem 4.1, Case (2), (c)).

Let us consider the normal exponential map $\exp^\nu : \nu(M) \to \mathbb{R}^N$, given by $\exp^\nu(\eta) = \pi(\eta) + \eta$. Let $\eta \in \nu_p(M)$ and identify, as usual, via $\mathrm{d}\,\pi$, $T_pM \simeq \mathcal{H}_\eta$. The vertical distribution $\nu_\eta = T_\eta\nu_p(M)$ is canonically identified to $\nu_p(M)$. With this identification one has the well-known expression for the differential of the normal exponential map:

$$\mathrm{d}(\exp^\nu)_{|\mathcal{H}_\eta} = (I - A_\eta), \qquad \mathrm{d}(\exp^\nu)_{|\nu_\eta} = Id_{\nu_p(M)}. \tag{C}$$

Then $\exp^\nu : \operatorname{Hol}_\xi(M) \to \mathbb{R}^N$ is an immersion. The image of this map is the so-called holonomy tube M_ξ of M by ξ. It is given by

$$M_\xi = \big\{c(1) + \bar\xi(1) : \bar\xi(t) \text{ is } \nabla^\perp\text{-parallel along } c(t) \text{ where } c(0) = p,\ \bar\xi(0) = \xi\big\}.$$

Many times, and in particular in the proof of Theorem 3.4, for the sake of simplifying the notation, the immersed submanifold $\exp^\nu : \mathrm{Hol}_\xi(M) \to \mathbb{R}^N$ will be also denoted by M_ξ.

One has that the Euclidean submanifold $\exp^\nu : \mathrm{Hol}_\xi(M) \to \mathbb{R}^N$, with the induced metric $\langle\ ,\ \rangle$, is a complete Riemannian manifold. In fact, let $(\ ,\ )$ be the Sasaki metric on $\mathrm{Hol}_\xi(M)$. In such a metric the horizontal distribution is perpendicular to the vertical one. Moreover, π is a Riemannian submersion and the metric in the vertical space $\Phi(p).\eta$ is that induced from the metric on the normal space $\nu_p(M)$. Since M is complete and the fibres are compact, then $(\ ,\ )$ is complete. Then, from (C), $a^2(\ ,\ ) \le \langle\ ,\ \rangle$. This implies that the induced metric is also complete.

4. The proof of the conjecture in dimension 3.

THEOREM 4.1. *Let $M^3 = H.p$ be a 3-dimensional homogeneous submanifold of the sphere S^{N-1} which is full and irreducible (as a submanifold of the Euclidean space $\mathbb{R}^N$). Assume that the normal holonomy group of M is non-transitive. Then M is an orbit of an s-representation.*

PROOF. Assume that $\mathrm{rank}(M) = 1$. Otherwise, by Theorem 2.2, M is an orbit of an s-representation. Then, by Lemma 4.2, the normal holonomy of M, as a submanifold of the sphere acts irreducibly and $N = 9 = 3 + 3(3+1)/2$. We have also that the first normal bundle, which coincides with the normal bundle, has maximal codimension.

Keeping the notation and general constructions in the proof of Theorem 3.4, we have that everything is still valid up to (II). The only difference is that the eigenvalue $\hat\lambda_2$ has multiplicity 1. So, we have the Dupin condition only for the eigendistribution E_1 but not for the 1-dimensional eigendistribution E_2.

Let $\bar M = M_\xi/\mathcal{E}_1$ be the quotient of the (partial) holonomy tube M_ξ by the (maximal) integral manifolds of the 2-dimensional integrable distribution E_1.

Observe that the (partial) holonomy tube M_ξ has dimension 5. In fact, from Lemma 4.2, any focal orbit of the restricted normal holonomy group $\Phi(p) \simeq SO(3)$ has dimension 2 (and it is isometric to the Veronese V^2).

By [**BCO**, Theorem 6.2.4 part (2)], one has that $H \subset SO(9)$ acts by (extrinsic) isometries on M_ξ. Moreover, the projection $\pi : M_\xi \to M$ is H-equivariant.

If $H.(p+\xi) = M_\xi$, then M_ξ is a full and irreducible homogeneous Euclidean submanifold which is of higher rank. Then, in this case, by the rank rigidity theorem for submanifolds, M_ξ is an orbit of an s-representation. Hence $M = (M_\xi)_{-\hat\xi}$ is an orbit of an s-representation.

So, we may assume that $H.(p+\xi) \subsetneq M_\xi$. Let $\mathfrak{h} = \mathrm{Lie}(H)$. Let us consider the subspace $\mathfrak{h}.(p+\xi)$ of $T_{p+\xi}M_\xi$. This subspace has dimension at least 3, since $\mathrm{d}\pi(\mathfrak{h}.(p+\xi)) = \mathfrak{h}.p = T_pM$. The horizontal subspace $\mathcal{H}_{(p+\xi)}$ of $T_{p+\xi}M_\xi$ has dimension 3. Since $T_{p+\xi}M_\xi$ has dimension 5, $\dim(\mathcal{H}_{(p+\xi)} \cap \mathfrak{h}.(p+\xi)) \ge 1$.

Case (1): $E_1(x) + (\mathcal{H}_x \cap \mathfrak{h}.x) = \mathcal{H}_x$, for some $x \in M_\xi$.

We may assume that $x = p + \xi$. Observe that if the above equality holds at $(p + \xi)$ then it also holds for q in some open neighbourhood U of $(p + \xi)$ in M_ξ.

Recall, continuing with the notation in the proof of Theorem 3.4, that the eigenvalues functions (which are differentiable) of the shape operator $\hat{A}_{\hat{\xi}}$ at q are: $\hat{\lambda}_1(q)$ with multiplicity 2, $\hat{\lambda}_2(q)$ with multiplicity 1 and $\hat{\lambda}_3(q) = -1$ with multiplicity 2 (whose associated eigenspace is the vertical distribution ν_q).

On one hand, from the Dupin condition, since $\dim(E_1) = 2$, and the equivalence (I) in the proof of the above mentioned theorem, we have that

$$0 = v(\hat{\lambda}_1) = v(\hat{\lambda}_2) = v(\hat{\lambda}_3)$$

for any $v \in E_1(q)$. Or, briefly,

$$\{0\} = E_1(q)(\hat{\lambda}_1) = E_1(q)(\hat{\lambda}_2) = E_1(q)(\hat{\lambda}_3).$$

On the other hand, if $X \in \mathfrak{h}$,

$$0 = (X.q)(\hat{\lambda}_1) = (X.q)(\hat{\lambda}_2) = (X.q).(\hat{\lambda}_3).$$

In fact, this follows from the fact that the parallel normal field $\hat{\xi}$ of M_ξ is H-invariant and that $\hat{A}_{h.\hat{\xi}(q)} = h.\hat{A}_{\hat{\xi}(q)}.h^{-1}$, for all $h \in H$.

Then, from the assumptions of this case,

$$\{0\} = \mathcal{H}_q(\hat{\lambda}_1) = \mathcal{H}_q(\hat{\lambda}_2) = \mathcal{H}_q(\hat{\lambda}_3) \tag{III}$$

for any $q \in U$.

Since M is (extrinsically) homogeneous, the local normal holonomy groups have all the same dimension. Then the local normal holonomy group at any $x \in M$ coincides with the restricted normal holonomy group $\Phi(x)$.

The $\nabla^\perp$-parallel transport along short loops, based at $p \in M$, produces a neighbourhood Ω of e in the local normal holonomy group (see [**CO**], [**DO**]). This implies, from (III), that the eigenvalues of $\hat{A}_{\hat{\xi}(p+\omega.\xi)}$ are the same as the eigenvalues $\hat{\lambda}_1(p + \xi)$, $\hat{\lambda}_2(p+\xi)$, $\hat{\lambda}_3(p+\xi) = -1$ of $\hat{A}_{\hat{\xi}(p+\xi)}$, for all $\omega \in \Omega$. From this it is standard to show that the eigenvalues of $\hat{A}_{\hat{\xi}(p+\phi.\xi)}$ are the same of those of $\hat{A}_{\hat{\xi}(p+\xi)}$, for all $\phi \in \Phi(p)$. Therefore, the eigenvalues of $\hat{A}_{\hat{\xi}}$ are constant on $p + \Phi(p).\xi = \pi^{-1}(\{p\})$. Since H acts transitively on M, then $H.\pi^{-1}(\{p\}) = M_\xi$. This implies, since $\hat{\xi}$ is H-invariant, that the eigenvalues of $\hat{A}_{\hat{\xi}}$ are constant on M_ξ.

Observe that the parallel normal field $\hat{\xi}$ is not umbilical, since $\hat{A}_{\hat{\xi}}$ has three distinct (constant) eigenvalues. Then, from [**DO**] (see Theorem 5.5.8 of [**BCO**]), M_ξ has constant principal curvatures. So, $M = (M_\xi)_{-\hat{\xi}}$ has constant principal curvatures. If $\tilde{M}$ is a principal holonomy tube of M, then $\tilde{M}$ is isoparametric [**HOT**]. Observe that $\tilde{M}$ is not a hypersurface of a sphere (since the normal holonomy group, in the Euclidean space, is not transitive on the orthogonal complement of the position vector), then by the theorem

of Thorbergsson [**Th**], [**O2**] $\tilde{M}$ is an orbit of an s-representation. Then M is an orbit of an s-representation, since it is a focal (parallel) manifold to $\tilde{M}$.

Case (2): $E_1(x) + (\mathcal{H}_x \cap \mathfrak{h}.x) \subsetneq \mathcal{H}_x$, for all $x \in M_\xi$

or equivalently, $(\mathcal{H}_x \cap \mathfrak{h}.x) \subset E_1(x)$, since $\dim(E_1(x)) = 2$ and $\dim(\mathcal{H}_x) = 3$.

This case splits into several sub-cases, depending on how big is the group H. Namely, depending on $\dim(H) \geq 3 = \dim(M)$. The most difficult case is the generic one where $\dim(H) = 3$. For this case we will have to use topological arguments.

Note that $\dim(H) \leq 6$. In fact, H acts effectively on M, since M is a full submanifold. Otherwise, if $h \in H$ acts trivially on M then it acts trivially on the (affine) span of M which is $\mathbb{R}^9$. But the dimension of the isometry group of an n-dimensional Riemannian manifold is bounded by $(n+1)n/2$ (the dimension of the isometry group of an n-dimensional space of constant curvature). In our case, since $n = 3$, $\dim(H) \leq 6$.

Observe that H cannot be abelian. In fact if H is abelian, since the dimension of the ambient space $N = 9$ is odd, the (connected) subgroup $H \subset SO(9)$ must fix a vector, let us say $v \neq 0$. So, no H-orbit $H.q$ is a full submanifold, since it is contained in $q + \{v\}^\perp$. A contradiction, since $M = H.p$ is full.

Observe that $\dim(H)$ cannot be 5. In fact, if $\dim(H) = 5$ then the isotropy H_p has dimension 2 and so it is abelian. We regard $H_p \subset SO(T_pM) \simeq SO(3)$, via the isotropy representation. But the rank of $SO(3)$ is 1 and so it has no abelian two dimensional subgroups. A contradiction.

(a) $\dim(H) = 6$.

In this case we must have that $(H_p)_0 = SO(3)$, since $\dim(H_p) = 3$. Since $SO(3)$ is simple, the slice representation sr of $(H_p)_0$ on the normal space $\nu_p(M)$ must be either trivial or its image has dimension 3. In the first case we obtain that all shape operators A_μ of M at p are a multiple of the identity, since they commute all with $(H_p)_0$. Note that $A_\mu = A_{h.\mu} = h.A_\mu.h^{-1}$. So $M = M^3$ is an umbilical submanifold of $S^8 \subset \mathbb{R}^9$. So, M is not full. A contradiction.

Let us deal with the case that the image of the slice representation has dimension 3. By [**BCO**, Corollary 6.2.6] $sr((H_p)_0) \subset \Phi(p)$ where $\Phi(p)$ is the restricted normal holonomy group of M as a Euclidean submanifold. Since $\dim(\Phi(p)) = 3$, we conclude that $sr((H_p)_0) = \Phi(p)$. Then, any holonomy tube of M is an H-orbit. In particular the principal ones, which have flat normal bundle. But the holonomy tubes are full and irreducible Euclidean submanifolds, which have codimension at least 3 (since $\Phi(p)$ acts on the 6-dimensional normal space $\nu_p(M)$ with cohomogeneity 3). Then, by the theorem of Thorbergsson [**Th**], [**O2**], any holonomy tube is an orbit of an s-representation and so M is an orbit of an s-representation. By Proposition 2.17 one has that $M = M^3$ is a Veronese submanifold.

(b) $\dim(H) = 4$.

In this case the isotropy H_p has dimension 1. If the slice representation sr of $(H_p)_0$ is trivial, then, as in (a), all shape operators at p commute with $(H_p)_0 \simeq S^1$. A contradiction, since the family of shape operators is $Sim(T_pM)$.

Let us then restrict to the case that the slice representation is not trivial. For this we have to use a result of [**OS**] (see [**BCO**, Theorem 6.2.7]). In fact, we need the following

weaker version, which was the main step in the proof of Simons holonomy theorem given in [**O6**]. Namely, Proposition 2.4 of [**O6**]: *for a full and irreducible H-homogeneous Euclidean submanifold M^n, $n \geq 2$, the projections, on the normal space $\nu_p(M)$, of the (Euclidean) Killing fields given by the elements of $\mathfrak{h} = Lie(H)$, belong to the normal holonomy algebra $\mathfrak{g}$.*

Then, in our situation, since $\dim(\mathfrak{h}) = 4$ and $\dim(\mathfrak{g}) = 3$, there must exist $0 \neq X \in \mathfrak{h}$ such that it projects trivially on the normal space. Such an X cannot be in the isotropy algebra, since we assume that the slice representation of $(H_p)_0 \simeq S^1$ is non-trivial. This implies that $0 \neq X.p \in T_pM$.

Let us consider the H-invariant parallel normal field $\hat{\xi}$ of M_ξ. Recall that $(M_\xi)_{-\hat{\xi}} = M$ (and so M is a parallel focal manifold of M_ξ).

Since X projects trivially on $\nu_p(M)$, $X.q \in \mathcal{H}_q$, for all $q \in (p + \Phi(p).\xi) = ((\pi)^{-1}(\{p\}))_q \subset M_\xi$.

Recall that we are in Case (2). Then, $X.q \in E_1(q)$, for all $q \in (p + \Phi(p).\xi)$. Let us consider the curve $\gamma(t) = \mathrm{Exp}(tX).p$ of M^3. One has that $\gamma'(0) = X.p \neq 0$. Let $q \in (p + \Phi(p).\xi)$ and let $\psi(t)$ be the normal parallel transport of $(q - p) \in \nu_p(M)$ along $\gamma(t)$. Then $\psi(t) = \hat{\xi}(\gamma(t) + \psi(t))$, as it is well known, from the construction of holonomy tubes [**HOT**], [**BCO**] (observe that $M_\xi = M_{q-p}$). From the tube formula of [**BCO**, Lemma 4.4.7] (the notation in this lemma permutes our objects),

$$A_{(q-p)} = \hat{A}_{(q-p)|\mathcal{H}}.((Id - \hat{A}_{(q-p)})_{|\mathcal{H}})^{-1}$$

one has that $E_1(q)$ is an eigenspace of the shape operator $A_{(q-p)}$ of M.

On the one hand, since $\pi(q) = q - \hat{\xi}(q)$,

$$\mathrm{d}\pi(E_1(q)) = (Id + \hat{A}_{\hat{\xi}})(E_1(q)) \subset E_1(q).$$

On the other hand, since $\hat{\xi}$ is H-invariant and $\hat{\xi}(q) = (q - p)$,

$$\begin{aligned} \mathrm{d}\pi(X.q) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 (\mathrm{Exp}(tX).q - \hat{\xi}(\mathrm{Exp}(tX).q)) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 (\mathrm{Exp}(tX).q - \mathrm{Exp}(tX).(q - p)) = X.p. \end{aligned}$$

Therefore, $X.p$ belongs to an eigenspace of any shape operator A_{q-p} of M, such that $q \in (p + \Phi(p).\xi)$ (recall that we have assumed, without loss of generality, that ξ is perpendicular to the position vector p).

Observe that $\Phi(p).\xi$ spans $\{p\}^\perp$, since $\Phi(p)$ acts irreducibly on $\{p\}^\perp$. So $X.p$ is an eigenvector of any shape operator A_η, where $\langle \eta, p \rangle = 0$.

Since $A_p = -Id$, we conclude that $X.p$ is an eigenvector of all shape operators of M at p. This is a contradiction, since the family of shape operators at p coincides with $Sim(T_pM)$.

(c) $\dim(H) = 3$.

Since we have excluded the case where H is abelian, then H must be simple, with universal cover the (compact) group $\mathrm{Spin}(3) \simeq S^3$. This case is the generic one where the isotropy is finite. Note that M must be compact.

Also note that the (full) normal holonomy group $\tilde{\Phi}(p)$ of M is compact. In fact, $(\tilde{\Phi}(p))_0$ coincides with the restricted normal holonomy group $\Phi(p)$. Moreover, $\tilde{\Phi}(p)$ is included in the compact group $N(\Phi(p))$, the normalizer of $\Phi(p)$ in $O(\nu_p(M))$. Observe that $(N(\Phi(p)))_0 = \Phi(p)$, since $\Phi(p)$ acts as an s-representation (see [**BCO**, Lemma 6.2.2]). Then $\tilde{\Phi}(p)$ has a finite number of connected components, as well as $\tilde{\Phi}(p).\xi$. This implies that M_ξ is compact.

Let us construct the so-called *caustic fibration*. The eigenvalues functions of $\hat{A}_{\hat{\xi}}$ are bounded on M_ξ. Since M is contained in a sphere, M_ξ is contained in a (different) sphere. If η is the position vector field of M_ξ, then η is an umbilical parallel normal field. In fact, $\hat{A}_\eta = -\mathrm{Id}$. By adding, eventually, to the parallel normal field $\hat{\xi}$ a (big) constant multiple of $-\eta$ we obtain a new parallel and H-invariant normal field, such that its associated shape operator has the same eigendistributions as $\hat{A}_{\hat{\xi}}$ and all of the three eigenvalues functions are everywhere positive and so nowhere vanishing. Just for the sake of simplifying the notation, we also denote this perturbed normal field by $\hat{\xi}$. The eigenvalues of $\hat{A}_{\hat{\xi}}$ are also denoted by $\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{\lambda}_3$, which differ from the original ones by a (same) constant c.

The *caustic* map ρ, from M_ξ into $\mathbb{R}^9$, $q \overset{\rho}{\mapsto} q + (\hat{\lambda}_1(q))^{-1}\hat{\xi}(q)$ has constant rank. In fact, $ker(\mathrm{d}\rho) = E_1$ has constant dimension 2, since from the Dupin condition, $\hat{\lambda}_1$ is constant along any integral manifold $Q_1(q)$ of E_1. Observe that $\hat{\lambda}_2$ is also constant along $Q_1(q)$, due to equivalence (I) in the proof of Theorem 3.4 (and the same is true, of course, for the third eigenvalue $\hat{\lambda}_3 \equiv -1 + c$).

Let $\bar{M} = M_\xi/\mathcal{E}_1$ be the quotient of M_ξ by the family $\mathcal{E}_1$ of (maximal) integral manifolds of E_1. From Lemma 4.3 we have that $\bar{M}$ is a compact 3-manifold immersed in $\mathbb{R}^9$, via the projection $\bar{\rho}$, of the caustic map ρ, to the compact quotient manifold $\bar{M}$. Moreover, $\bar{\pi} : M_\xi \to \bar{M}$ is a fibration, where $\bar{\pi} : M_\xi \to \bar{M}$ is the projection. The distribution E_1 is H-invariant, since $\hat{\xi}$ is so. So, the action of H on M_ξ projects down to an action on $\bar{M}$. So, $\bar{\pi}$ is H-equivariant.

Observe that ρ is H-equivariant, since $\hat{\xi}$ is H-invariant. Then, since $\bar{\pi}$ is H-equivariant, the immersion $\bar{\rho} : \bar{M} \to \mathbb{R}^9$ is H-equivariant.

We have the following two H-equivariant fibrations on M_ξ:

$$0 \to \Phi(p).\xi \to M_\xi \overset{\tilde{\pi}}{\to} \tilde{M} \to 0 \qquad \text{(holonomy tube fibration)}$$
$$0 \to Q \to M_\xi \overset{\bar{\pi}}{\to} \bar{M} \to 0 \qquad \text{(caustic fibration)}$$

where Q is any integral manifold of E_1 and $\tilde{M}$ is the quotient manifold of M_ξ over the connected component of the fibres of $\pi : M_\xi \to M$, which are orbits of the restricted normal holonomy groups $\Phi(p)$, $p \in M$. We have that $\tilde{M}$ is a finite cover of M.

Recall that we are under the assumptions of Case (2)

We will derive a *topological* contradiction. This is by using that the holonomy tube M_ξ is the total space of above two different fibrations.

On the one hand the holonomy tube has a finite fundamental group $\pi_1(M_\xi)$. This follows from the long exact sequence of homotopies, associated to the holonomy tube fibration. In fact, the fibres are (real) projective 2-spaces (which have a finite fundamental group). Moreover, the base space $\tilde{M}$ has also a finite fundamental group, since it is an orbit, with finite isotropy, of the group $\mathrm{Spin}(3) \simeq S^3$. Since the fibres of the caustic fibration are connected and the total space M_ξ has finite fundamental group, then the caustic (base) manifold $\bar{M}$ has a finite fundamental group.

On the other hand, from Lemma 4.4 we have that the fundamental group of the caustic manifold $\bar{M}$ is not finite (this is by showing that H acts with cohomogeneity 1 and without singular orbits on $\bar{M}$).

A contradiction. So we can never be under the assumptions of Case (2) if $H \simeq \mathrm{Spin}(3)$.

This finishes the proof that M is an orbit of an s-representation. □

Lemma 4.2. *We are in the assumptions of Theorem* 4.1. *Then, if* $\mathrm{rank}(M) = 1$, *the* (*restricted*) *normal holonomy group* $\Phi(p)$, *as a submanifold of the sphere, acts irreducibly and* $N = 9$. *Moreover, the* (*restricted*) *normal holonomy acts as the action of* $SO(3)$, *by conjugation, on the traceless* 3×3*-symmetric matrices. Furthermore, the traceless shape operator* $\tilde{A}$ *of* M *at* p *is* $SO(3)$*-equivariant.*

Proof. Let us regard M^3 as a submanifold of the Euclidean space $\mathbb{R}^N$. If M is not of higher rank one has, from Proposition 6.1, that the (restricted) normal holonomy group $\Phi(p)$ acts irreducibly on $\bar{\nu}(p)$ (the orthogonal complement of the position vector p). Since $\Phi(p)$ is non-transitive (on the unit sphere of $\bar{\nu}_p(M)$), the first normal space, as a submanifold of the Euclidean space, coincides with the normal space (see Remark 2.11). Then, the codimension $k = N - 3$ satisfies $k \leq 6 = 3(3+1)/2$. Then the normal holonomy group representation coincides with the isotropy representation of an irreducible symmetric space of rank at least 2 and dimension at most 5. Then, by Remark 4.6, the normal holonomy representation is equivalent to the isotropy representation of $SL(3)/SO(3)$. So the codimension of M, in the sphere, is 5 and hence $N = 9$. The equivariance follows from Lemma 3.1. □

Lemma 4.3 (Caustic fibration lemma). *Let* $\hat{M}$ *be a compact immersed submanifold of* $\mathbb{R}^N$ *which is contained in the sphere* S^{N-1}. *Let* $\hat{\xi}$ *be a parallel normal field to* $\hat{M}$ *such that the eigenvalues of the shape operator* $A_{\hat{\xi}}$ *have constant multiplicities on* $\hat{M}$. *Let* $\hat{\lambda} : \hat{M} \to \mathbb{R}$ *be an eigenvalue function of* $A_{\hat{\xi}}$ *whose associated* (*integrable*) *eigendistribution* E *has* (*constant*) *dimension at least* 2. *Let* $\mathcal{E}$ *be the family of* (*maximal*) *integral manifolds of* E. *Assume that the eigenvalue function* $\hat{\lambda}$ *never vanishes* (this can always be assumed by adding to $\hat{\xi}$ an appropriate constant multiple of the umbilical position vector). *Then*

(i) *Any integral manifold* $Q \in \mathcal{E}$ *is compact.*
(ii) *The quotient space* $\bar{M} = \hat{M}/\mathcal{E}$ *is a* (*compact*) *manifold and the projection* $\pi : \hat{M} \to \bar{M}$ *is a fibration* (*in particular, a submersion*).
(iii) *The caustic map* $\rho : \hat{M} \to \mathbb{R}^N$, $\rho(q) = q + (\hat{\lambda}(q))^{-1}\hat{\xi}(q)$, *projects down to an immersion* $\bar{\rho} : \bar{M} \to \mathbb{R}^N$ (*i.e.* $\rho = \bar{\rho} \circ \pi$).

PROOF. From the Dupin condition, see Lemma 3.3, one has that $\hat{\lambda}$ is constant along any integral manifold Q of E.

Consider the caustic map $\rho(q) = q + (\lambda(q))^{-1}\hat{\xi}(q)$ (see the proof of Theorem 4.1, Case (2)(c)). Then $ker(\mathrm{d}\rho) = E$ and so $\mathrm{d}\rho$ has constant rank. From the local form of a map with constant rank and the compactness of $\hat{M}$ one has that there exists a finite open cover $V_1, \ldots, V_d$ of $\hat{M}$ such that, for any $i = 1, \ldots, d$ and $q, q' \in V_i$, the following equivalence holds:

$$\rho(q) = \rho(q') \iff q \text{ and } q' \text{ belong both to a same integral manifold of } E.$$

This implies that any (maximal) integral manifold Q of E must be a closed subset of $\hat{M}$ and hence compact. Moreover, the above equivalence implies that the foliation $\mathcal{E}$ is a regular foliation in the sense of Palais [**P**].

In order to prove that the quotient is a manifold we need to prove that this quotient is Hausdorff. But this can be done as follows: let $E^{\perp}$ be the distribution which is perpendicular, with respect to the metric, induced by the ambient space, on $\hat{M}$. Let us define a new Riemannian metric $\langle\ ,\ \rangle$ on $\hat{M}$ by changing the induced metric $(\ ,\ )$ on the distribution $E^{\perp}$ in such a way that ρ is locally a Riemannian submersion onto its image. Namely,

- $\langle E, E^{\perp}\rangle = 0$.
- $\langle\ ,\ \rangle$ coincides with $(\ ,\ )$ when restricted to E
- $\mathrm{d}_{|q}\rho$ is a linear isometry from $(E^{\perp})_q$ onto its image.

Such a metric is a bundle-like metric in the sense of Reinhart [**Re**]. Since $\hat{M}$ is compact, $\langle\ ,\ \rangle$ is a complete Riemannian metric. Then, [**Re**, Corollary 3, p. 129], the quotient space $\bar{M}$ is Hausdorff and π is a fibration (cf. [**DO**, Proposition 2.4, p. 83]).

Then one has that the map ρ projects down to an immersion $\bar{\rho} : \bar{M} \to \mathbb{R}^N$ and $\rho = \bar{\rho} \circ \pi$. □

LEMMA 4.4. *We keep the assumptions of Theorem* 4.1. *Moreover, we are in the assumptions and notation of Case* (2)(c)*, inside the proof of this theorem* (in particular, $H \simeq \mathrm{Spin}(3)$, up to a cover). *Then*:

(i) *All orbits of the action of H on $\bar{M}$ have dimension* 2.
(ii) *The universal cover $\tilde{M}$ of $\bar{M}$ splits off a line and hence the fundamental group of $\bar{M}$ is not finite* (*since $\bar{M}$ is compact*).

PROOF. The action of H on M_ξ projects down to $\bar{M}$, since $\hat{\xi}$ is H-invariant and so any eigendistribution of $\hat{A}_{\hat{\xi}}$ is H-invariant. Let $q \in M_\xi$. Then the 3-dimensional subspace $\mathfrak{h}.q \subset T_qM_\xi$ intersects the 3-dimensional horizontal subspace $\mathcal{H}_q$ in a non-trivial subspace, since $\dim(M_\xi) = 5$. Since we are in Case (2),

$$\{0\} \neq (\mathfrak{h}.q \cap \mathcal{H}_q) \subset E_1(q).$$

Let $H_{\bar{q}}$ be the isotropy group of H at the point $\bar{q} = \bar{\pi}(q) \in \bar{M}$. Let $\mathfrak{h}_{\bar{q}} = \mathrm{Lie}(H_{\bar{q}})$. Then one has that

$$\mathfrak{h}_{\bar{q}} = \{X \in \mathfrak{h} : X.q' \in E_1(q')\}$$

where it is independent of $q' \in S_1(q) = (\bar{\pi})^{-1}(\bar{\pi}(\{q\}))$, since a Killing field that is tangent to an integral manifold $S_1(q)$ of E_1, at some point, must be always tangent to it (since the action projects down to the quotient).

If $\dim(\mathfrak{h}_{\bar{q}}) = 3$. Then $\mathfrak{h}_{\bar{q}} = \mathfrak{h}$. Then H leaves invariant the 2-dimensional integral manifold $S_1(q)$ of E_1 by q. Then the isotropy H_q has positive dimension. But $H_q \subset H_{\pi(q)}$, where $H_{\pi(q)}$ is the isotropy group of H at the point $\pi(q) \in M^3 = H.p$. A contradiction, since $\dim(H) = 3$.

Observe that $\dim(\mathfrak{h}_{\bar{q}}) \neq 2$. In fact, if this dimension is 2, then $\mathfrak{h}_{\bar{q}}$ is an ideal of the 3-dimensional (compact type) Lie algebra $\mathfrak{h}$. A contradiction, since $\mathfrak{h}$ is simple. We have used that a Lie subalgebra of codimension 1 of a Lie algebra which admits a bi-invariant metric must be an ideal. (Also, this 2-dimensional Lie subalgebra should be abelian, in contradiction with $rank(\mathfrak{h}) = 1$).

Then $\dim(\mathfrak{h}_{\bar{q}}) = 1$ for all $q \in M_\xi$. This implies that all H-orbits in $\bar{M}$ have dimension 2. Since H acts with cohomogeneity 1 on $\bar{M}$ then, the universal cover of $\bar{M}$ cannot be compact. Otherwise, as it is well-known, there would exist a singular orbit (after lifting the action to the universal cover).

For the sake of self-completeness we will show the argument of this assertion.

We define an auxiliary Riemannian metric on $\bar{M}$, by changing, along the H-orbits, the metric $\langle\ ,\ \rangle$ induced by the immersion $\bar{\rho}$.

Since H acts with cohomogeneity 1 on $\bar{M}$, H acts locally polarly. In particular, the one dimensional distribution $\mathcal{D}$ on $\bar{M}$, perpendicular to the H-orbits, is an autoparallel distribution. If $\bar{q} \in \bar{M}$ then we put on the orbit $H.\bar{q}$ the normal homogeneous metric. That is, the metric associated to the reductive decomposition

$$\mathfrak{h} = \mathfrak{h}_{\bar{q}} \oplus (\mathfrak{h}_{\bar{q}})^{\perp}$$

where the orthogonal complement is taken with respect to a (fixed) bi-invariant metric on $\mathfrak{h}$.

We define $\langle\ ,\ \rangle'$ by:

a) $\langle\ ,\ \rangle'_{|\mathcal{D}} = \langle\ ,\ \rangle_{|\mathcal{D}}$.
b) $\langle \mathcal{U}, \mathcal{D} \rangle' = 0$, where $\mathcal{U}$ is the distribution given by the tangent spaces of the H-orbits on $\bar{M}$.
c) $\langle\ ,\ \rangle'_{|\mathcal{U}_{\bar{q}}}$ coincides with the normal homogeneous metric of $H.\bar{q}$, for any $\bar{q}$ on $\bar{M}$.

Since $\bar{M}$ is compact, the metric $\langle\ ,\ \rangle'$ is complete. Let $\langle\ ,\ \rangle'$ also denote the lift of the Riemannian metric $\langle\ ,\ \rangle'$ to the universal cover $\tilde{M}$ of $\bar{M}$. Then $(\tilde{M}, \langle\ ,\ \rangle')$ is a complete Riemannian manifold. Let us denote by $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{D}}$ the lifts to $\tilde{M}$ of the distributions $\mathcal{U}$ and $\mathcal{D}$, respectively. Let us also lift the H-action on $\bar{M}$ to $\tilde{M}$. Then, since $\tilde{M}$ is simply connected, the one dimension distribution $\tilde{\mathfrak{D}}$ is parallelizable. Namely, there exists a nowhere vanishing vector field $\tilde{X}$ of $\tilde{M}$ such that $\mathbb{R}.\tilde{X} = \tilde{\mathfrak{D}}$. Let $\tilde{Z} = (1/\|\tilde{X}\|)\tilde{X}$, where the norm is with the metric $\langle\ ,\ \rangle'$. Then, the flow ϕ_t, associated to $\tilde{Z}$, is by isometries. So $\tilde{Z}$ is a Killing field. Then $\langle \nabla_{.}\tilde{Z}, . \rangle'$ is skew-symmetric. So, in particular, $\langle \nabla_v \tilde{Z}, v \rangle' = 0$, for any vector v that lies in $\tilde{\mathcal{U}}$. But, if $\tilde{A}_{\tilde{Z}}$ is the shape operator of the orbit $H.x$,

$x \in \tilde{M}$, then $\langle \tilde{A}_{\tilde{Z}} v, v\rangle' = \langle \nabla_v \tilde{Z}, v\rangle' = 0$. Then $\tilde{\mathcal{U}} = \tilde{\mathcal{D}}^{\perp}$ is an autoparallel distribution. The distribution $\tilde{\mathcal{D}}$ is also autoparallel, since the Killing fields induced by H are always perpendicular to it. But two complementary perpendicular autoparallel distribution must be parallel. Then, by the de Rham decomposition theorem, $\tilde{M}$ is a Riemannian product. Since one of the parallel distributions is one dimensional then $\tilde{M} = \mathbb{R} \times M'$. □

REMARK 4.5. In this paper, for dealing with homogeneous submanifolds of dimension 3, we need to know which are the compact Lie groups G of dimension at most 4. For the sake of self-completeness we will briefly show, without using classification results, which are these compact Lie groups G (up to covering spaces).

We will use the following fact that it is well-known and standard to show: *a codimension* 1 *subgroup, of a Lie group with a bi-invariant metric, must be a normal subgroup.*

(i) $\dim(G) \leq 2$.
In this case, from the above fact, one has that G must be abelian.

(ii) $\dim(G) = 3$.
If $\operatorname{rank}(G) \geq 2$ then, from the above fact, G must be abelian. If $\operatorname{rank}(G) = 1$, then G is, up to a cover, Spin(3). This well-known result follows from a topological argument that proves that a rank 1 simply connected compact group is isomorphic to Spin(3) (a proof can be found in Remark 2.6 of [**OR**]).

(iii) $\dim(G) = 4$.
If G is neither simple nor abelian, then, from the previous cases, we have that a finite cover of G splits as $S^1 \times \mathrm{Spin}(3)$.
If G is simple then $\operatorname{rank}(G) \leq 2$. Otherwise G would have a codimension 1 (abelian) subgroup (which must be normal).
If G is simple, then $\operatorname{rank}(G) > 1$. Otherwise, $G = \mathrm{Spin}(3)$ which has dimension 3.
Let G be simple and $\operatorname{rank}(G) = 2$. Then the Ad-representation of G on $\mathfrak{g} = \mathrm{Lie}(G)$ must have a focal (non-trivial) orbit $G.v$. Such an orbit must have codimension 3. The 3-dimensional normal space $\nu_v(G.v)$ is Lie triple system, since it coincides with the commutator of v. Then $\nu_v(G.v)$ is an ideal of $\mathfrak{g}$. A contradiction.

REMARK 4.6. Let $X = G/K$ be an irreducible simply connected symmetric space of the non-compact type and rank at least 2, where G is the connected component of the full isometry group of X. Assume that the dimension of X is at most 5. Then, $X \simeq SL(3)/SO(3)$.

We will next outline a classification free proof of this fact.

Observe, since $\operatorname{rank}(X) \geq 2$, that the isotropy representation of K on T_pX has a non-trivial focal orbit $M = K.v$ ($p = [e]$). Such an orbit M must have dimension 2. In fact, M cannot have dimension 3. Otherwise, a principal K-orbit must have dimension 4 and so K would act transitively on the sphere. Observe also that the dimension of M cannot be 1. In fact, since K acts irreducibly on T_pX, then K acts effectively on any non-trivial orbit. If $\dim(M) = 1$, then $\dim(K) = 1$. Then, since $\dim(X) > 2$, K does not act irreducibly on T_pX. A contradiction.

Observe that the isotropy K_v of the focal orbit $M^2 = K.v$ at v must have positive dimension. Moreover, since M is not a principal orbit, the image under the slice representation of K_v is not trivial. So, by Corollary 2.5, the restricted normal holonomy

group $\Phi(v)$ of M at v is not trivial. Then $\Phi(v)$ must act irreducibly on the 2-dimensional space $\bar{\nu}_v(M) = \{v\}^\perp \cap \nu_v(M)$. Observe that the codimension of M^2 is $3 = 2(2+1)/2$. Then, by Proposition 2.17, M is a Veronese submanifold, i.e. orthogonally equivalent to a Veronese-type orbit V^2 of $SO(3)$ on $Sim_0(3)$ (the action is by conjugation). So, one may assume that $Sim_0(3) = T_pX$ and that $M = V^2$. Then both K and $SO(3)$ are Lie subgroups of $\tilde{K} = \{g \in SO(Sim_0(3)) : g.M = M\}$. Observe that $\tilde{K}$ is not transitive on the unit sphere of $Sim_0(3)$ since the codimension of M is 3. Let R' and R be the curvature tensors at $p = [e]$ of X and $SL(3)/SO(3)$. Then we have the following irreducible non-transitive holonomy systems: $[Sim_0(3), R, \tilde{K}]$ and $[Sim_0(3), R', \tilde{K}]$.

Then by the holonomy theorem of Simons 2.12, R is unique up to scalar multiple and $\tilde{K} = K = SO(3)$, since its Lie algebra is spaned by R. This implies that the symmetric space X is homothetical to $SL(3)/SO(3)$.

Remark 4.7. Let $M^3 = K.v \subset \mathbb{R}^N$ be a 3-dimensional full and irreducible homogeneous (Euclidean) submanifold. Assume that $\operatorname{rank}(M) \geq 2$. In this case, by the rank rigidity theorem, M is an orbit of an s-representation. So, we may assume, that K acts as an s-representation.

Let ξ be a K-invariant parallel normal field to M which is not umbilical. If the shape operator A_ξ has two different (constant) eigenvalues then its associated eigendistributions, let us say E_1 and E_2 are autoparallel distributions that are invariant under the shape operators of M (recall that A_ξ commutes with any other shape operator due to Ricci equality). Then, by the so-called Moore's lemma [**BCO**, Lemma 2.7.1], M is product of submanifolds. A contradiction.

If A_ξ has three eigenvalues, then the multiplicities of any of them are 1. Since A_ξ commutes with any other shape operator, all shape operators of M must commute. Then, by the Ricci identity, M has flat normal bundle. Then M is isoparametric, since it is an orbit of an s-representation.

Therefore, a full irreducible and homogeneous Euclidean 3-dimensional submanifold M^3, of higher rank, must be isoparametric with exactly three curvature normals. This implies that the irreducible Coxeter group associated to M [**Te**], [**PT**] has exactly three reflection hyperplanes. This is only possible if the dimension of the normal space is 2. Otherwise, the curvature normals must be mutually perpendicular and hence M would be a product of circles.

This implies that $N = 5$ and that M is an isoparametric hypersurface of the sphere S^4. Moreover, from Remark 4.6, M is a principal orbit of the isotropy representation of $SL(3)/SO(3)$.

Proof of Theorem A. If M^n is a (full) Veronese submanifold, $n \geq 3$, then the normal holonomy, as a submanifold of the sphere, acts irreducibly and non-transitively (see Facts 2.16, (iii)).

For the converse observe that M must be a full and irreducible Euclidean submanifold, since the normal holonomy group (as a submanifold of the sphere) acts irreducibly (see the beginning of Section 3). Then, from Theorem 3.4, Theorem 4.1 and Proposition 2.17, M is a Veronese submanifold. □

Proof of Theorem B. From Theorem 4.1 M is an orbit of an s-representation.

Assume that $\text{rank}(M) = 1$. Then, by Lemma 4.2, the (restricted) normal holonomy group of M, as a submanifold of the sphere, acts irreducibly and $N = 9 = 3+3(3+1)/2$. Then, by Proposition 2.17, M is a Veronese submanifold.

If M is of higher rank, then, by Remark 4.7, M is a principal orbit of the isotropy representation of $SL(3)/SO(3)$. □

5. Minimal submanifolds with non-transitive normal holonomy.

In this section we prove Theorem C of the Introduction.

We use many of the ideas used for the homogeneous case, when $n > 3$. But now the situation is much more simple, for $n = 3$.

Proof of Theorem C. Observe that M must be full and irreducible as a Euclidean submanifold (since the normal holonomy group, as a submanifold of the sphere, acts irreducibly; see Section 3). Note, by the minimality, that the traceless shape operators coincide with the shape operators (of vectors which are perpendicular to the position vector).

We keep the notation in the proof of Theorem 3.4.

Let $p \in M$ be such that the adapted normal curvature tensor $\mathcal{R}^\perp(p) \neq 0$, or equivalently, $R^\perp(p) \neq 0$. Let us consider the irreducible and non-transitive holonomy systems $[\bar\nu_p(M), \mathcal{R}^\perp(p), \Phi(p)]$ and $[Sim_0(T_pM), R, SO(T_pM)]$.

We have, from formula $(****)$ of Section 3 and Proposition 2.21, that the shape operator at p, $A^p : \bar\nu_p(M) \to Sim_0(T_pM)$ is a homothecy and $A^p\Phi(p)(A^p)^{-1} = SO(T_pM)$. This implies, if $\phi \in \Phi(p)$, that the eigenvalues of A^p_η coincide with the eigenvalues of $A^p_{\phi(\eta)}$.

Let U be a contractible neighbourhood of p in M such that $\mathcal{R}^\perp$ never vanishes on U. Let now $p' \in U$ be arbitrary and let $\gamma : [0,1] \to U$ be a piece-wise differentiable curve from p to p'. Let τ_t be the $\nabla^\perp$-parallel transport along $\gamma_{[0,t]}$. We have that $\tau_t\Phi(p)(\tau_t)^{-1} = \Phi(\gamma(t))$.

Let us choose $\xi \in \bar\nu_p(M)$ such that $A^p_\xi \in Sim_0(T_pM)$ has exactly two eigenvalues $\lambda_1 = 1/2$ of multiplicity 2 and $\lambda_2 = -1/(n-2)$ of multiplicity $(n-2)$.

Recall that the shape operator $A^{\gamma(t)} : \bar\nu_{\gamma(t)} \to Sim_0(T_{\gamma(t)})$ maps $\Phi(\gamma(t))$ into $SO(T_{\gamma(t)})$. Then, the homothecy $g_t := A^{\gamma(t)} \circ \tau_t \circ (A^p)^{-1} : Sim_0(T_pM) \to Sim_0(T_{\gamma(t)}M)$ maps the group $SO(T_pM)$ into $SO(T_{\gamma(t)}M)$. Then g_t maps the isotropy subgroup $SO(T_pM)_{A^p_\xi} \simeq S(O(2) \times O(n-2))$ into the isotropy subgroup $SO(T_{\gamma(t)}M)_{B(t)}$, where $B(t) = A^{\gamma(t)}_{\tau_t(\xi)}$. This implies, as it is not difficult to see, that $B(t)$ has two eigenvalues, let us say λ^t_1 of multiplicity 2 and λ^t_2 of multiplicity $n-2$. Since $B(t) \in Sim_0(T_{\gamma(t)})$, $\lambda^t_2 = -(2/(n-2))\lambda^t_1$.

Then the two eigenvalues of $B(t)$ are constant up to the multiplication by $a(t) = \lambda^t_1 \neq 0$. Note, if γ is a loop by p, that $\tau_1 \in \Phi(p)$. Then, as we have previously observed, the eigenvalues of A^p_ξ are the same as those of $B(1)$. Then $a(t)$ depends only on $\gamma(t)$. So there is a non-vanishing $f : U \to \mathbb{R}$ such that $a(t) = f(\gamma(t))$. It is standard to show that f must be C^∞. Note that $f(p) = 1/2$.

Let us consider (eventually, by making U smaller) the holonomy tube U_ξ. We use the notation in the proof to Theorem 3.4. We will modify the arguments in this proof.

We have the parallel normal field $\hat{\xi}$ of U_ξ. The eigenvalues of the shape operator $\hat{A}_{\hat{\xi}}$ at $q \in U_\xi$ are given by

$$\hat{\lambda}_1(q) = \frac{f(\pi(q))}{1 - f(\pi(q))}$$

associated to the (horizontal) eigendistribution E_1 of dimension 2,

$$\hat{\lambda}_2(q) = \frac{-f(\pi(q))/(n-2)}{1 + (f(\pi(q))/(n-2))}$$

associated to the (horizontal) eigendistribution E_2 of dimension $n-2$.

The third eigenvalue of $\hat{A}_{\hat{\xi}}$, is $\hat{\lambda}_3 = -1$, associated to the vertical distribution ν, tangent to the normal holonomy orbits.

By the Dupin condition, $\mathrm{d}(\hat{\lambda}_1)(E_1) = 0$ which implies that

$$\mathrm{d}(f \circ \pi)(E_1) = 0. \tag{J}$$

If $n > 3$ this is also true for the eigendistribution E_2, since it has dimension at least 2. But we will not assume this and the proof will also work for $n = 3$.

From the tube formula, as we have observed in the proof of Theorem 4.1, Case (2)(b), $\mathrm{d}\pi(E_1(q)) = E_1(q)$, as linear subspaces. Moreover, $E_1(q)$ is an eigenspace of $A_{q-\pi(q)} = A_{\hat{\xi}(q)}$, where A is the shape operator of M (we drop the supra-index $\pi(q)$ of A). Let now $q \in U_\xi$ with $\pi(q) = p$ and let $\mathbb{V}$ be the subspace of T_pM which is generated by $E_1(q')$, with $q' \in \Phi(p).q = (\pi^{-1}(\{p\}))_q$. If $\mathbb{V} = T_pM$, then, from formula (J), $\mathrm{d}f(T_pM) = \{0\}$. If $\mathbb{V}$ is properly contained in T_pM, then let $0 \neq v \in \mathbb{V}^\perp$. We will derive, in this case, a contradiction. In fact, since any shape operator $A_{q'-p}$ has only two eigenvalues and v is perpendicular to the eigenspace $E_1(q')$ of $A_{q'-p}$, then v is an eigenvector of this shape operator, for any $q' \in \Phi(p).q$. Observe that the linear span of $\Phi(p).q$ is $\bar{\nu}_p(M)$, since $q' \neq 0$ and $\Phi(p)$ acts irreducibly on this normal space. Then v is a common eigenvector for all shape operators A_η, $\eta \in \nu_p(M)$. But $A : \bar{\nu}_p(M) \to Sim_0(T_pM)$ is an isomorphism. This is a contradiction. Then $\mathrm{d}f(T_pM) = \{0\}$ and the same is valid for all $p' \in U$. Then $f = f(p) = 1/2$ is constant on U.

Then the eigenvalues $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3$ are constant on U_ξ. Then $\hat{\xi}$ is a (non-umbilical) parallel normal isoparametric field of U_ξ. Then, by [**CO**] (see [**BCO**, Theorem 5.5.2]), U_ξ and hence U has constant principal curvatures. But this is true provided one shows that U_ξ is full and locally irreducible around some point $q \in \pi^{-1}(\{p\})$.

Let us show that the local normal holonomy group of M at p coincides with the restricted normal holonomy group. In fact, the holonomy system $[\bar{\nu}_p(M), \mathcal{R}^\perp(p), \Phi(p)]$ is irreducible and non-transitive. Then, by the holonomy theorem of Simons [**S**], it is symmetric. Moreover, $\mathrm{Lie}(\Phi(p))$ is linearly generated by the endomorphisms $\{\mathcal{R}^\perp_{\xi,\eta}(p)\}$. This implies that the local normal holonomy at p coincides with $\Phi(p)$. Then the local rank of M, as submanifold of the Euclidean space, is 1. This implies that M is full and locally irreducible around p. Hence U_ξ is full and irreducible around any point $q \in \pi^{-1}(\{p\})$. Then U is a submanifold with constant principal curvatures.

Since the normal holonomy of M is not transitive on the unit sphere, of the normal space to the sphere, any principal holonomy tube (which is isoparametric) has codimension at least 3 in the Euclidean space. Then, by the theorem of Thorbergsson [**Th**], U is locally an orbit of an s-representation. Then $\|\mathcal{R}^{\perp}\|$ is constant on U. From this one obtains that $\|\mathcal{R}^{\perp}\|$ is constant on Ω, where Ω is a connected component of the open subset $\{p \in M : \mathcal{R}^{\perp}(p) \neq 0\}$. But if $p' \in M$ is a limit point of Ω then, $\mathcal{R}^{\perp}(p') \neq 0$. This implies that Ω can be enlarged unless $p' \in \Omega$. This shows that the open subset Ω is also closed in M. Then M has constant principal curvatures. Hence, the image of M (under the isometric immersion), is an embedded submanifold with constant principal curvatures. Moreover, it is an orbit of an s-representation. From Proposition 2.17, the image of M is a Veronese submanifold.

The converse is true by Facts 2.16, (i) and (iii). □

Remark 5.1. We keep the notation of the proof of Theorem C. The fact that f is constant can also be proved in the following way. Let $p \in M$ be such that $\mathcal{R}^{\perp}(p) \neq 0$. Then, since the shape operator A maps $\Phi(p)$-orbits into $SO(T_pM)$-orbits of $Sim_0(T_pM)$, one obtains that the second fundamental form is λ-isotropic. That is, the length of $\alpha(X, X)$ is $\lambda(p)$ independent of X in the unit sphere of T_pM, where α is the second fundamental form. The function λ must be a constant multiple of f. Then, by Proposition 4.1. of [**IO**], λ, and hence f, must be constant ($n \geq 3$).

6. The number of factors of the normal holonomy.

In this section we will prove a sharp linear bound, depending on the dimension n of the submanifold, of the number of irreducible factors of the local normal holonomy representations. This improves, substantially, the quadratic bound $n(n-1)/2$ given in Theorem 4.5.1 of [**BCO**].

Proposition 6.1. *Let M^n be a submanifold of the Euclidean space $\mathbb{R}^N$. Assume that at any point of M the local normal holonomy group and the restricted normal holonomy group coincide (or, equivalently, the dimensions of the local normal holonomy groups are constant on M). Let $p \in M$ and let r be the number of irreducible (non-abelian) subspaces of the representation of the restricted normal holonomy group $\Phi(p)$ on $\nu_p(M)$. Then $r \leq n/2$. Moreover, this bound is sharp for all $n \in \mathbb{N}$ (also in the class of irreducible submanifolds).*

Proof. Let us decompose $\nu_p(M) = \nu_p^0(M) \oplus \nu_p^1(M) \oplus \cdots \oplus \nu_p^r(M)$, where $\Phi(p)$ acts trivially on $\nu_p^0(M)$ and irreducibly on $\nu_p^i(M)$, for $i = 1, \ldots, r$. From the assumptions we obtain that $\nu_p^i(M)$ extends to a $\nabla^{\perp}$-parallel subbundle ν^i of the normal bundle $\nu(M)$, $i = 0, \ldots, r$ (eventually, by making M smaller around p). Note that we have the decomposition $\nu(M) = \nu^0(M) \oplus \nu^1(M) \oplus \cdots \oplus \nu^r(M)$. Moreover, we obtain from the assumptions, for any $q \in M$, that the local normal holonomy group $\Phi(q)$ acts trivially on $\nu_q^0(M)$ and irreducibly on $\nu_q^i(M)$, for any $i = 1, \ldots, r$.

Let $\mathcal{R}^{\perp}_{\xi,\xi'}$ be the adapted normal curvature tensor (see Section 2.1). From the expression of $\mathcal{R}^{\perp}$ in terms of shape operator A, one has that $\mathcal{R}^{\perp}_{\xi,\xi'} = 0$ if and only if $[A_\xi, A_{\xi'}] = 0$.

Observe, if $i \neq j$, that $\mathcal{R}^{\perp}_{\xi_i,\xi'_j} = 0$ if ξ_i, ξ'_j are normal sections that lie in $\nu^i(M)$ and $\nu^j(M)$, respectively.

There must exist $q \in M$, arbitrary close to p, such that $\mathcal{R}^{\perp}_{\nu^i_q,\nu^i_q} \neq \{0\}$, for all $i = 1, \ldots, r$. In fact, there exists $q_1 \in M$, arbitrary close to p such that $\mathcal{R}^{\perp}_{\nu^1_{q_1},\nu^1_{q_1}} \neq \{0\}$ (otherwise, $\nu^1(M)$ would be flat). The above inequality must be true in a neighbourhood V_1 of q_1. Now choose $q_2 \in V_1$ such that $\mathcal{R}^{\perp}_{\nu^2_{q_2},\nu^2_{q_2}} \neq \{0\}$. Continuing with this procedure we find $q := q_r$ with the desired properties.

Let us show that for any $i = 1, \ldots, r$ there exist ξ_i, ξ'_i in $\nu^i_q(M)$ such that $[A_{\xi_i}, A_{\xi'_i}]$ does not belong to the algebra of endomorphisms generated by $\{A_{\eta^i}\}$, where $\eta^i \in \nu_q(M)$ has no component in $\nu^i_q(M)$. In fact, if this is not true, then, for any ξ_i, ξ'_i in $\nu^i_q(M)$, $[A_{\xi_i}, A_{\xi'_i}]$ commutes with A_{ξ_i} (since the shape operators of elements of the subspaces $\nu^j_q(M)$ commute with A_{ξ_i}, if $j \neq i$). Then

$$\langle [[A_{\xi_i}, A_{\xi'_i}], A_{\xi_i}], A_{\xi'_i} \rangle = 0 = -\langle [A_{\xi_i}, A_{\xi'_i}], [A_{\xi_i}, A_{\xi'_i}] \rangle$$

and hence $[A_{\xi_i}, A_{\xi'_i}] = 0$. A contradiction, since $\mathcal{R}^{\perp}_{\nu^i_q,\nu^i_q} \neq \{0\}$. This proves our assertion.

Observe that $[A_{\xi_1}, A_{\xi'_1}], \ldots, [A_{\xi_r}, A_{\xi'_r}]$ are linearly independent and commuting skew-symmetric endomorphisms of T_qM. Then $r \leq rank(SO(T_pM)) = [n/2]$ (the integer part of $n/2$). This proves the inequality.

Let us see that it is sharp. For $M^2 \subset S^{k_1-1}, \bar{M}^3 \subset S^{k_2-1}$ be a surface and a 3-dimensional submanifold such that the normal holonomies have one irreducible factor (for example, the Veronese V^2 and V^3). Let $n > 3$ and write $n = 2d$ if n is even or $n = 2d + 3$ if n is odd.

Let M^n be the product of d times M^2 or M^n be the product of d times M^2 by $\bar{M}^3$. Such submanifolds are contained in the product of Euclidean ambient spaces. Moreover, the number of irreducible factors of the normal holonomy group (representation) of M^n is exactly the upper bound $[n/2]$. Moreover, since M^n is contained in a sphere, we can apply to M^n a conformal transformation of the sphere (the normal holonomy group is a conformal invariant) in such a way that M^n is an irreducible (Riemannian) submanifold of the Euclidean space. □

7. Further comments.

REMARK 7.1. There is a beautiful result of Little and Pohl [**LP**] which characterizes Veronese submanifolds M^n, modulo projective diffeomorphisms, by the *two-piece property* and the fact that the codimension is the maximal one $n(n+1)/2$ (for submanifolds with the two-piece property). Note that a tight submanifold has the two-piece property. This result generalizes the well-known result of Kuiper for $n = 2$. A projective transformation, in general, does not preserve the normal holonomy (unless it induces a conformal transformation of the ambient sphere).

REMARK 7.2. A natural question that arises, since the normal holonomy group is a conformal invariant, is the following: *is a compact submanifold $M^n \subset S^{n-1+n(n+1)/2}$, with irreducible and non-transitive* (*restricted*) *normal holonomy, equivalent, modulo con-*

formal transformations of the sphere, to a Veronese submanifold?

REMARK 7.3. The symmetric space $X = SU(4)/SO(4)$, dual to $SL(4)/SO(4)$, is isometric to the Grassmannian $SO(6)/SO(3) \times SO(3)$. In this last model, $T_{[e]}X = \mathbb{R}^{3\times 3}$ and the isotropy representation is given by $(g,h).T = gTh^{-1}$, $(g,h) \in SO(3) \times SO(3)$. The Veronese submanifold V^3 is given by

$$SO(3) \times SO(3).Id = SO(3) \times \{Id\}.Id = SO(3) \subset \mathbb{R}^{3\times 3}.$$

Thus V^3 is also an orbit of the smaller group $SO(3) \simeq SO(3) \times \{Id\}$. The other orbits $SO(3).A$, where A is invertible and near Id, must be full and irreducible submanifolds of $\mathbb{R}^{3\times 3}$, since V^3 is so. Note that the action of $SO(3)$ on $\mathbb{R}^{3\times 3}$ is reducible. In fact, it is the sum of three times the standard representation of $SO(3)$ on $\mathbb{R}^3$. The orbit, $SO(3).A$ is not minimal in the sphere, for a generic A. So, the normal holonomy group of this orbit must be transitive on the unit sphere (of the normal space to the sphere).

Observe that the linear isomorphism $r_{A^{-1}}$ of $\mathbb{R}^{3\times 3}$, $r_{A^{-1}}(T) = TA^{-1}$, transforms $SO(3).A$ into V^3. In particular, since V^3 is a tight submanifold, that orbit is so. Hence, as it is well known, $SO(3).A$ is a taut submanifold, since it lies in a sphere (see [**CR**], [**G**]).

8. Appendix.

8.1. The Veronese embedding.

We recall here some basic definitions and facts about the well-known Veronese submanifolds.

Let S^n, $n \geq 2$, be the unit sphere of the Euclidean space $\mathbb{R}^{n+1}$ and let $\mathbb{R}^{n+1}\otimes_s\mathbb{R}^{n+1}$ be the space of symmetric 2-tensors of $\mathbb{R}^{n+1}$. Let $h : \mathbb{R}^{n+1}\otimes_s\mathbb{R}^{n+1} \to Sim(n+1)$ the usual isomorphism onto the symmetric matrices of $\mathbb{R}^{n+1}$. Namely, let $e_1, \ldots, e_{n+1}$ be the canonical basis of $\mathbb{R}^{n+1}$. Then, $h(e_i \otimes e_j + e_j \otimes e_i)$ is the matrix whose coefficients $a_{k,l}$ are all zero except:

$$a_{i,j} = a_{j,i} = 1, \text{ if } i \neq j; \quad a_{i,i} = 2, \text{ if } i = j.$$

The Veronese map $Q : S^n \to Sim(n+1)$ is defined by

$$\mathrm{Q}(v) = h(v \otimes v).$$

Observe that $(\mathrm{Q}(v))_{i,j} = v_i v_j$, where $v = (v_1, \ldots, v_{n+1})$. Let $\langle\,,\,\rangle$ be the inner product on $Sim(n+1)$ given by $\langle A, B\rangle = (1/2)trace(AB)$. Then Q is an isometric immersion. Observe that $trace(\mathrm{Q}(v)) = 1$, for all $v \in S^n$. So, the image of Q is contained in the affine hyperplane of $Sim(n+1)$, given by the linear equation

$$\langle\, \cdot\, , Id\rangle = \frac{1}{2}.$$

Let $\tilde{\rho} : S^n \to Sim_0(n+1)$ be defined by $\tilde{\rho}(v) = Q(v) - (1/(n+1))Id$, where

$Sim_0(n+1)$ are the symmetric traceless matrices. The map $\tilde{\rho}$ is called the *Veronese Riemannian immersion* of the sphere S^n into $Sim_0(n+1)$. One has that $\tilde{\rho}$, (as well as Q) is $O(n+1)$-equivariant. Namely, if $g \in O(n+1)$, then

$$\tilde{\rho}(g.v) = g.\tilde{\rho}(v).g^{-1}.$$

In fact, if we regard $v \in \mathbb{R}^{n+1}$ as a column vector, then

$$\tilde{\rho}(v) = v.v^t - \frac{1}{n+1}Id.$$

From the above formula it follows easily $O(n+1)$-equivariance of $\tilde{\rho}$. It is also not difficult to verify, as it is well known, that $\tilde{\rho}(v) = \tilde{\rho}(w)$ if and only if $w = \pm v$. Therefore, $\tilde{\rho}$ projects down to an isometric $O(n+1)$-equivariant embedding $\rho : \mathbb{R}P^n \to Sim_0(n+1)$, the so-called *Veronese Riemannian embedding*.

Let us consider the simple symmetric pair $(SL(n+1), SO(n+1))$ of the non-compact type. The Cartan decomposition associated to such a pair is

$$\mathfrak{sl}(n+1) = \mathfrak{so}(n+1) \oplus Sim_0(n+1).$$

Then the (irreducible) isotropy representation of $X = SL(n+1)/SO(n+1)$ is naturally identified with the action, by conjugation, of $SO(n+1)$ on $Sim_0(n+1)$. Then, the image of the Veronese embedding, is the orbit

$$M = SO(n+1).S$$

where $S \in Sim_0(n+1)$ is the diagonal matrix with exactly two eigenvalues. Namely, $1 - 1/(n+1)$ and $-1/(n+1)$. The first one, with multiplicity 1, is associated to the eigenspace $\mathbb{R}e_1$ and the second one, with multiplicity n, is associated to the eigenspace $(\mathbb{R}e_1)^{\perp}$.

Let $S' \in Sim_0(n+1)$ with exactly two eigenvalues λ_1 of multiplicity 1 and λ_2 with multiplicity n. Assume that $\|S'\| = \|S\|$ (i.e. S and S' have the same length). It is easy to verify that either $\lambda_1 = 1 - 1/(n+1)$, $\lambda_2 = -1/(n+1)$ or $\lambda_1 = -1 + 1/(n+1)$, $\lambda_2 = 1/(n+1)$. In the first case one has that $S' \in SO(n+1).S = \rho(\mathbb{R}P^n)$. In the second case, $-S' \in SO(n+1).S$.

Observe that S' and $-S'$ cannot be both in the image of the Veronese embedding, since the respective eigenvalues of multiplicity 1 are different. In general, if $\bar{S} \in Sim_0(n+1)$ has two different eigenvalues, one of multiplicity 1 and the other of multiplicity n, then $\bar{S} = \lambda S$, for some $0 \neq \lambda \in \mathbb{R}$. The orbit $SO(n+1).\bar{S}$ is called a *Veronese-type orbit* (see Section 1.1). Observe that there are exactly two Veronese-type orbits in any given sphere, centered at 0, of $Sim_0(n+1)$. Moreover, any of these two Veronese-type orbits is isometric to the other, via the isometry $-Id_{Sim_0(n+1)}$ of $Sim_0(n+1)$.

We have the following well-known fact.

LEMMA 8.1. *Let $SO(r)$ acts by conjugation on $Sim_0(r)$, the traceless symmetric $r \times r$-matrices, and let $M = SO(r).A$ be an orbit, $A \neq 0$. Then $r - 1 \leq \dim(M)$.*

Moreover, the equality holds if and only if M is an orbit of Veronese-type.

PROOF. Let us assume that M has minimal dimension. We will first prove that A has exactly two eigenvalues. If not, let $\lambda_1, \dots, \lambda_d$ be the different eigenvalues of A with associated eigenspaces $E_1, \dots, E_d$ ($d \geq 3$). Then the isotropy subgroup $SO(r)_A = S(SO(E_1) \times \cdots \times SO(E_d))$ has less dimension than $S(SO(E_1) \times SO(E_2 \oplus \cdots \oplus E_d))$, which is the isotropy group of some $A' \in Sim_0(r)$ with two different eigenvalues whose associated eigenspaces are E_1 and $E_2 \oplus \cdots \oplus E_d$. Then $\dim(M) > \dim(SO(r).A')$. A contradiction. Therefore, $d = 2$. (Observe that $d = 1$ implies that $A = 0$, since it is traceless).

Let now $k = \dim(E_1)$ and so $r - k = \dim(E_2)$.

We have the well-known formula for the dimension of the Grassmannians,

$$\dim(M) = \dim(SO(r)) - \dim(SO(k)) - \dim(SO(r-k)) = k(r-k).$$

But the quadratic $q(x) = x(r-x)$, $x \in [0, r]$, is increasing in the interval $[0, r/2)$ and it is decreasing in $(r/2, r]$. So, the minimum of q, restricted to the finite set $\{1, \dots, r-1\}$ is attained at both, $x = 1$ and $x = r - 1$. Then $k = 1$ or $k = r - 1$, in which case M is a Veronese-type orbit (of dimension $r - 1$). □

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