PHYSICAL REVIEW D **92,** 024017 (2015)

Strong cosmic censorship and Misner spacetime

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(Received 7 May 2015; published 9 July 2015)

Misner spacetime is among the simplest solutions of Einstein's equation that exhibits a Cauchy horizon with a smooth extension beyond it. Besides violating strong cosmic censorship, this extension contains closed timelike curves. We analyze the stability of the Cauchy horizon and prove that neighboring spacetimes in one parameter families of solutions through Misner's in pure gravity, gravity coupled to a scalar field, or Einstein-Maxwell theory end at the Cauchy horizon developing a curvature singularity.

DOI: 10.1103/PhysRevD.92.024017 PACS numbers: 04.70.Bw, 04.30.-w

I. INTRODUCTION

The possibility of smoothly extending a solution of Einstein's equations beyond the maximal Cauchy development of compact or asymptotically simple data is an undesirable feature of general relativity (GR). From a $3+1$ viewpoint, the evolution of the three-metric ceases to be nonunique at the Cauchy horizon, predictability being lost in a classical theory. Strong cosmic censorship (SCC) is the conjecture that generic solutions of GR cannot be extended beyond a Cauchy horizon. Notably, these pathologies occur among the most important solutions of GR: all double horizon black holes in the Kerr Newman family, those where either charge or angular momentum, or both, is nonzero. In these black holes the inner horizon is a Cauchy horizon for any Cauchy surface connecting both copies of spatial infinity i_o, and the standard analytic extension beyond it is unique only if we enforce the nonphysical requirement of analyticity. In the rotating case, moreover, causality is completely lost in the analytic extension since it is possible to connect any two given events in this region with a future directed timelike curve [1]; in particular, there are closed timelike curves through any point. A simple argument first given by Penrose in [2] (see also [3]) suggests that any perturbation of these solutions will actually end at the Cauchy horizon with a curvature singularity. The instability of the Cauchy horizon was illustrated for the Reissner Nordström spacetime in [4], using a model with a cross flow of outgoing and ingoing lightlike fluxes. An instability of transverse derivatives of test scalar fields along the Cauchy horizon of an extremal Reissner-Nordström black hole was recently found in [5,6]; the analogous result for extremal Kerr black holes is given in [7].

Misner spacetime is obtained from the half $x^0 < x^1$ of Minkowski space by identifying points connected by a fixed boost. In spite of its simplicity, the resulting spacetime has a rich structure that includes a Cauchy horizon with closed timelike curves beyond it. Since it is a flat spacetime, it is possible to obtain explicit solutions for scalar and Maxwell test fields and use these results in perturbation theory to order higher than one for the coupled scalar-gravity and Maxwell-gravity systems. We use these results, as well as perturbations in pure gravity, to show that Misner spacetime is an isolated solution in any of these theories. More precisely, we prove that given a one parameter family of solutions through Misner's in any of these theories, neighboring solutions develop a curvature singularity that truncates the spacetime at the Cauchy horizon except for fine-tuned cases.

We review the construction of Misner space in Sec. II where we also analyze in detail the null geodesics, as they provide insight into the evolution of massless fields. In Sec. III we prove that a zero scalar field on a Misner background is a nongeneric solution within the Einstein-scalar field theory: except for fine-tuned cases, perturbations of this solution within Einstein-scalar field theory develop a curvature horizon that truncates the spacetime at the Cauchy horizon. The analogous result is proven for Einstein-Maxwell theory in Sec. IV, and then for pure gravity in Sec. V.

For simplicity, we have performed calculations in compactified Misner space $\mathcal{M}_2 \times \mathbb{T}^2$, where $\mathcal{M}_2$ is two-dimensional Misner space, and $\mathbb{T}^2$ a 2-torus. We can recover the noncompact case by taking the limit $a, b \to \infty$ of the periods a and b of the spatial coordinates y and z. This amounts to a few changes in the test field expressions, such as replacing Fourier series in (y, z) with Fourier transforms.

II. MISNER SPACETIME

This section is a review of Misner spacetime. It serves the double purpose of introducing its key features and setting the notation we use in the following sections. Most of the material presented in this section can be found elsewhere (see, e.g., Ref. [8]).

Consider the half-space $\tilde{\mathcal{M}}_2$ of two-dimensional Minkowski spacetime,

$$ds^2 = -(dx^0)^2 + (dx^1)^2 = -du dv,$$
$$u = x^0 - x^1, \qquad v = x^0 + x^1, \tag{1}$$

defined by the condition $v < 0$. We introduce coordinates

$$\psi = -\ln\left(\frac{v}{v_o}\right), \qquad t = -uv, \tag{2}$$

with $v_o < 0$ a constant used for dimensional purposes. The line element in these coordinates is

$$ds^2 = -d\psi dt - t d\psi^2, \tag{3}$$

and the boost

$$\mathcal{B}\colon\ (u, v) \to (\exp(\gamma)u, \exp(-\gamma)v), \qquad \gamma > 0, \tag{4}$$

is given by

$$(\psi, t) \to (\psi + \gamma, t). \tag{5}$$

Two-dimensional Misner space $\mathcal{M}_2$ is defined as the quotient of $\tilde{\mathcal{M}}_2$ under the action of the subgroup $G = \{\mathcal{B}^n | n \in \mathbb{Z}\}$ of the Lorentz group in 1 + 1 dimensions; that is, points in $\tilde{\mathcal{M}}_2$ which are related by $\mathcal{B}^n$ for some $n \in \mathbb{Z}$ are considered equivalent, with $\mathcal{M}_2$ being the set of equivalence classes. Since (ψ, t) and $(\psi + n\gamma, t), n \in \mathbb{Z}$ represent the same point of $\mathcal{M}_2$, and t extends from minus to plus infinity, two-dimensional Misner space has the manifold structure of a cylinder $S^1_\psi \times \mathbb{R}_t$, with $2\pi\psi/\gamma$ being an angular coordinate of S^1 (Fig. 1), on which the flat Lorentzian metric (3) is defined. Since the nonvanishing vector field $\partial/\partial t$ is always null, it gives a time orientation on $\mathcal{M}_2$: we define the future null half-cone as that where $\partial/\partial t$ belongs. This is consistent with the time orientation $\partial/\partial x^0$ on the covering $\tilde{\mathcal{M}}_2$, as can be seen by lifting $\partial/\partial t$ to $\tilde{\mathcal{M}}_2$, which gives $-(2v)^{-1}(\partial/\partial x^0 - \partial/\partial x^1)$, a vector field that lies in the same half-cone of $\partial/\partial x^0$ since $v < 0$ on $\tilde{\mathcal{M}}_2$.

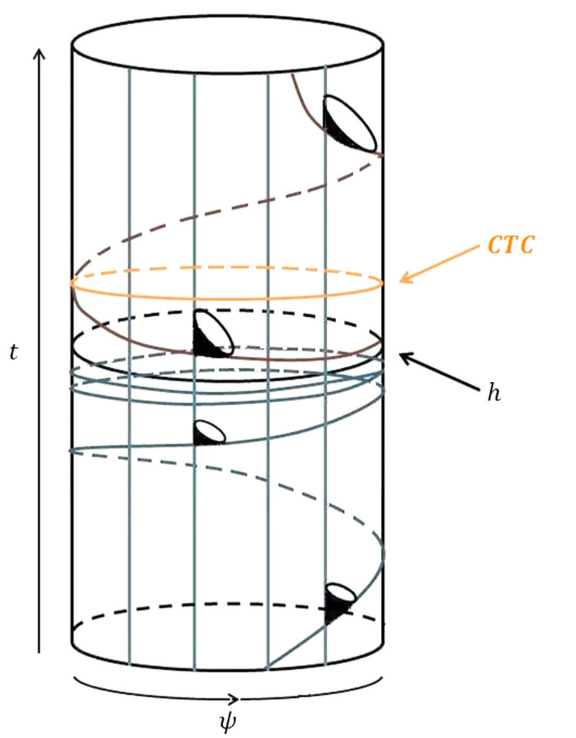

FIG. 1 (color online). Two-dimensional Misner space.

A. Null geodesics

The image under $\mathcal{B}$ of the Minkowskian null geodesic $v = v_o < 0$ is the geodesic $v = \exp(-\gamma)v_o$; $\mathcal{M}_2$ can therefore be regarded as the strip $\mathcal{S}_0 \subset \tilde{\mathcal{M}}_2$ limited by $\ell = \{(u, v_o), u \in \mathbb{R}\}$ and $\mathcal{B}\ell = \{(u, \exp(-\gamma)v_o), u \in \mathbb{R}\}$, with the boundary points (u, v_o) and $(\exp(\gamma)u,\ \exp(-\gamma)v_o)$ identified for every $u \in \mathbb{R}$. This construction is shown in Fig. 2, where some of the points to be identified are marked with circles. A geodesic segment in $\tilde{\mathcal{M}}_2$ connecting identified points maps onto a closed curve in $\mathcal{M}_2$ of square length

$$\Delta s^2 = -\Delta u \Delta v = -2t(\cosh(\gamma) - 1). \tag{6}$$

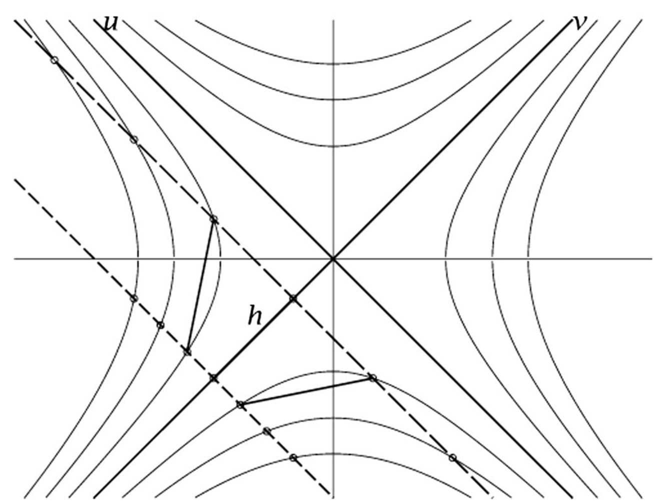

FIG. 2. Two-dimensional Minkowski space: x^0 and x^1 are the vertical and horizontal axes. A few orbits of the Lorentz group are shown, including the u and v axes. Misner space $\mathcal{M}_2$ can be regarded as the strip between ℓ (null, dashed line) and $\mathcal{B}\ell$ (null, long-dashed line) with points in ℓ identified with their image under $\mathcal{B}$ in $\mathcal{B}\ell$ (this is the t axis in Fig. 1). Some of these pairs of identified points are marked with circles. The three geodesic segments shown are, from left to right, timelike, null, and spacelike; they become closed geodesics in $\mathcal{M}_2$. The closed null geodesic h at $t = 0$ ($u = 0$) separates the noncausal region $M_2^>$ above it ($t > 0, u > 0$) from the causal region $M_2^<$ below ($t < 0, u < 0$).

These closed curves are timelike in the $t > 0$ sector and spacelike in the $t < 0$ sector. The $t_o = 0$ segment h connecting $(u = 0, v = v_o)$ with $(u = 0, v = \exp(-\gamma)v_o)$ corresponds to a closed null geodesic which is a horizon separating the causally pathological $t > 0$ region from the globally hyperbolic $t < 0$ region.

The vector fields

$$N_1 = \frac{\partial}{\partial t}, \qquad \tilde{N}_2 = -t\frac{\partial}{\partial t} + \frac{\partial}{\partial \psi} \tag{7}$$

are geodesic, null and, since $N_1^a N_2 b = -\frac{1}{2}$, future oriented; this explains the arrangement of future half-cones in Fig. 1, from where it is readily seen that any future causal curve crossing h (i.e., $\dot{t} \neq 0$ at $t = 0$) must satisfy $\dot{t} > 0$ at $t = 0$. Note that N_1 in (7) is affine but $\tilde{N}_2$ is not. In fact, there is no globally defined affine geodesic field proportional to $\tilde{N}_2$. It is, however, possible to rescale $\tilde{N}_2$ separately in the $t > 0$ and $t < 0$ open sets to obtain a future null affine geodesic field N_2 in each of these regions:

$$N_2 = \begin{cases} -\frac{\partial}{\partial t} + \frac{1}{t}\frac{\partial}{\partial \psi} & t > 0 \\ \frac{\partial}{\partial t} - \frac{1}{t}\frac{\partial}{\partial \psi} & t < 0. \end{cases} \tag{8}$$

The integral curves of N_1 starting at (t_0, ψ_0) at the affine parameter $s = 0$ are

$$t = t_0 + s, \qquad \psi = \psi_0, \qquad -\infty < s < \infty. \tag{9}$$

These geodesics are complete and cross h. The integral curves of $\tilde{N}_2$, starting at (t_0, ψ_0) at the affine parameter $s = 0$ are the future incomplete null geodesics

$$t = t_0 - s, \qquad \psi = \psi_0 - \ln\left(\frac{t_0 - s}{t_0}\right), \qquad -\infty < s < t_0 \quad \text{if } t_0 > 0 \tag{10}$$

$$t = 0, \qquad \psi = -\ln\left(e^{-\psi_0} - \frac{s}{s_0}\right), \qquad -\infty < s < s_0 e^{-\psi_0}, \qquad s_0 > 0 \quad \text{if } t_0 = 0 \tag{11}$$

$$t = t_0 + s, \qquad \psi = \psi_0 - \ln\left(\frac{t_0 + s}{t_0}\right), \qquad -\infty < s < -t_0 \quad \text{if } t_0 < 0. \tag{12}$$

For $t_0 \neq 0$ the above equations imply $\psi = \psi_0 - \ln(\frac{t}{t_0})$, with $t \to 0^+$ $(t \to 0^-)$ at the geodesic future end if $t_0 > 0$ $(t_0 < 0)$. These geodesics spiral, asymptotically approaching h as $s \to |t_0|^-$; see Fig. 1. This behavior can be understood using the construction in Fig. 2: a future directed segment along the v direction, starting at a point $p \in \ell$, will reach $\mathcal{B}\ell$ at q and emerge at the equivalent point $q' \in \ell$ which lies closer to h than p, and this process repeats itself indefinitely. The affine geodesic h starting at $(t_0 = 0, \psi_0)$ in the direction of $\tilde{N}_2$ (increasing ψ) has a tangent vector $(e^{\psi}/s_0)\frac{\partial}{\partial \psi}$ [see Eq. (11)], which, being nonperiodic in ψ, does *not* define a vector field on h. This is a peculiar situation, which is allowed for closed affine null geodesics in a Lorentzian geometry, for which the vector can scale in every turn and still have a constant norm. It can be understood by noting that h lifts to the (future affine null) geodesic $\tilde{h}$ of $\tilde{M}_2$ given by $u = 0, v = v_o[e^{-\psi_0} - (s/s_0)]$. The nth turn of h $(n = 0, 1, 2, \ldots)$ is lifted to the intersection $\tilde{h}_n$ of $\tilde{h}$ with the strip $\mathcal{S}_n \subset \tilde{\mathcal{M}}_2$ limited by $\ell_n = \{(u, \exp(-n\gamma)v_o), u \in \mathbb{R}\}$ and $\mathcal{B}\ell_n = \ell_{n+1}$. The affine parameter span of this geodesic segment is $\exp(-n\gamma)$ times that of the segment h_1, yet the ψ span is the same; therefore, $\dot{\psi}$ scales as $\exp(+n\gamma)$ relative to the first turn. An extension of M_2 can be constructed that is geodesically complete, but it fails to be a Hausdorff manifold [8].

The $\mathcal{M}_2^<$ subset defined by the condition $t < 0$ is globally hyperbolic; any $t =$ constant surface is a Cauchy surface with Cauchy horizon h. The region $t > 0$ violates causality in any possible form: given any two points p and q in this region, there is a future oriented timelike curve from p to q (these curves can be easily constructed with the help of Fig. 2.) Thus, $\mathcal{M}_2^<$ is a two-dimensional example of spacetime smoothly extensible beyond a Cauchy horizon, i.e., violating SCC, with the extension violating causality in any possible form.

Four-dimensional Misner spacetime $\mathcal{M}$ is the quotient of the $v < 0$ half of Minkowski spacetime $ds^2 = -dudv + dy^2 + dz^2$ by the boosts (4). The metric is

$$n_{ab}dx^a dx^b = -d\psi dt - t d\psi^2 + dy^2 + dz^2. \tag{13}$$

The similarity of the notation n_{ab} and the standard notation η_{ab} for the metric of Minkowski spacetime is reminiscent of the fact that (13) is locally flat, as it comes from a quotient of a sector of Minkowski spacetime $(\mathbb{R}^4, \eta_{ab})$.

As a Lorentzian manifold, $\mathcal{M} = \mathcal{M}_2 \times \mathbb{R}^2$ with the flat metric $dy^2 + dz^2$ on the $\mathbb{R}^2$ factor. We may consider x and y periodic with periods a and b, that is, work instead with $\mathcal{M}_c = \mathcal{M}_2 \times \mathbb{T}^2$ with the flat metric on the torus. In any case, the open set defined by $t < 0$ is globally hyperbolic, the Cauchy surfaces $t =$ (a negative) constant have the null hypersurface $\mathcal{H}$ defined by the condition $t = 0$ as a future Cauchy horizon, and this four-dimensional spacetime violates a strong form of the cosmic censorship conjecture, as it admits a smooth extension beyond a Cauchy horizon. This is precisely what happens for the Kerr (also Kerr-Newman and Reissner-Nordström) black holes, with the inner horizon replacing $\mathcal{H}$. According to the SCC, small departures in the initial data of these spacetimes will develop a globally hyperbolic spacetime with a null-like curvature singularity in place of the Cauchy horizon, and therefore admitting no extension beyond it.

The purpose of this paper is to test SCC for $\mathcal{M}^<$, the $t < 0$ sector of $\mathcal{M}$ (or its compact slice version $\mathcal{M}_c^<$ with periodic x and y) which, in spite of its simplicity (flat metric), has all possible pathologies in an exact solution of Einstein's equation. In the following sections we consider Misner spacetime as an exact solution in Einstein-scalar, Einstein-Maxwell and pure RG theories, and prove that in any of these theories it is an isolated solution.

III. INSTABILITY OF THE CAUCHY HORIZON IN EINSTEIN-SCALAR FIELD THEORY

Let Φ be a massless scalar field minimally coupled to gravity on the manifold $\mathcal{M}_c^< = S^1_\psi \times \mathbb{R}_{t<0} \times T^2_{(y,z)}$. The field equations are

$$G_{ab} = 8\pi\left[(\partial_a\Phi)(\partial_b\Phi) - \frac{1}{2}g_{ab}g^{cd}(\partial_c\Phi)(\partial_d\Phi)\right], \quad (14)$$

$$0 = g^{cd}\nabla_d\nabla_c\Phi, \quad (15)$$

where ∇_a is the Levi-Civita derivative of g_{ab}. These equations admit the solution

$$\Phi_o = 0, \qquad g_{ab} = n_{ab}. \quad (16)$$

In this section we prove that, although this solution can be extended to $\mathcal{M}_c = S^1_\psi \times \mathbb{R}_t \times T^2_{(y,z)}$ (i.e., add the $t \geq 0$ sector), any neighboring solution of the system (14)–(15) on the manifold $\mathcal{M}_c^<$ develops a curvature singularity as $t \to 0^-$. To this end, consider a monoparametric family of solutions $(\Phi_\lambda, (g_\lambda)_{ab})$ such that

$$(g_{\lambda=0})_{ab} = n_{ab}, \qquad \Phi_{\lambda=0} = \Phi_o = 0. \quad (17)$$

With n overdots, we denote the nth derivative with respect to the parameter λ, evaluated at $\lambda = 0$; then

$$\Phi_\lambda = \lambda\dot{\Phi} + \frac{1}{2}\lambda^2\ddot{\Phi} + \cdots \quad (18)$$

$$(g_\lambda)_{ab} = n_{ab} + \lambda\dot{g}_{ab} + \frac{1}{2}\lambda^2\ddot{g}_{ab} + \cdots \quad (19)$$

and similarly for any other tensor field. From (14)–(15) we obtain for the Ricci scalar $\mathcal{R}$

$$\mathcal{R}_\lambda = \frac{1}{2}\lambda^2\ddot{\mathcal{R}} + \mathcal{O}(\lambda^3) \quad (20)$$

where

$$\ddot{\mathcal{R}} = 16\pi n^{ab}(\partial_a\dot{\Phi})(\partial_b\dot{\Phi}), \quad (21)$$

and

$$0 = n^{ab}\partial_a\partial_b\dot{\Phi}. \quad (22)$$

Note that $\dot{\Phi}$ satisfies the equation of a *test* scalar field on the Misner background (that is, without backreaction effects). Yet, it gives information on the Ricci scalar up to order two for the coupled scalar-gravity system (14)–(15). Motivated by this observation, we devote the following subsections to the study of test scalar fields on $\mathcal{M}^<$, starting with (y,z) independent fields, that is, scalar fields on $\mathcal{M}_2^<$. This will be used to prove that $\mathcal{R}$ in (20) diverges as $t \to 0^-$ for generic solutions of the theory (14)–(15) near (17).

A. Massless test scalar fields on $\mathcal{M}_2$

Massless scalar fields $\tilde{\Phi}$ on $\tilde{\mathcal{M}}_2$ satisfy $\partial_u\partial_v\tilde{\Phi} = 0$; they are a superposition $\tilde{\Phi} = R(u) + L(v)$ of left- and right-moving waves. For fields defined on $\mathcal{M}_2$, the extra condition

$$R(e^\gamma u) - R(u) = L(v) - L(e^{-\gamma}v) = c \quad (23)$$

should be imposed, where c is a constant and $c = 0$ if $\lim_{u\to 0}R(u)$ exists. Using the inverse of (2),

$$v = v_o e^{-\psi}, \qquad u = \begin{cases} -v_o e^{\psi + \ln(|t|/v_o^2)} & t > 0 \\ v_o e^{\psi + \ln(|t|/v_o^2)} & t < 0, \end{cases} \quad (24)$$

and introducing $L(v) = L(v_o e^{-\psi}) =: l(\psi)$, condition (23) reads $l(\psi) - l(\psi + \gamma) = c$, which implies that there exists a periodic function $\hat{l}$, $\hat{l}(\psi + \gamma) = \hat{l}(\psi)$, such that

$$l(\psi) = \hat{l}(\psi) - \frac{c}{\gamma}\psi. \quad (25)$$

A similar analysis for $R(u)$ in (23) using (24) leads to

$$\Phi = \begin{cases} \hat{l}_<(\psi) + \hat{r}_<(\psi + \ln(|t|/v_o^2)) + \frac{c_<}{\gamma}\ln(|t|/v_o^2) & t < 0 \\ \hat{l}_>(\psi) + \hat{r}_>(\psi + \ln(|t|/v_o^2)) + \frac{c_>}{\gamma}\ln(|t|/v_o^2) & t > 0, \end{cases} \quad (26)$$

where all hatted functions are periodic with period γ, and therefore bounded if they are to be smooth in the corresponding $t > 0$ or $t < 0$ half-space. Note that $\lim_{t\to 0}\Phi(t,\psi)$

along curves in the open set $\mathcal{M}_2^<$ cannot exist unless $c_< = 0$ and $\hat{r}_<$ is a constant, which may then be absorbed into $\hat{l}_<$ to set $\hat{r}_< = 0$ (a similar analysis applies for $\mathcal{M}_2^>$). Continuity across h $(t = 0)$ would furthermore require $\hat{l}_< = \hat{l}_> =: \hat{l}$. Thus, the only solutions that are continuous through $\mathcal{M}_2$ are the left-moving waves $\Phi = \hat{l}(\psi)$ for all t. These conditions and fields are dealt with in [9].

B. Scalar fields on $(\mathcal{M}_c^<, n_{ab})$

The massless scalar field Φ equation on $(\mathcal{M}^<, n_{ab})$ is

$$0 = 4t\frac{\partial^2\Phi}{\partial t^2} - 4\frac{\partial^2\Phi}{\partial\psi\partial t} + 4\frac{\partial\Phi}{\partial t} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = \Box_2\Phi + \Delta_2\Phi, \tag{27}$$

where $\Delta_2 = \partial_y^2 + \partial_z^2$ and $\Box_2$ is the massless scalar field operator on $\mathcal{M}_2^<$. Solutions of (27) can be written as

$$\Phi = \phi_{(0)}(\psi, t) + \phi_{(1)}(\psi, t, y, z), \tag{28}$$

where

$$\phi_{(0)}(\psi, t) \equiv \frac{1}{ab}\int_0^a dy \int_0^b dz \Phi(\psi, t, y, z) \tag{29}$$

is a (y, z)-independent solution of $\Box_2\phi_0 = 0$,

$$\phi_{(0)}(\psi, t) = \hat{l}(\psi) + \hat{r}(\psi + \ln(|t|/v_o^2)) + \frac{c}{\gamma}\ln(|t|/v_o^2), \tag{30}$$

and

$$\phi_{(1)} = \sum_{\substack{k,l,n=-\infty \\ (l,n)\neq(0,0)}}^{\infty} C_{(k,l,n)}(t) e^{2\pi i k\psi/\gamma} e^{2\pi i l y/a} e^{2\pi i n z/b}, \tag{31}$$

with

$$C_{(k,l,n)}(t) := \frac{1}{\gamma ab}\int_0^\gamma d\psi \int_0^a dy \times \int_0^b dz \phi_{(1)} e^{-2\pi i k\psi/\gamma} e^{-2\pi i l y/a} e^{-2\pi i n z/b}. \tag{32}$$

Equation (27) reduces to

$$t\frac{d^2C_{(k,l,n)}}{dt^2} + (1 - i\nu)\frac{dC_{(k,l,n)}}{dt} - \left(\frac{m}{2}\right)^2 C_{(k,l,n)} = 0, \tag{33}$$

where

$$m \equiv 2\pi\sqrt{\frac{l^2}{a^2} + \frac{n^2}{b^2}}, \qquad \nu \equiv \frac{2\pi k}{\gamma}. \tag{34}$$

Introducing

$$x \equiv m\sqrt{-t} \in (0, \infty), \; C_{(k,l,n)} = e^{i\nu\ln(x)} D_{(k,l,n)}, \tag{35}$$

Eq. (33) gives a Bessel equation of imaginary order for $D_{(k,l,n)}$:

$$x^2\frac{d^2D_{(k,l,n)}}{dx^2} + x\frac{dD_{(k,l,n)}}{dx} + (x^2 + \nu^2)D_{(k,l,n)} = 0, \tag{36}$$

which admits the following two real, bounded, linearly independent $\mathbb{C}^\infty$ solutions for $x \in (0, \infty)$ (we follow the notation and conventions in [10]):

$$\tilde{J}_\nu(x) := \mathrm{sech}\left(\frac{1}{2}\pi\nu\right)\mathrm{Re}(J_{i\nu}(x))$$
$$\tilde{Y}_\nu(x) := \mathrm{sech}\left(\frac{1}{2}\pi\nu\right)\mathrm{Re}(Y_{i\nu}(x)). \tag{37}$$

Thus

$$C_{(k,l,n)} = (A_{(k,l,n)}\tilde{J}_\nu(x) + B_{(k,l,n)}\tilde{Y}_\nu(x))e^{i\nu\ln(x)} \tag{38}$$

and $A_{(-k,-l,-n)} = A^*_{(k,l,n)}$ for real Φ. The functions (37) satisfy

$$\tilde{J}_\nu(x) = \tilde{J}_{-\nu}(x), \qquad \tilde{Y}_\nu(x) = \tilde{Y}_{-\nu}(x) \tag{39}$$

and have the following asymptotic behavior: as $x \to \infty$ $(t \to -\infty)$,

$$\tilde{J}_\nu(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4}\right) + \mathcal{O}(x^{-3/2}), \tag{40}$$

$$\tilde{Y}_\nu(x) = \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\pi}{4}\right) + \mathcal{O}(x^{-3/2}); \tag{41}$$

as $x \to 0^+$ $(t \to 0^-)$,

$$\tilde{J}_\nu(x) = \sqrt{\frac{2\tanh(\pi\nu/2)}{\pi\nu}}\cos(\nu\ln(x/2) - \gamma_\nu) + \mathcal{O}(x^2), \tag{42}$$

$$\tilde{Y}_\nu(x) = \sqrt{\frac{2\coth(\pi|\nu|/2)}{\pi|\nu|}}\sin(|\nu|\ln(x/2) - \gamma_{|\nu|}) + \mathcal{O}(x^2), \tag{43}$$

where γ_ν is defined by

$$\exp(i\gamma_\nu) = \left(\frac{\sinh(\pi\nu)}{\pi\nu}\right)^{1/2}\Gamma(1 + i\nu). \tag{44}$$

C. Instability of the Cauchy horizon in $\mathcal{M}_c^<$

The instability of the Cauchy horizon in $\mathcal{M}_c^<$ is expressed as follows:

Theorem 1: Let $((g_\lambda)_{ab}, \Phi_\lambda)$ be a one-parametric family of solutions for the Einstein real scalar field equations (14)–(15) on the manifold $\mathcal{M}_c^< = S^1_\psi \times \mathbb{R}_{t<0} \times T^2_{(y,z)}$. Assume that $\lambda = 0$ corresponds to Misner spacetime (16). Let $\mathcal{R}_\lambda$ be the Riccci scalar of g_λ, then Eqs. (20)–(22) hold and, generically, $\ddot{\mathcal{R}} \sim 1/t$ as $t \to 0^-$.

Proof: It only remains to prove that, with the exception of fine-tuned solutions, $\ddot{\mathcal{R}} \sim 1/t$ as $t \to 0^-$. From (22),

$$\ddot{\mathcal{R}} = \ddot{\mathcal{R}}_{(0)(0)} + \ddot{\mathcal{R}}_{(1)(1)} + 2\ddot{\mathcal{R}}_{(0)(1)} \tag{45}$$

where

$$\ddot{\mathcal{R}}_{(i)(j)} = -4\frac{\partial\phi_{(i)}}{\partial\psi}\frac{\partial\phi_{(j)}}{\partial t} - 4\frac{\partial\phi_{(i)}}{\partial t}\frac{\partial\phi_{(j)}}{\partial\psi} + 8\frac{\partial\phi_{(i)}}{\partial t}\frac{\partial\phi_{(j)}}{\partial t} + 2\frac{\partial\phi_{(i)}}{\partial y}\frac{\partial\phi_{(j)}}{\partial y} + 2\frac{\partial\phi_{(i)}}{\partial z}\frac{\partial\phi_{(j)}}{\partial z} \tag{46}$$

and $\phi_{(i)}, i = 0, 1$ were defined in (28)–(31).

From (30), we obtain

$$\ddot{\mathcal{R}}_{(0)(0)} = \frac{4}{\gamma t}\left[\hat{r}'(\psi + \ln(|t|/v_o^2)) + \frac{c}{\gamma}\right]\left[\frac{c}{\gamma} - \hat{l}'(\psi)\right]. \tag{47}$$

Since the derivative of a periodic function cannot be a nonzero constant, $\ddot{\mathcal{R}}_{00}$ can only vanish identically if $c = 0$ and either $\hat{r}$ or $\hat{l}$ vanish identically. For generic one-parametric solutions of the Einstein-scalar field theory, $\ddot{\mathcal{R}}_{00} \sim 1/t$ near the Cauchy horizon. This divergence could only be canceled by the $1/t$ contribution from $\ddot{\mathcal{R}}_{11}$ [implied by the asymptotic behavior (42)–(43)] by fine-tuning the constants in these independent pieces of the scalar field.

If we restrict ourselves to fields that decay along past directed causal curves, we need to set $c = 0$. This does not prevent the divergence (47) except, once again, for the fine-tuned case of pure left- or right-moving waves.

IV. INSTABILITY OF THE CAUCHY HORIZON IN EINSTEIN-MAXWELL THEORY

Let F_{ab} be a Maxwell field coupled to gravity on the manifold $\mathcal{M}_c^< = S^1_\psi \times \mathbb{R}_{t<0} \times T^2_{(y,z)}$. The Einstein-Maxwell field equations

$$R_{ab} = 2F_{ac}F_{bd}g^{cd} - \frac{1}{2}g_{ab}F_{cd}F_{ef}g^{ec}g^{df}, \tag{48}$$

$$\nabla_{[a}F_{bc]} = 0, \qquad \nabla^a F_{ab} = 0 \tag{49}$$

admit the solution

$$F_{ab} = 0, \qquad g_{ab} = n_{ab}, \tag{50}$$

which can be extended to $\mathcal{M}_c = S^1_\psi \times \mathbb{R}_t \times T^2_{(y,z)}$. In this section we prove that generic neighboring solutions of the system (48)–(49) on the manifold $\mathcal{M}_c^<$ develop a curvature singularity as $t \to 0^-$. The Ricci scalar vanishes identically for the Einstein-Maxwell system; the singularity arises in the quadratic curvature invariant

$$\mathcal{Q} = R_{ab}R_{cd}g^{ac}g^{bd}. \tag{51}$$

Consider a monoparametric family of solutions $((F_\lambda)_{ab}, (g_\lambda)_{ab})$ such that

$$(g_{\lambda=0})_{ab} = n_{ab}, \qquad (F_{\lambda=0})_{ab} = 0. \tag{52}$$

As in the previous section, n overdots are used to indicate the nth derivative with respect to the parameter λ, evaluated at $\lambda = 0$. We have

$$(F_\lambda)_{ab} = \lambda\dot{F}_{ab} + \frac{1}{2}\lambda^2\ddot{F}_{ab} + \cdots, \tag{53}$$

$$(g_\lambda)_{ab} = n_{ab} + \lambda\dot{g}_{ab} + \frac{1}{2}\lambda^2\ddot{g}_{ab} + \cdots, \tag{54}$$

and

$$\mathcal{Q}_\lambda = \frac{1}{4!}\lambda^4\ddddot{\mathcal{Q}} + \mathcal{O}(\lambda^5) \tag{55}$$

where

$$\ddddot{\mathcal{Q}} = \ddot{R}_{ab}\ddot{R}_{cd}n^{ac}n^{bd}, \tag{56}$$

$$\ddot{R}_{ab} = 2\dot{F}_{ac}\dot{F}_{bd}n^{cd} - \frac{1}{2}n_{ab}\dot{F}_{cd}\dot{F}_{ef}n^{ec}n^{df}. \tag{57}$$

Note that $\dot{F}$ satisfies the equations of a test Maxwell field *on the Misner background*, that is,

$$\nabla_{[a}\dot{F}_{bc]} = 0, \qquad n^{bc}\nabla_c\dot{F}_{ab} = 0, \tag{58}$$

where ∇_c is the covariant derivative of n_{ab}, and that this test Maxwell field gives information on the leading term of the curvature scalar $\mathcal{Q}$, which is fourth order in λ.

A. Maxwell fields on $\mathcal{M}_c^<$

Test Maxwell fields on the $M_c^<$ background are relevant to the Cauchy horizon stability problem because, according to Eqs. (56) and (58), they give the leading order contribution to the $\mathcal{Q} = R_{ab}R^{ab}$ curvature scalar in Einstein-Maxwell theory on this manifold.

The second Betti number of $\mathcal{M}_c^< = S^1_\psi \times \mathbb{R}_{t<0} \times T^2_{(y,z)}$ is 3; the three-dimensional space of closed nonexact two-forms is generated by $d\psi \wedge dy, d\psi \wedge dz$ and $dy \wedge dz$.

Since these two-forms are divergence free for the flat metric n, the general solution of the Maxwell equations on the background n_{ab} is

$$F = K d\psi \wedge dy + L d\psi \wedge dz + M dy \wedge dz + dA^{(0)} + dA^{(1)} \tag{59}$$

where, as done with the scalar field, we have split $A_b = A_b^{(0)} + A_b^{(1)}$ with $\pounds_{\partial/\partial y} A_b^{(0)} = \pounds_{\partial/\partial z} A_b^{(0)} = 0$. We chose the one-forms $A_b^{(j)}$ in the Lorenz gauge $\nabla^b A_b^{(j)} = 0$; then Maxwell equations reduce to

$$\nabla^b A_b^{(j)} = 0, \qquad n^{ab}\nabla_a \nabla_b A_c^{(j)} = 0. \tag{60}$$

Introducing $A_b^{(0)}(\psi, t) = \sum_{k\in\mathbb{Z}} C_b^k(t)\exp(2\pi i k\psi/\gamma)$ in (60) we find, after treating separately the $k = 0$ and $k \neq 0$ terms and then summing up the series, that

$$\begin{aligned}
A_\psi^{(0)}(\psi,t) &= 2at + \hat{l}(\psi) + \hat{r}(\psi + \ln(-t/v_o^2))\\
A_t^{(0)}(\psi,t) &= a + t^{-1}b + t^{-1}\hat{r}(\psi + \ln(-t/v_o^2))\\
A_y^{(0)}(\psi,t) &= c_y \ln(-t/v_o^2) + \hat{l}_y(\psi) + \hat{r}_y(\psi + \ln(-t/v_o^2)\\
A_z^{(0)}(\psi,t) &= c_z \ln(-t/v_o^2) + \hat{l}_z(\psi) + \hat{r}_z(\psi + \ln(-t/v_o^2)).
\end{aligned}$$

This can be simplified using the residual gauge freedom $A_c^{(0)} \to A_c^{(0)} + \partial_c\chi, n^{ab}\partial_a\partial_b\chi = 0$. Taking an appropriate χ of the form (30), we get a vector potential of the form

$$\begin{aligned}
A_\psi^{(0)}(\psi,t) &= 2at + \hat{l}_0\\
A_t^{(0)}(\psi,t) &= a\\
A_y^{(0)}(\psi,t) &= c_y \ln(-t/v_o^2) + \hat{l}_y(\psi) + \hat{r}_y(\psi + \ln(-t/v_o^2))\\
A_z^{(0)}(\psi,t) &= c_z \ln(-t/v_o^2) + \hat{l}_z(\psi) + \hat{r}_z(\psi + \ln(-t/v_o^2))
\end{aligned} \tag{61}$$

with $a, \hat{l}_0, c_y$ and c_z constants [the irrelevant constant in $A_t^{(0)}(\psi, t)$ can be gauged away using the nonperiodic harmonic function $\chi_0 = -a\psi$]. For the Maxwell field we obtain

$$F^{(0)} = dA^{(0)} = \begin{pmatrix} 0 & -2a & \hat{l}_y{}' + \hat{r}_y{}' & \hat{l}_z{}' + \hat{r}_z{}' \\ 2a & 0 & (c_y + \hat{r}_y{}')/t & (c_z + \hat{r}_z{}')/t \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix} \tag{62}$$

where we have omitted the arguments in the periodic (hatted) functions and the order of coordinates is (ψ, t, y, z). The two field invariants for (62) are

$$F_{ab}^{(0)}F^{(0)ab} = -32a^2 + 8t^{-1}[c_y{}^2 + c_z{}^2 + c_y(\hat{r}_y{}' - \hat{l}_y{}') + c_z(\hat{r}_z{}' - \hat{l}_z{}') - \hat{l}_y{}'\hat{r}_y{}' - \hat{l}_z{}'\hat{r}_z{}'] \tag{63}$$

and

$$\epsilon^{abcd}F_{ab}^{(0)}F_{cd}^{(0)} = -16t^{-1}[l_z' r_y' - l_y' r_z' + c_y(l_z' + r_z') - c_z(l_y' + r_y')]. \tag{64}$$

B. Instability of the Cauchy horizon in $\mathcal{M}_c^<$

The instability of the Cauchy horizon of $\mathcal{M}_c^<$ in the Einstein-Maxwell theory is expressed in the following.

Theorem 2: Let $((g_\lambda)_{ab}, (F_\lambda)_{ab})$ be a one-parametric family of solutions for the Einstein-Maxwell field equations (48)-(49) on the manifold $\mathcal{M}_c^< = S_\psi^1 \times \mathbb{R}_{t<0} \times T^2_{(y,z)}$. Assume that $\lambda = 0$ corresponds to Misner spacetime (50). Let $\mathcal{Q}_\lambda$ be the square Ricci scalar (51) of g_λ; then Eqs. (55)–(58) hold and, generically, $\dddot{\mathcal{Q}}$ diverges at least as $\sim 1/t^2$ as $t \to 0^-$.

Proof: According to Eqs. (56) and (57), $\dddot{\mathcal{Q}}$ is quartic on $\dot{F}_{ab}$. Since $\dot{F}_{ab}$ satisfies Maxwell equations on the flat Misner background [see Eq. (58)], it is of the form (59). We focus on the contribution $\dddot{\mathcal{Q}}'$ to $\dddot{\mathcal{Q}}$ that is quartic in $F^{(0)} = dA^{(0)}$ in (59). Note from (62) that the general $F^{(0)}$ field is finite on any Cauchy slice in $\mathcal{M}_c^<$. A stronger condition of decay as $t \to -\infty$ can be enforced by requiring $a = 0$ [see Eqs. (63) and (64)]. In any case, the contribution $\dddot{\mathcal{Q}}'$, obtained by replacing $\dot{F}$ with (62) in (56) and (57), is

$$\dddot{\mathcal{Q}}' = 512a^4 - 256a^2t^{-1}[c_y^2 + c_z^2 + c_y(\hat{r}_y{}' - \hat{l}_y{}') + c_z(\hat{r}_z{}' - \hat{l}_z{}') - \hat{l}_y{}'\hat{r}_y{}' - \hat{l}_z{}'\hat{r}_z{}'] + 16t^{-2}[\hat{l}_z'^2\hat{r}_y'^2 + \hat{l}_y'^2\hat{r}_z'^2 + 2\hat{l}_z'^2\hat{r}_z'^2 + 2\hat{l}_y'^2\hat{r}_y'^2 + 2\hat{l}_y{}'\hat{l}_z{}'\hat{r}_y{}'\hat{r}_z{}' + \cdots + 2c_y{}^4 + 2c_z{}^4] \tag{65}$$

where the missing terms in the t^{-2} coefficient involve growing powers of c_y and c_z times derivatives of periodic functions. As $t \to 0^-$, Eq. (65) behaves as a bounded function times t^{-2}. This divergence could (in principle) be canceled out by the remaining contributions to $\dddot{\mathcal{Q}}$, but this could only be done by fine-tuning and will not be the case for generic monoparametric solutions of the Einstein-Maxwell system.

V. INSTABILITY OF THE CAUCHY HORIZON IN PURE GRAVITY

The Cauchy horizon of $M_c^<$ can also be seen to be unstable in the context of pure gravity. Consider a mono-parametric family of Ricci flat metrics through n_{ab} in (13):

$$(g_\lambda)_{ab} = n_{ab} + \lambda \dot{g}_{ab} + \frac{1}{2}\lambda^2 \ddot{g}_{ab} + \cdots. \tag{66}$$

As is well known, any algebraic curvature scalar for a vacuum metric is a polynomial on $K \coloneqq R_{abcd}R^{abcd}$ and $L \coloneqq \epsilon_{abpq}R^{pq}_{cd}R^{abcd}$. For (66) we obtain

$$K_\lambda = \lambda^2 \dot{R}_{abcd}\dot{R}_{efgh}n^{ae}n^{bf}n^{cg}n^{dh} + \cdots \tag{67}$$

and similarly for L_λ; that is, knowledge of a linearized solution $\dot{g}_{ab}$ of Einstein's equation provides information on the dominant contributions to K and L, which are second order in λ.

Linear gravity on the background (13) can be approached using the formalism in [11], which applies to warped metrics of any dimensions with an Einstein compact Riemannian manifold factor which, in our case, is the trivial 2-torus flat metric $dy^2 + dz^2$. Three different families of modes arise—tensor, vector, and scalar—which satisfy decoupled equations. Among them, the simplest contributions are the two zero modes in the tensor sector, which are constructed using the divergence-free, trace-free, harmonic symmetric tensors $dx \otimes dx - dy \otimes dy$ and $dx \otimes dy + dy \otimes dx$ on $\mathbb{T}^2$. For these, the metric perturbation is [the order of coordinates is (ψ, t, y, z)]

$$\dot{g}_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{H}(\psi,t) & \hat{P}(\psi,t) \\ 0 & 0 & \hat{P}(\psi,t) & -\hat{H}(\psi,t) \end{pmatrix} \tag{68}$$

and Einstein's linearized equation $\dot{R}_{ab} = 0$ reduces to $\Box_2 \hat{H} = \Box_2 \hat{P} = 0$ [Eq. (4.2) in [11]], whose solution is [see Eq. (30)]

$$\hat{H} = \hat{H}_l(\psi) + \hat{H}_r(\psi + \ln(|t|/v_o^2)) + \frac{C_H}{\gamma}\ln(|t|/v_o^2), \tag{69}$$

$$\hat{P} = \hat{P}_l(\psi) + \hat{P}_r(\psi + \ln(|t|/v_o^2)) + \frac{C_P}{\gamma}\ln(|t|/v_o^2). \tag{70}$$

We set $C_H = C_P = 0$ to keep the perturbation bounded as $t \to -\infty$ (note that $\hat{H}$ and $\hat{P}$ are gauge-invariant fields in the linearized gravity theory [11]). For the perturbation (68)–(69) we obtain

$$\ddot{K} = \dot{R}_{ab}{}^{cd}\dot{R}_{cd}{}^{ab} = 32t^{-2}[\hat{H}_r''\hat{H}_l'' + \hat{P}_r''\hat{P}_l'' + \hat{H}_r''\hat{H}_l' + \hat{P}_r''\hat{P}_l' - \hat{H}_l''\hat{H}_r' - \hat{P}_l''\hat{P}_r' - \hat{H}_r'\hat{H}_l' - \hat{P}_r'\hat{P}_l'], \tag{71}$$

which decays along past oriented causal curves and diverges as the future Cauchy horizon is approached. Once again, this divergence could possibly be canceled from (y,z)-independent contributions to $\ddot{K}$ from the (y,z)-dependent piece of $\dot{g}_{ab}$, but this could only happen after fine-tuning, and not for generic solutions around n_{ab}.

VI. DISCUSSION

Penrose's heuristic argument anticipating a curvature singularity at the Cauchy horizon of a Kerr-Newman black hole applies to the horizon h of Misner spacetime. This is readily seen by inspecting Fig. 1: an observer crossing the horizon is exposed to the information traveling, in the geometric optics approximation, along the infinitely many geodesics of the form (12) originating in his past. He is thus expected to measure a divergent energy density, as it is easily checked, e.g., in our simplest example: the (y,z)-independent scalar field (30) with $c = 0$. The stress-energy-momentum tensor of this field is

$$T_{ab} = \begin{pmatrix} \hat{r}'^2 + \hat{l}'^2 & \hat{r}'^2/t & 0 & 0 \\ \hat{r}'^2/t & \hat{r}'^2/t^2 & 0 & 0 \\ 0 & 0 & 2\hat{r}'\hat{l}'/t & 0 \\ 0 & 0 & 0 & 2\hat{r}'\hat{l}'/t \end{pmatrix} \tag{72}$$

where we have suppressed the arguments in $\hat{l}'(\psi)$ and $\hat{r}'(\psi + \ln(|t|/v_o^2))$, and the order of coordinates above is (ψ, t, y, z). An observer crossing the horizon with four-velocity $u = \dot{\psi}\partial/\partial\psi + \dot{t}\partial/\partial t + \dot{y}\partial/\partial y + \dot{z}\partial/\partial z$ has $\dot{t} \neq 0$ at $t = 0$. Note that these coordinates are valid beyond the horizon, and hence $\dot{\psi}, \dot{t}, \dot{y}$, and $\dot{z}$ must all be finite at $t = 0$. The energy density the observer measures is, after using the condition $u^c u_c$ to eliminate the $\dot{\psi}\,\dot{t}$ term,

$$\rho = T_{ab}u^a u^b = \dot{\psi}^2(\hat{l}'^2 - \hat{r}'^2) + \frac{2\hat{r}'^2}{t}(1 + \dot{y}^2 + \dot{z}^2) + \frac{2\hat{r}'\hat{l}'}{t}(\dot{y}^2 + \dot{z}^2) + \left(\frac{\hat{r}'}{t}\right)^2 \dot{t}^2. \tag{73}$$

Only the first term on the right-hand side above remains finite as $t \to 0^-$; the others all diverge except for the trivial $\hat{r} = 0$ case. It is interesting to note, however, that there is no curvature singularity in the full Einstein-scalar field theory unless *both* $\hat{r}$ and $\hat{l}$ are different from zero. This is seen by setting $c = 0$ in Eq. (47), which gives

$$\ddot{\mathcal{R}}_{(0)(0)} = \frac{-4\hat{l}'(\psi)}{\gamma t}\hat{r}'(\psi + \ln(|t|/v_o^2)). \tag{74}$$

Thus, there are situations where the energy density measured by an observer at the horizon diverges while no curvature singularity forms. This happens because in this highly relativistic regime, the pressure or tension cancels the energy density effect on $T^a{}_a \propto R$ unless both left- and right-moving waves are present. This can easily be seen from (72): the trace of the two-by-two (ψ, t) block vanishes, and only the (y, z) tensions or pressures, which contain $\hat{l}'\hat{r}'$ products, contribute to $T^a{}_a$. Note that this happens without violating energy conditions; as is well known, the stress-energy-momentum tensor of a scalar field satisfies the strong as well as the dominant energy conditions. $T^a{}_b$ above can indeed be diagonalized to the form $T^b{}_a = 2/t\,\mathrm{diag}(|\hat{r}'\hat{l}'|, -|\hat{r}'\hat{l}'|, \hat{r}'\hat{l}', \hat{r}'\hat{l}')$ in a specific orthonormal tetrad.

ACKNOWLEDGMENTS

We thank O. V. Groshev for providing us a copy of [9] and an anonymous referee for carefully reading the manuscript and pointing out errors that led to an improved version of this paper. This work was partially funded by Grants No. PIP 11220080102479 (Conicet, Argentina) and No. Secyt-UNC 05/B498 (Universidad Nacional de Córdoba, Argentina).

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