Existence of Strictly Positive Solutions for Sublinear Elliptic Problems in Bounded Domains

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Abstract

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and let m be a possibly discontinuous and unbounded function that changes sign in Ω. Let $f : [0, \infty) \to [0, \infty)$ be a nondecreasing continuous function such that $k_1 \xi^p \le f(\xi) \le k_2 \xi^p$ for all $\xi \ge 0$ and some $k_1, k_2 > 0$ and $p \in (0, 1)$. We study existence and nonexistence of strictly positive solutions for nonlinear elliptic problems of the form $-\Delta u = m(x) f(u)$ in Ω, $u = 0$ on $\partial\Omega$.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \ge 1$, be a $C^{1,1}$ bounded domain. Our aim in this paper is to consider the question of existence of solutions for nonlinear problems of the form

$$\begin{cases} -\Delta u = mf(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $m : \Omega \to \mathbb{R}$ is a function that changes sign in Ω and $f : [0, \infty) \to [0, \infty)$ is a continuous function satisfying

H1. f is nondecreasing, and there exist $k_1, k_2 > 0$ and $p \in (0, 1)$ such that $k_1 \xi^p \le f(\xi) \le k_2 \xi^p$ for all $\xi \ge 0$.

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As pointed out in [**?**], the existence of strictly positive solutions for sublinear problems with indefinite nonlinearities as (**??**) raises many interesting questions and is intriguing even in the one-dimensional case for various reasons. One of them is that the existence of (nontrivial) nonnegative solutions does not guarantee the strict positivity of such solutions (in contrast for example to superlinear problems, where they even belong to the interior of the positive cone). In fact, there are situations in which there exist nonnegative solutions which actually vanish in a subset of Ω (see e.g. [**?**]). Another one is for instance that several *non-comparable* sufficient conditions on m can be established for the existence of solutions for (**??**) in the one-dimensional case under some evenness assumptions on m (see [**?**], Section 2), and these solutions may not be in the interior of the positive cone.

The present work is a natural continuation of the research started in [**?**], where m was considered (when $N > 1$) to be radially symmetric. Let us note that the nonlinearity studied there was $f(\xi) = \xi^p$. One of the most important differences between ξ^p and the nonlinearities treated in this paper is that here (**??**) is no longer homogeneous in m (i.e. (**??**) may admit a solution but km may not ($k > 0$ constant), and viceversa), and the homogeneity was crucial in the existence proofs given in [**?**].

We shall primarily rely on the well-known sub- and supersolution method in the presence of weak sub and supersolutions (see e.g. [**?**], Theorem 4.9). One of the reasons is that the existence of supersolutions represent no difficulty, see Remark 2.3 below. In order to supply (strictly positive) subsolutions, we shall divide the domain in parts and construct "subsolutions" in each of them, and later check that they can be joined appropriately to get a subsolution in the entire domain. This last fact depends on obtaining estimates for the normal derivatives of these subsolutions on the boundaries of the subdomains. In [**?**] these bounds could be computed rather explicitly making use of the radial symmetry of m (and the fact that Ω was a ball) but in the present situation those computations cannot be done any more. Let us mention that here the key tool will be an estimate due to Morel and Oswald, see Lemma 2.1 below.

In Theorem 3.1 we shall state a sufficient condition on m for the existence of solutions of (**??**), while in Theorem 3.2 we shall provide a "local" necessary condition and a "global" one in Corollary 3.3 under an additional assumption on m. We observe that this last condition is of similar type to the one in Theorem 3.1. In order to relate these results to others already existing, we mention that two necessary conditions were proved for some particular radial functions in [**?**], Theorem 3.4 (see also Remark 3.5 there), and as far as we know there are no other results (other than the obvious condition $m^+ \not\equiv 0$ implied by the maximum principle). Concerning the matter of sufficient conditions, the only theorem we found in the literature, apart from the ones proved in [**?**] for m radial, is that there exists a solution for (**??**) provided that the solution of the linear problem $-\Delta\phi = m$ in Ω, $\phi = 0$ on $\partial\Omega$, satisfies $\phi > 0$ in Ω (see [**?**], Theorem 4.4, or [**?**], Theorem 10.6). As a matter of fact, this even holds for linear second order elliptic operators with nonnegative zero order coefficient. We note however that the aforementioned condition is far from being necessary in the sense that there are examples of (**??**) having a solution but with the corresponding ϕ satisfying $\phi < 0$ in Ω (cf. [**?**]). Let us finally mention that for m smooth an $p \in (0, 1)$ it is known that the problem $-\Delta u = mu^p$ in Ω, $u = 0$ on $\partial\Omega$ admits a (nontrivial) nonnegative solution if and only if $m(x_0) > 0$ for some $x_0 \in \Omega$ (see e.g. [**?**] or [**?**]).

We conclude this introduction with some few words on the case of a general second order elliptic operator. We believe that at least some of the results presented here should still be true when $-\Delta$ is replaced by such differential operators. In fact, one can verify that except the use of Lemma 2.1, the proof of Theorem 3.1 can be carried out exactly as it is done here (with the obvious changes) in the case of a general operator. Hence, if a similar version of the aforementioned lemma holds for these operators (which a priori it is not clear since the proof makes use of the mean value properties for superharmonic functions), then an analogue of Theorem 3.1 can be proved in this case.

2 Preliminaries

The following estimate appeared first in an unpublished work by Morel and Oswald ([**?**]), and a nice proof can be found in the paper of Brezis and Cabré, [**?**], Lemma 3.2.

Lemma 2.1 *Let $0 \le h \in L^r(\Omega)$, $r > N$, and let u be the solution of*

$$\begin{cases} -\Delta u = h & in\ \Omega \\ u = 0 & on\ \partial\Omega. \end{cases} \tag{2.2}$$

Then there exists some $c = c(\Omega) > 0$ such that

$$u(x) \ge c\delta_\Omega(x) \int_\Omega h\delta_\Omega \qquad \text{for all } x \in \Omega,$$

$$\text{where } \delta_\Omega(x) := dist(x, \partial\Omega).$$

The next result is also known (see e.g. Theorem 3.4 in [**?**]). We present a brief sketch of the proof for the sake of completeness. Let us note that the following proof is much simpler than the one given in [**?**]. We set

$$P^\circ \doteq \text{interior of the positive cone of } C^{1,\alpha}\left(\overline{\Omega}\right), \alpha \in (0, 1).$$

Lemma 2.2 *Let $m \in L^r(\Omega)$ with $r > N$ and such that $0 \not\equiv m \ge 0$, and let f satisfying H1. Then there exists $v \in W^{2,r}(\Omega) \cap P^\circ$ solution of*

$$\begin{cases} -\Delta v = mf(v) & in\ \Omega \\ v > 0 & in\ \Omega \\ v = 0 & on\ \partial\Omega. \end{cases} \tag{2.3}$$

Proof. Let $\phi > 0$ be the solution of $-\Delta\phi = m$ in Ω and $\phi = 0$ on $\partial\Omega$. Then using the second inequality in H1 one can verify that for every $k > 0$ large enough it holds that $k(\phi + 1)$ is a supersolution of (**??**). On the other side, let $\varphi > 0$ with $\|\varphi\|_\infty = 1$ satisfying

$$\begin{cases} -\Delta\varphi = \lambda_1(m, \Omega)\,m\varphi & in\ \Omega \\ \varphi = 0 & on\ \partial\Omega, \end{cases}$$

where $\lambda_1(m, \Omega)$ denotes the (unique) positive principal eigenvalue for m. It is easy to check employing the first inequality in H1 that $\varepsilon\varphi$ is a subsolution of (**??**) for all $\varepsilon > 0$ sufficiently small, and the lemma follows. ∎

Remark 2.3 Let us mention that the construction of the supersolution made in the first part of the above proof still works if m changes sign in Ω, taking there ϕ as the solution of $-\Delta\phi = m^+$ in Ω and $\phi = 0$ on $\partial\Omega$ (where as usual we write $m = m^+ - m^-$ with $m^+ = \max(m, 0)$ and $m^- = \max(-m, 0)$). Furthermore, this is also true for a strongly uniformly elliptic differential operator with nonnegative zero order coefficient. ∎

3   Main results

Theorem 3.1 *Let Ω_0 be a $C^{1,1}$ domain with $\overline{\Omega}_0 \subset \Omega$, and let $m \in L^r(\Omega)$ with $r > N$ and $0 \not\equiv m \geq 0$ in Ω_0. Let k_1, k_2 be given by H1. There exist some $C_0, C_1 > 0$ depending only on Ω and Ω_0 such that if*

$$\left\| m^- \right\|_{L^r\left(\Omega - \overline{\Omega}_0\right)} \leq \frac{k_1 C_0}{k_2 C_1^{1-p}} \int_{\Omega_0} m \delta_{\Omega_0}^{p+1}$$

*then (**??**) has a solution $u \in W^{2,r}(\Omega)$.*

Proof. Let $\Omega - \overline{\Omega}_0 := \Omega_1$. For $M > 0$, we start constructing some $0 \leq w \in W^{2,r}(\Omega_1)$ solution of

$$\begin{cases} -\Delta w = -m^- f(w) & \text{in } \Omega_1 \\ w = 0 & \text{on } \partial\Omega \\ w = M & \text{on } \partial\Omega_0. \end{cases} \tag{3.4}$$

Let us first note that since by H1 $f(0) = 0$, it holds that $\underline{w} := 0$ is a subsolution of (**??**), and also since f is nonnegative we have that $\overline{w} := M$ is a supersolution of (**??**). It follows from Theorem 4.9 in [**?**] that there exists some w weak solution of (**??**) satisfying $0 \leq w \leq M$. Furthermore, by standard arguments we may conclude that $w \in W^{2,r}(\Omega_1)$ (indeed, it is enough to note that if $z \in W^{2,r}(\Omega_1)$ is the unique solution of the problem $-\Delta z = -m^- f(w)$ in Ω_1, $z = 0$ on $\partial\Omega$ and $z = M$ on $\partial\Omega_0$, then the maximum principle implies that $z = w$).

We claim now that there exists some $C > 0$ depending only on Ω_1 such that if $M := \left[C k_2 \left\| m^- \right\|_{L^r}\right]^{1/(1-p)}$ then $w > 0$ in Ω_1 (k_2 given by H1). To confirm this, let $\theta, \psi \in W^{2,r}(\Omega_1)$ be the unique solutions of

$$\begin{cases} \Delta\theta = 0 & \text{in } \Omega_1 \\ \theta = 0 & \text{on } \partial\Omega \\ \theta = 1 & \text{on } \partial\Omega_0, \end{cases} \qquad \begin{cases} -\Delta\psi = m^- & \text{in } \Omega_1 \\ \psi = 0 & \text{on } \partial\Omega_1. \end{cases}$$

From the Sobolev imbedding theorems and the $W^{2,r}$-theory for elliptic equations (e.g. [**?**], Theorem 2.4.2.5) we derive that

$$|\psi| \leq \|\nabla\psi\|_{L^\infty} \delta_{\Omega_1} \leq \|\psi\|_{C^1} \delta_{\Omega_1} \leq c_0 \|\psi\|_{W^{2,r}} \delta_{\Omega_1} \leq c_1 \left\| m^- \right\|_{L^r} \delta_{\Omega_1}$$

for some $c_1 = c_1(\Omega_1) > 0$, and we also have that $\theta > c_2 \delta_{\Omega_1}$ in Ω_1 for some $c_2 = c_2(\Omega_1) > 0$.

On the other hand, since $w \leq M$, recalling H1 we get that in Ω_1

$$-\Delta\left(M\theta - k_2 M^p \psi\right) = -m^- k_2 M^p \leq -m^- k_2 w^p \leq -m^- f(w) = -\Delta w$$

and so

$$w \geq M\theta - k_2 M^p \psi > \left(c_2 M - c_1 k_2 M^p \left\| m^- \right\|_{L^r}\right) \delta_{\Omega_1} \qquad \text{in } \Omega_1$$

and the claim is proved. We fix for rest of the proof M as in the aforementioned claim.

Let ν denote the outward unit normal to $\partial\Omega_0$. Let us observe now that

$$\left| \frac{\partial w}{\partial \nu} \right| \leq \|w\|_{C^1} \leq c_0 \|w\|_{W^{2,r}} \leq c_1 \left(M + \left\| m^- \right\|_{L^r} \|f(w)\|_{L^\infty}\right) \leq \tag{3.5}$$

$$c_1 \left(M + k_2 M^p \left\| m^- \right\|_{L^r}\right) \leq 2c_1 \left[\max\{1, C\} k_2 \left\| m^- \right\|_{L^r}\right]^{1/(1-p)} :=$$

$$c_3 \left[c_4 k_2 \left\| m^- \right\|_{L^r}\right]^{1/(1-p)},$$

with c_3 and c_4 depending only on Ω_1.

On the other side, let $v > 0$ be the solution of (??) with Ω_0 in place of Ω. Taking into account H1 and Lemma 2.1, there exists $c_5 = c_5(\Omega_0) > 0$ such that $v \geq c_5 k_1 \delta_{\Omega_0} \int_{\Omega_0} m v^p \delta_{\Omega_0}$ and so raising this inequality to the power p, multiplying by $m\delta_{\Omega_0}$ and integrating over Ω_0 we obtain $\left(\int_{\Omega_0} m v^p \delta_{\Omega_0} \right)^{1-p} \geq$ $(c_5 k_1)^p \int_{\Omega_0} m\delta_{\Omega_0}^{1+p}$ and hence

$$v \geq \left[c_5 k_1 \int_{\Omega_0} m\delta_{\Omega_0}^{1+p} \right]^{1/(1-p)} \delta_{\Omega_0}.$$

Define now $u := M + v$. Then $\partial u / \partial \nu \leq - \left[c_5 k_1 \int_{\Omega_0} m\delta_{\Omega_0}^{p+1} \right]^{1/(1-p)}$ and $u = w$ on $\partial\Omega_0$. Hence, if we set $\omega := u$ in $\overline{\Omega}_0$ and $\omega := w$ in $\overline{\Omega} - \Omega_0$ it follows applying the divergence theorem (as stated e.g. in [?], p. 742) that ω is a weak subsolution of (??) if $\partial u / \partial \nu \leq \partial w / \partial \nu$. Recalling (??) this occurs if

$$c_3^{1-p} c_4 k_2 \left\| m^- \right\|_{L^r} \leq c_5 k_1 \int_{\Omega_0} m\delta_{\Omega_0}^{p+1}$$

and thus, taking into account Remark 2.3, this ends the proof. ∎

We denote with $B_R(x_0)$ the open ball in $\mathbb{R}^N$ centered at x_0 with radius R, and we write $(-\Delta)^{-1} : L^r(\Omega) \to L^\infty(\Omega)$ for the solution operator of (??). We also set

$$C_{N,p} := \frac{(1-p)^2}{2(N(1-p) + 2p)}. \tag{3.6}$$

Theorem 3.2 *Let $m \in L^r(\Omega)$ with $r > N$, let $C_{N,p}$ be given by (??) and let k_1, k_2 be given by H1. If there exists a solution $u \in C\left(\overline{\Omega}\right)$ of (??), then*

$$\frac{C_{N,p}}{\left\| (-\Delta)^{-1} \right\|} \sup_{B_R(x_0) \in \mathfrak{B}} \left[m_R R^2 \right] < \frac{k_2}{k_1} \left\| m^+ \right\|_{L^r(\Omega)}, \qquad where \tag{3.7}$$

$$\mathfrak{B} := \{ B_R(x_0) \subset \Omega : m \leq 0 \text{ in } B_R(x_0) \}, \qquad m_R := \inf_{B_R(x_0)} m^-.$$

Proof. We proceed by contradiction. If (??) does not hold, then there exists some $B_R(x_0) \in \mathfrak{B}$ such that

$$\frac{C_{N,p} m_R R^2}{\left\| (-\Delta)^{-1} \right\|} \geq \frac{k_2}{k_1} \left\| m^+ \right\|_{L^r(\Omega)}. \tag{3.8}$$

Let $\beta := 1/(1-p)$, and for $x \in \overline{B}_R(x_0)$ define

$$w(x) := \left[k_1 C_{N,p} m_R |x - x_0|^2 \right]^\beta.$$

After some computations one can verify that $\Delta w \leq k_1 m^- w^p$ in $B_R(x_0)$. Let u be a solution of (??). In particular, it holds that $\Delta u \geq k_1 m^- u^p$ in $B_R(x_0)$. Also, taking into account H1, from (??) we deduce that

$$\| u \|_{L^\infty(\Omega)} \leq \left[k_2 \left\| (-\Delta)^{-1} \right\| \left\| m^+ \right\|_{L^r(\Omega)} \right]^\beta. \tag{3.9}$$

Moreover, if $x \in \partial B_R(x_0)$, employing (??) and (??) we derive that

$$w(x) = \left(k_1 C_{N,p} m_R R^2 \right)^\beta \geq \left[k_2 \left\| (-\Delta)^{-1} \right\| \left\| m^+ \right\|_{L^r(\Omega)} \right]^\beta \geq \| u \|_{L^\infty(\Omega)} \geq u(x).$$

It follows by the comparison principle that $w \geq u$ in $B_R(x_0)$, but $w(x_0) = 0$, contradicting the fact that $u > 0$ in Ω. ∎

Corollary 3.3 *Let $\Omega_1 \subset \Omega$ be a convex domain and let $m \in L^r(\Omega)$ with $r > N$ and such that in Ω_1 m is convex and $m \leq 0$. If there exists a solution $u \in C(\overline{\Omega})$ of (**??**), then*

$$\frac{4C_{N,p}}{27 |\Omega_1| \left\|(-\Delta)^{-1}\right\|} \int_{\Omega_1} m^- \delta_{\Omega_1}^2 < \frac{k_2}{k_1} \left\|m^+\right\|_{L^r\left(\Omega - \overline{\Omega}_1\right)}. \tag{3.10}$$

Proof. Let $\alpha := 2/3$ and let $x_1 \in \Omega_1$. We set $R_1 := \alpha \delta_{\Omega_1}(x_1)$ and let $y \in B_{R_1}(x_1)$. Observe that $z_y(t) := x_1 + t(y - x_1) \in \Omega_1$ for every $t \in [0, 1/\alpha]$ since $\left|z_y(t) - x_1\right| < \delta_{\Omega_1}(x_1)$. Define $M(t) := m^-\left(z_y(t)\right)$. Then $M(t)$ is concave in $[0, 1/\alpha]$ and hence

$$m^-(y) = M(1) \geq \alpha M(1/\alpha) + (1 - \alpha) M(0) \geq (1 - \alpha) M(0) = m^-(x_1)/3.$$

It follows that $\inf_{B_{R_1}(x_1)} m^- \geq m^-(x_1)/3$. Now, if (**??**) possesses a solution $u \in C(\overline{\Omega})$, by Theorem 3.2 we obtain that

$$\frac{k_2}{k_1} \left\|m^+\right\|_{L^r(\Omega)} > \frac{C_{N,p}}{\left\|(-\Delta)^{-1}\right\|} \sup_{B_R(x_0) \in \mathfrak{B}} \left[m_R R^2\right] \geq$$

$$\frac{C_{N,p}}{3 \left\|(-\Delta)^{-1}\right\|} m^-(x_1) R_1^2 = \frac{4C_{N,p}}{27 \left\|(-\Delta)^{-1}\right\|} m^-(x_1) \delta_{\Omega_1}^2(x_1)$$

for every $x_1 \in \Omega_1$. Integrating this inequality in Ω_1 with respect to x_1 gives (**??**) and thus the corollary is proved. ∎

Remark 3.4 We observe that $C_{N,p} \to 0$ when $p \to 1$ and thus (**??**) and (**??**) are satisfied for any m provided that p is close enough to 1. Let us mention that this must occur since, at least when $m^- \in L^\infty(\Omega)$, $f(\xi) = \xi^p$, and either $N = 1$ or $N > 1$ and m is radial with $0 \not\equiv m \geq 0$ in some $B_r(0)$, it is known that (**??**) has a solution if p is sufficiently close to 1 (cf. [**?**], Theorems 2.1 (i) and 3.2). ∎

References

[1] C. Bandle, M. Pozio, A. Tesei, *The asymptotic behavior of the solutions of degenerate parabolic equations,* Trans. Amer. Math. Soc. **303** (1987), 487-501.

[2] H. Brezis, X. Cabré, *Some simple nonlinear PDE's without solutions*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **1** (1998), 223–262.

[3] M. Cuesta, P. Takáč, *A strong comparison principle for positive solutions of degenerate elliptic equations*, Differential Integral Equations **13** (2000), 721–746.

[4] Y. Du, *Order structure and topological methods in nonlinear partial differential equations. Vol. 1. Maximum principles and applications*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.

[5] T. Godoy, U. Kaufmann, *On the existence of positive solutions for periodic parabolic sublinear problems,* Abstr. Appl. Anal. **2003** (2003), 975-984.

[6]   T. Godoy, U. Kaufmann, *Periodic parabolic problems with nonlinearities indefinite in sign*, Publ. Mat. **51** (2007), 45-57.

[7]   T. Godoy, U. Kaufmann, *On strictly positive solutions for some semilinear elliptic problems*, NoDEA Nonlinear Differ. Equ. Appl. **20** (2013), 779-795.

[8]   P. Grisvard, Elliptic problems in nonsmooth domains. Monographs and Studies in Mathematics, 24. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[9]   J. Hernández, F. Mancebo, *Singular elliptic and parabolic equations*, M. Chipot (ed.) et al., Handbook of differential equations: Stationary partial differential equations. Vol. III. Amsterdam: Elsevier/North Holland. Handbook of Differential Equations, 317-400 (2006).

[10] J. Hernández, F. Mancebo, J. Vega, *On the linearization of some singular, nonlinear elliptic problems and applications*, Ann. Inst. H. Poincaré Anal. Non Linéaire **19** (2002), 777–813.

[11] J. Morel, L. Oswald, *A uniform formulation for the Hopf maximum principle*, preprint, 1985.