Spherical Functions of Fundamental K-Types Associated with the n-Dimensional Sphere

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Abstract. In this paper, we describe the irreducible spherical functions of fundamental K-types associated with the pair $(G,K) = (\mathrm{SO}(n+1), \mathrm{SO}(n))$ in terms of matrix hypergeometric functions. The output of this description is that the irreducible spherical functions of the same K-fundamental type are encoded in new examples of classical sequences of matrix-valued orthogonal polynomials, of size 2 and 3, with respect to a matrix-weight W supported on $[0,1]$. Moreover, we show that W has a second order symmetric hypergeometric operator D.

Key words: matrix-valued spherical functions; matrix orthogonal polynomials; the matrix hypergeometric operator; n-dimensional sphere

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1 Introduction

The theory of spherical functions dates back to the classical papers of É. Cartan and H. Weyl; they showed that spherical harmonics arise in a natural way from the study of functions on the n-dimensional sphere $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$. The first general results in this direction were obtained in 1950 by Gel'fand, who considered zonal spherical functions of a Riemannian symmetric space G/K. In this case we have a decomposition $G = KAK$. When the Abelian subgroup A is one dimensional, the restrictions of zonal spherical functions to A can be identified with hypergeometric functions, providing a deep and fruitful connection between group representation theory and special functions. In particular when G is compact this gives a one to one correspondence between all zonal spherical functions of the symmetric pair (G,K) and a sequence of orthogonal polynomials.

In light of this remarkable background it is reasonable to look for an extension of the above results, by considering matrix-valued irreducible spherical functions on G of a general K-type. This was accomplished for the first time in the case of the complex projective plane $P_2(\mathbb{C}) = \mathrm{SU}(3)/\mathrm{U}(2)$ in [5]. This seminal work gave rise to a series of papers including [6, 7, 8, 10, 14, 15, 16, 17, 18, 19], where one considers matrix valued spherical functions associated to a compact symmetric pair (G,K) of rank one, arriving at sequences of matrix valued orthogonal polynomials of one real variable satisfying an explicit three-term recursion relation, which are also eigenfunctions of a second order matrix differential operator (bispectral property).

The very explicit results contained in this paper are obtained for certain K-types, namely the fundamental K-types. Also, the detailed construction of sequences of matrix orthogonal polynomials out of these irreducible spherical functions, following the general pattern established in [5], gives new examples of *classical sequences* of matrix-valued orthogonal polynomials of size 2 and 3. For the general notions concerning matrix-valued orthogonal polynomials see [9].

Interesting generalizations of these sequences are given in [20], where the coefficients of the three term recursion relation satisfied by them is exhibited.

The present paper is an outgrowth of the results of [25, Chapter 5] and we are currently working on the extension of these results for the spherical functions of any K-type associated with the n-dimensional sphere. Using [23], one can obtain the corresponding results for the spherical functions of any K-type associated with n-dimensional real projective space. The starting point is to describe the irreducible spherical functions associated with the pair $(G,K) = (\mathrm{SO}(n{+}1),\mathrm{SO}(n))$ in terms of eigenfunctions of a matrix linear differential operator of order two. The output of this description is that the irreducible spherical functions of the same fundamental K-type are encoded in a sequence of matrix valued orthogonal polynomials.

Briefly the main results of this paper are the following. After some preliminaries, in Section 3 we study the eigenfunctions of an operator Δ on G, which is closely related to the Casimir operator. Every spherical function Φ has to be eigenfunction of this operator Δ; considering the KAK-decomposition

$$\mathrm{SO}(n+1) = \mathrm{SO}(n)\mathrm{SO}(2)\mathrm{SO}(n)$$

and choosing an appropriate coordinate y on an open subset of A, we translate the condition $\Delta\Phi = \lambda\Phi$, $\lambda \in \mathbb{C}$, into a matrix valued differential equation $\widetilde{D}H = \lambda H$ on the open interval $(0,1)$, where H is the restriction of Φ to $\mathrm{SO}(2)$. The property of the spherical functions

$$\Phi(xgy) = \pi(x)\Phi(g)\pi(y), \qquad g \in G, \qquad x,y \in K,$$

tell us that Φ is determined by its K-type and the function H.

In Section 4 we first explicitly describe all the irreducible spherical functions of the symmetric pair $(G,K) = (\mathrm{SO}(n+1),\mathrm{SO}(n))$ with M-irreducible K-types, with $M = \mathrm{SO}(n-1)$, the centralizer of the subgroup A in K; we give these expressions in terms of the hypergeometric function ${}_2F_1$.

In Section 5 the operator $\widetilde{D}$ is studied in detail when the K-types correspond to fundamental representations. Certain K-fundamental types are M irreducible, and therefore they were already considered en Section 4; besides, when n is odd there is a particular fundamental K-type which has three M-submodules, this case is studied in the last section of this work. For the rest of the cases we considered separately when n is even and when n is odd. Although, in both cases we worked with the concrete realizations of the fundamental representations considering the exterior powers of the standard representation of $\mathrm{SO}(n)$:

$$\Lambda^1(\mathbb{C}^n),\ \Lambda^2(\mathbb{C}^n),\ \ldots,\ \Lambda^{\ell-1}(\mathbb{C}^n),$$

with $n = 2\ell$ or $n = 2\ell+1$.

In Section 6 we conjugate the operator $\widetilde{D}$, by using the polynomial function

$$\Psi(y) = \begin{pmatrix} 2y-1 & 1 \\ 1 & 2y-1 \end{pmatrix},$$

whose columns correspond to irreducible spherical functions, in order to obtain a matrix-valued hypergeometric operator $D = \Psi^{-1}\widetilde{D}\Psi$:

$$DP = y(1-y)P'' + (C - yU)P' - VP,$$

with

$$C = \begin{pmatrix} (n/2+1) & 1 \\ 1 & (n/2+1) \end{pmatrix}, \qquad U = (n+2)I, \qquad V = \begin{pmatrix} p & 0 \\ 0 & n-p \end{pmatrix}.$$

Then, we study all the possible eigenvalues corresponding to irreducible spherical functions and all the polynomial eigenfunctions of D.

In Section 7, for any fundamental K-type $(\Lambda^k(\mathbb{C}^n))$ with $1 \leq p \leq \ell - 1$, we find a matrix-weight W, which is a scalar multiple of

$$W = (y(1-y))^{n/2-1} \begin{pmatrix} p(2y-1)^2 + n - p & n(2y-1) \\ n(2y-1) & (n-p)(2y-1)^2 + p \end{pmatrix},$$

such that D is a symmetric operator with respect to the inner product defined among continuous vector-valued functions on $[0,1]$ by

$$\langle P_1, P_2 \rangle_W = \int_0^1 P_2^*(y) W(y) P_1(y) dy.$$

Also we prove that every spherical function gives a vector polynomial eigenfunction P of D. Therefore we obtain the following explicit expression of P in terms of the matrix hypergeometric function for any irreducible spherical function

$$P(y) = \sum_{j=0}^{w} \frac{y^j}{j!} [C; U; V + \lambda]_j P(0),$$

see Theorem 7.6.

In Section 8 for each pair (n, p) we construct a sequence of matrix orthogonal polynomials $\{P_w\}_{w \geq 0}$ of size 2 with respect to the weight function W, which are eigenfunctions of the symmetric differential operator D. Namely,

$$DP_w = P_w \begin{pmatrix} \lambda(w,0) & 0 \\ 0 & \lambda(w,1) \end{pmatrix},$$

where

$$\lambda(w,\delta) = \begin{cases} -w(w+n+1) - p & \text{if} \quad \delta = 0, \\ -w(w+n+1) - n + p & \text{if} \quad \delta = 1. \end{cases}$$

Finally, in Section 9 we develop the same techniques in order to obtain analogous results for irreducible spherical functions of the particular K-fundamental type $\Lambda^\ell(\mathbb{C}^n)$ for which we have three M-submodules instead of only two. This only occurs when n is of the form $2\ell + 1$.

It is worth to notice that, unlike the other cases, the 3×3 matrix-weight built here does reduce to a smaller size.

2 Preliminaries

2.1 Spherical functions

Let G be a locally compact unimodular group and let K be a compact subgroup of G. Let $\hat{K}$ denote the set of all equivalence classes of complex finite dimensional irreducible representations of K; for each $\delta \in \hat{K}$, let ξ_δ denote the character of δ, $d(\delta)$ the degree of δ, i.e. the dimension of any representation in the class δ, and $\chi_\delta = d(\delta)\xi_\delta$. We shall choose once and for all the Haar measure dk on K normalized by $\int_K dk = 1$.

We shall denote by V a finite dimensional vector space over the field $\mathbb{C}$ of complex numbers and by of all linear transformations of V into V. Whenever we refer to a topology on such a vector space we shall be talking about the unique Hausdorff linear topology on it.

Definition 2.1. A spherical function Φ on G of type $\delta \in \hat{K}$ is a continuous function on G with values in $\mathrm{End}(V)$ such that

i) $\Phi(e) = I$ (I is the identity transformation);

ii) $\Phi(x)\Phi(y) = \int_K \chi_\delta(k^{-1})\Phi(xky)dk$ for all $x, y \in G$.

The reader can find a number of general results in [21] and [4]. For our purpose it is appropriate to recall the following facts.

Proposition 2.2 ([21, Proposition 1.2])**.** *If $\Phi : G \longrightarrow \mathrm{End}(V)$ is a spherical function of type δ then:*

i) $\Phi(k_1gk_2) = \Phi(k_1)\Phi(g)\Phi(k_2)$*, for all* $k_1, k_2 \in K$*,* $g \in G$*;*

ii) $k \mapsto \Phi(k)$ *is a representation of* K *such that any irreducible subrepresentation belongs to* δ*.*

Concerning the definition, let us point out that the spherical function Φ determines its type univocally (Proposition 2.2) and let us say that the number of times that δ occurs in the representation $k \mapsto \Phi(k)$ is called the height of Φ.

A spherical function $\Phi : G \longrightarrow \mathrm{End}(V)$ is called irreducible if V has no proper subspace invariant by $\Phi(g)$ for all $g \in G$.

If G is a connected Lie group, it is not difficult to prove that any spherical function $\Phi : G \longrightarrow \mathrm{End}(V)$ is differentiable (C^∞), and moreover that it is analytic. Let $D(G)$ denote the algebra of all left invariant differential operators on G and let $D(G)^K$ denote the subalgebra of all operators in $D(G)$ which are invariant under all right translations by elements in K.

In the following proposition (V, π) will be a finite dimensional representation of K such that any irreducible subrepresentation belongs to the same class $\delta \in \hat{K}$.

Proposition 2.3. *A function $\Phi : G \longrightarrow \mathrm{End}(V)$ is a spherical function of type δ if and only if*

i) Φ *is analytic;*

ii) $\Phi(k_1gk_2) = \pi(k_1)\Phi(g)\pi(k_2)$*, for all* $k_1, k_2 \in K$*,* $g \in G$*, and* $\Phi(e) = I$*;*

iii) $[D\Phi](g) = \Phi(g)[D\Phi](e)$*, for all* $D \in D(G)^K$*,* $g \in G$*.*

Moreover, we have that the eigenvalues $[D\Phi](e)$, $D \in D(G)^K$, characterize the spherical functions Φ as stated in the following proposition.

Proposition 2.4 ([21, Remark 4.7])**.** *Let $\Phi, \Psi : G \longrightarrow \mathrm{End}(V)$ be two spherical functions on a connected Lie group G of the same type $\delta \in K$. Then $\Phi = \Psi$ if and only if $(D\Phi)(e) = (D\Psi)(e)$ for all $D \in D(G)^K$.*

Let us observe that if $\Phi : G \longrightarrow \mathrm{End}(V)$ is a spherical function, then $\Phi : D \mapsto [D\Phi](e)$ maps $D(G)^K$ into $\mathrm{End}_K(V)$ ($\mathrm{End}_K(V)$ denotes the space of all linear maps of V into V which commutes with $\pi(k)$ for all $k \in K$) defining a finite dimensional representation of the associative algebra $D(G)^K$. Moreover, the spherical function is irreducible if and only if the representation $\Phi : D(G)^K \longrightarrow \mathrm{End}_K(V)$ is irreducible. We quote the following result from [19].

Proposition 2.5 ([19, Proposition 2.5])**.** *Let G be a connected reductive linear Lie group. Then the following properties are equivalent:*

i) $D(G)^K$ *is commutative;*

ii) every irreducible spherical function of (G, K) *is of height one.*

In this paper the pair (G, K) is $(\mathrm{SO}(n+1), \mathrm{SO}(n))$. Then, it is known that $D(G)^K$ is an Abelian algebra; moreover, $D(G)^K$ is isomorphic to $D(G)^G \otimes D(K)^K$ (see in [13, Theorem 10.1] or [1]), where $D(G)^G$ (resp. $D(K)^K$) denotes the subalgebra of all operators in $D(G)$ (resp. $D(K)$) which are invariant under all right translations by elements in G (resp. K).

An immediate consequence of this is that all irreducible spherical functions of our pair (G, K) are of height one.

Spherical functions of type δ (see in [21, Section 3]) arise in a natural way upon considering representations of G. If $g \mapsto U(g)$ is a continuous representation of G, say on a finite dimensional vector space E, then

$$P_\delta = \int_K \chi_\delta(k^{-1}) U(k) dk$$

is a projection of E onto $P_\delta E = E(\delta)$. If $P_\delta \neq 0$ the function $\Phi : G \longrightarrow \mathrm{End}(E(\delta))$ defined by

$$\Phi(g)a = P_\delta U(g)a, \qquad g \in G, \qquad a \in E(\delta), \tag{2.1}$$

is a spherical function of type δ. In fact, if $a \in E(\delta)$ we have

$$\begin{aligned}\Phi(x)\Phi(y)a = P_\delta U(x) P_\delta U(y) a &= \int_K \chi_\delta(k^{-1}) P_\delta U(x) U(k) U(y) a dk \\ &= \left(\int_K \chi_\delta(k^{-1}) \Phi(xky) dk \right) a.\end{aligned}$$

If the representation $g \mapsto U(g)$ is irreducible then the associated spherical function Φ is also irreducible. Conversely, any irreducible spherical function on a compact group G arises in this way from a finite dimensional irreducible representation of G.

2.2 Root space structure of $\mathfrak{so}(n, \mathbb{C})$

Let E_{ik} denote the square matrix with a 1 in the ik-entry and zeros elsewhere; and let us consider the matrices

$$I_{ki} = E_{ik} - E_{ki}, \qquad 1 \leq i, k \leq n.$$

Then, the set $\{I_{ki}\}_{i<k}$ is a basis of the Lie algebra $\mathfrak{so}(n)$. These matrices satisfy the following commutation relations

$$[I_{ki}, I_{rs}] = \delta_{ks} I_{ri} + \delta_{ri} I_{sk} + \delta_{is} I_{kr} + \delta_{rk} I_{is}.$$

If we assume that $k > i$, $r > s$ then we have

$$[I_{ki}, I_{is}] = I_{sk}, \qquad [I_{ki}, I_{rk}] = I_{ri}, \qquad [I_{ki}, I_{ri}] = I_{kr}, \qquad [I_{ki}, I_{ks}] = I_{is},$$

and all the other brackets are zero. From this it easily follows that the set

$$\{I_{p,p-1} : 2 \leq p \leq n\}$$

generates the Lie algebra $\mathfrak{so}(n)$.

Proposition 2.6. *Given $n \in \mathbb{N}$, we have that the operator*

$$Q_n = \sum_{1 \leq i, k \leq n} I_{ki}^2 \in D(\mathrm{SO}(n))$$

is right invariant under $\mathrm{SO}(n)$, *i.e.*

$$Q_n \in D(\mathrm{SO}(n))^{\mathrm{SO}(n)}, \qquad \forall\, n \in \mathbb{N}_0.$$

Proof. To prove that Q_n is right invariant under $\mathrm{SO}(n)$ it is enough to prove that $\dot I_{p,p-1}(Q_n)=0$ for all $2\le p\le n$. We have

$$\dot I_{p,p-1}(Q_n)=\sum_{1\le i,k\le n}\big([I_{p,p-1},I_{ki}]I_{ki}+I_{ki}[I_{p,p-1},I_{ki}]\big).$$

Then

$$\begin{aligned}\dot I_{p,p-1}(Q_n)&=\sum_{1\le i\le n}(I_{ip}I_{p-1,i}+I_{p-1,i}I_{ip})+\sum_{1\le k\le n}(I_{k,p-1}I_{kp}+I_{kp}I_{k,p-1})\\&\quad+\sum_{1\le k\le n}(I_{pk}I_{k,p-1}+I_{k,p-1}I_{pk})+\sum_{1\le i\le n}(I_{p-1,i}I_{p,i}+I_{p,i}I_{p-1,i})=0.\end{aligned}$$

This proves the proposition. ∎

2.3 The operator $Q_{2\ell}$

Let us assume that $n=2\ell$. We look at a root space decomposition of $\mathfrak{so}(n)$ in terms of the basis elements I_{ki}, $1\le i<k\le n$.

The linear span

$$\mathfrak{h}=\langle I_{21},I_{43},\dots,I_{2\ell,2\ell-1}\rangle_{\mathbb{C}}$$

is a Cartan subalgebra of $\mathfrak{so}(n,\mathbb{C})$. To find the root vectors it is convenient to visualize the elements of $\mathfrak{so}(n,\mathbb{C})$ as $\ell\times\ell$ matrices of 2×2 blocks. Thus $\mathfrak{h}$ is the subspace of all diagonal matrices of 2×2 skew-symmetric blocks. The subspaces of all matrices A with a block A_{jk} of size two, $1\le j<k\le\ell$, in the place (j,k) and $-A_{jk}^t$ in the place (k,j) with zeros in all other places, are $\mathrm{ad}(\mathfrak{h})$-stable. Let

$$H=i(x_1I_{21}+\dots+x_\ell I_{2\ell,2\ell-1})\in\mathfrak{h},$$

for $x_1,\dots,x_\ell\in\mathbb{R}$. Then $[H,A]=\lambda(H)A$, $\forall\,H\in\mathfrak{h}$, if and only if for every A_{jk} we have

$$x_j(H)iI_{2j,2j-1}A_{jk}-x_k(H)iA_{jk}I_{2k,2k-1}=\lambda(H)A_{jk},\qquad\forall\,H\in\mathfrak{h}.$$

Up to a scalar, the nontrivial solutions of these linear equations are the following:

$$\begin{aligned}A_{jk}&=\begin{pmatrix}1&\pm i\\\pm i&-1\end{pmatrix}\qquad\text{with corresponding}\quad\lambda=\mp(x_j+x_k),\\A_{jk}&=\begin{pmatrix}1&\mp i\\\pm i&1\end{pmatrix}\qquad\text{with corresponding}\quad\lambda=\mp(x_j-x_k).\end{aligned}$$

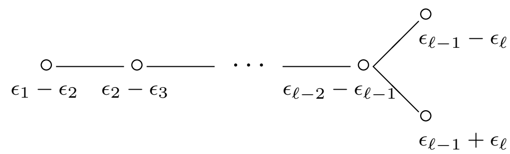
Let $\epsilon_j\in\mathfrak{h}^*$ be defined by $\epsilon_j(H)=x_j$ for $1\le j\le\ell$. Then for $1\le j<k\le\ell$, the following matrices are root vectors of $\mathfrak{so}(2\ell,\mathbb{C})$:

$$\begin{aligned}X_{\epsilon_j+\epsilon_k}&=I_{2k-1,2j-1}-I_{2k,2j}-i(I_{2k-1,2j}+I_{2k,2j-1}),\\X_{-\epsilon_j-\epsilon_k}&=I_{2k-1,2j-1}-I_{2k,2j}+i(I_{2k-1,2j}+I_{2k,2j-1}),\\X_{\epsilon_j-\epsilon_k}&=I_{2k-1,2j-1}+I_{2k,2j}-i(I_{2k-1,2j}-I_{2k,2j-1}),\\X_{-\epsilon_j+\epsilon_k}&=I_{2k-1,2j-1}+I_{2k,2j}+i(I_{2k-1,2j}-I_{2k,2j-1}).\end{aligned}\tag{2.2}$$

Thus, if we choose the following set of positive roots

$$\Delta^+=\{\epsilon_j+\epsilon_k,\epsilon_j-\epsilon_k:1\le j<k\le\ell\},$$

then the Dynkin diagram of $\mathfrak{so}(2\ell,\mathbb{C})$ is D_ℓ:



By looking at the 2×2 blocks A_{jk} of the different roots, namely

$$X_{\epsilon_j+\epsilon_k} = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \qquad X_{-\epsilon_j-\epsilon_k} = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix},$$
$$X_{\epsilon_j-\epsilon_k} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \qquad X_{-\epsilon_j+\epsilon_k} = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix},$$

it is easy to obtain the following inverse relations

$$I_{2k-1,2j-1} = \tfrac{1}{4}\big(X_{\epsilon_j+\epsilon_k} + X_{-\epsilon_j-\epsilon_k} + X_{\epsilon_j-\epsilon_k} + X_{-\epsilon_j+\epsilon_k}\big),$$
$$I_{2k,2j} = \tfrac{1}{4}\big(- X_{\epsilon_j+\epsilon_k} - X_{-\epsilon_j-\epsilon_k} + X_{\epsilon_j-\epsilon_k} + X_{-\epsilon_j+\epsilon_k}\big),$$
$$I_{2k,2j-1} = \tfrac{i}{4}\big(X_{\epsilon_j+\epsilon_k} - X_{-\epsilon_j-\epsilon_k} - X_{\epsilon_j-\epsilon_k} + X_{-\epsilon_j+\epsilon_k}\big),$$
$$I_{2k-1,2j} = \tfrac{i}{4}\big(X_{\epsilon_j+\epsilon_k} - X_{-\epsilon_j-\epsilon_k} + X_{\epsilon_j-\epsilon_k} - X_{-\epsilon_j+\epsilon_k}\big).$$

From this it follows that

$$I^2_{2k-1,2j-1} + I^2_{2k,2j} + I^2_{2k,2j-1} + I^2_{2k-1,2j}$$
$$= \tfrac{1}{4}\big(X_{\epsilon_j+\epsilon_k}X_{-\epsilon_j-\epsilon_k} + X_{-\epsilon_j-\epsilon_k}X_{\epsilon_j+\epsilon_k} + X_{\epsilon_j-\epsilon_k}X_{-\epsilon_j+\epsilon_k} + X_{-\epsilon_j+\epsilon_k}X_{\epsilon_j-\epsilon_k}\big).$$

Therefore

$$Q_{2\ell} = \sum_{1\le j\le \ell} I^2_{2j,2j-1} + \frac{1}{4}\sum_{1\le j<k\le\ell}\big(X_{\epsilon_j+\epsilon_k}X_{-\epsilon_j-\epsilon_k} + X_{-\epsilon_j-\epsilon_k}X_{\epsilon_j+\epsilon_k}$$
$$+ X_{\epsilon_j-\epsilon_k}X_{-\epsilon_j+\epsilon_k} + X_{-\epsilon_j+\epsilon_k}X_{\epsilon_j-\epsilon_k}\big).$$

Now using the expressions in (2.2) we get

$$[X_{\epsilon_j+\epsilon_k}, X_{-\epsilon_j-\epsilon_k}] = -4i(I_{2j,2j-1} + I_{2k,2k-1}),$$
$$[X_{\epsilon_j-\epsilon_k}, X_{-\epsilon_j+\epsilon_k}] = -4i(I_{2j,2j-1} - I_{2k,2k-1}).$$

Thus $Q_{2\ell}$ becomes

$$Q_{2\ell} = \sum_{1\le j\le\ell} I^2_{2j,2j-1} - 2\sum_{1\le j\le\ell}(\ell-j)iI_{2j,2j-1}$$
$$+ \sum_{1\le j<k\le\ell}\tfrac{1}{2}\big(X_{-\epsilon_j-\epsilon_k}X_{\epsilon_j+\epsilon_k} + X_{-\epsilon_j+\epsilon_k}X_{\epsilon_j-\epsilon_k}\big). \tag{2.3}$$

2.4 The operator $Q_{2\ell+1}$

Now we look at a root space decomposition of $\mathfrak{so}(n)$ in terms of the basis elements I_{ki}, $1\le i< k\le n$ when $n = 2\ell+1$.

The linear span

$$\mathfrak{h} = \langle I_{21}, I_{43}, \dots, I_{2\ell,2\ell-1}\rangle_{\mathbb{C}}$$

is a Cartan subalgebra of $\mathfrak{so}(n,\mathbb{C})$. To find the root vectors it is convenient to visualize the elements of $\mathfrak{so}(n,\mathbb{C})$ as $\ell\times\ell$ matrices of 2×2 blocks occupying the left upper corner of the square matrices of size $2\ell+1$, with the last column (respectively row) made up of ℓ columns (respectively rows) of size two and a zero in the place $(2\ell+1, 2\ell+1)$. The subspaces of all matrices A with a block A_{jk}, $1 \le j < k \le \ell$, in the place (j,k), with the block $-A^t_{jk}$ in the place (k,j) and with zeros in all other places, are $\mathrm{ad}(\mathfrak{h})$-stable. Also the subspaces of all matrices B with a column B_j of size two, $1 \le j \le \ell$, in the place $(j, \ell+1)$, with the row $-B^t_j$ in the place $(\ell+1, j)$ and with zeros in all other places, are $\mathrm{ad}(\mathfrak{h})$-stable.

On the other hand $[H,B] = \lambda B$ if and only if

$$x_j i I_{2j,2j-1} B_j = \lambda B_j.$$

Up to a scalar this linear equation has two linearly independent solutions:

$$B_j = \begin{pmatrix} 1 \\ \pm i \end{pmatrix} \qquad \text{with corresponding} \quad \lambda = \mp x_j,$$

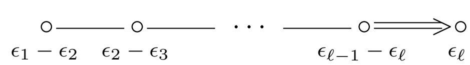
Let $\epsilon \in \mathfrak{h}^*$ be defined by $\epsilon(H) = x_j$ for $1 \le j \le \ell$. Then for $1 \le j < k \le \ell$ and $1 \le r \le \ell$, the following matrices are root vectors of $\mathfrak{so}(2\ell+1,\mathbb{C})$:

$$\begin{aligned}
X_{\epsilon_j+\epsilon_k} &= I_{2k-1,2j-1} - I_{2k,2j} - i(I_{2k-1,2j} + I_{2k,2j-1}),\\
X_{-\epsilon_j-\epsilon_k} &= I_{2k-1,2j-1} - I_{2k,2j} + i(I_{2k-1,2j} + I_{2k,2j-1}),\\
X_{\epsilon_j-\epsilon_k} &= I_{2k-1,2j-1} + I_{2k,2j} - i(I_{2k-1,2j} - I_{2k,2j-1}),\\
X_{-\epsilon_j+\epsilon_k} &= I_{2k-1,2j-1} + I_{2k,2j} + i(I_{2k-1,2j} - I_{2k,2j-1}),\\
X_{\epsilon_r} &= I_{n,2r-1} - iI_{n,2r},\\
X_{-\epsilon_r} &= I_{n,2r-1} + iI_{n,2r}.
\end{aligned}$$

Thus, if we choose the following set of positive roots

$$\Delta^+ = \{\epsilon_r, \epsilon_j + \epsilon_k, \epsilon_j - \epsilon_k : 1 \le r \le \ell, 1 \le j < k \le \ell\},$$

then the Dynkin diagram of $\mathfrak{so}(2\ell+1,\mathbb{C})$ is B_ℓ:



By looking at the 2×1 columns of the different roots, namely

$$X_{\epsilon_j} = \begin{pmatrix} 1 \\ -i \end{pmatrix}, \qquad X_{-\epsilon_j} = \begin{pmatrix} 1 \\ i \end{pmatrix},$$

it is easy to obtain the following inverse relations

$$I_{n,2r-1} = \tfrac{1}{2}(X_{\epsilon_r} + X_{-\epsilon_r}), \qquad I_{n,2r} = \tfrac{i}{2}(X_{\epsilon_r} - X_{-\epsilon_r}).$$

From this it follows that

$$I^2_{n,2r-1} + I^2_{n,2r} = \tfrac{1}{2}(X_{\epsilon_r}X_{-\epsilon_r} + X_{-\epsilon_r}X_{\epsilon_r}) = -iI_{2r,2r-1} + X_{-\epsilon_r}X_{\epsilon_r},$$

since $[X_{\epsilon_r}, X_{-\epsilon_r}] = -2iI_{2r,2r-1}$. Therefore we have that

$$Q_{2\ell+1} = \sum_{1\le j\le 2\ell} I^2_{n,j} + Q_{2\ell} = \sum_{1\le r\le 2\ell} (-iI_{2r,2r-1} + X_{-\epsilon_r}X_{\epsilon_r}) + Q_{2\ell}.$$

Then

$$\begin{aligned}
Q_{2\ell+1} = {}& \sum_{1\le j\le \ell} I^2_{2j,2j-1} - \sum_{1\le j\le \ell} (2(\ell-j)+1) i I_{2j,2j-1}\\
&+ \sum_{1\le j<k\le \ell} \tfrac{1}{2}\big(X_{-\epsilon_j-\epsilon_k}X_{\epsilon_j+\epsilon_k} + X_{-\epsilon_j+\epsilon_k}X_{\epsilon_j-\epsilon_k}\big) + \sum_{1\le r\le 2\ell} X_{-\epsilon_r}X_{\epsilon_r}.
\end{aligned} \tag{2.4}$$

2.5 Gel'fand–Tsetlin basis

For any n we identify the group $\mathrm{SO}(n)$ with a subgroup of $\mathrm{SO}(n+1)$ in the following way: given $k \in \mathrm{SO}(n)$ we have

$$k \simeq \begin{pmatrix} k & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} \in \mathrm{SO}(n+1).$$

Let $T_{\mathbf{m}}$ be an irreducible unitary representation of $\mathrm{SO}(n)$ with highest weight $\mathbf{m}$ and let $V_{\mathbf{m}}$ be the space of this representation. Highest weights $\mathbf{m}$ of these representations are given by the ℓ-tuples of integers $\mathbf{m} = \mathbf{m}_n = (m_{1n}, \dots, m_{\ell n})$ for which

$$\begin{aligned} & m_{1n} \geq m_{2n} \geq \cdots \geq m_{\ell-1,n} \geq |m_{\ell n}| \qquad \text{if} \quad n = 2\ell, \\ & m_{1n} \geq m_{2n} \geq \cdots \geq m_{\ell n} \geq 0 \qquad \text{if} \quad n = 2\ell+1, \end{aligned}$$

and m_{jn} are all integers.

The restriction of the representation $T_{\mathbf{m}}$ of the group $\mathrm{SO}(2\ell+1)$ to the subgroup $\mathrm{SO}(2\ell)$ decomposes into the direct sum of all representations $T_{\mathbf{m}'}$, $\mathbf{m}' = \mathbf{m}_{n-1} = (m_{1,n-1}, \dots, m_{\ell,n-1})$ for which the betweenness conditions

$$m_{1,2\ell+1} \geq m_{1,2\ell} \geq m_{2,2\ell+1} \geq m_{2,2\ell} \geq \cdots \geq m_{\ell,2\ell+1} \geq m_{\ell,2\ell} \geq -m_{\ell,2\ell+1}$$

are satisfied. For the restriction of the representations $T_{\mathbf{m}}$ of $\mathrm{SO}(2\ell)$ to the subgroup $\mathrm{SO}(2\ell-1)$ the corresponding betweenness conditions are

$$m_{1,2\ell} \geq m_{1,2\ell-1} \geq m_{2,2\ell} \geq m_{2,2\ell-1} \geq \cdots \geq m_{\ell-1,2\ell} \geq m_{\ell-1,2\ell-1} \geq |m_{\ell,2\ell}|.$$

All multiplicities in the decompositions are equal to one (see [24, p. 362]).

If we continue this procedure of restriction of irreducible representations successively to the subgroups

$$\mathrm{SO}(n-2) > \mathrm{SO}(n-3) > \cdots > \mathrm{SO}(2),$$

then we finally get one dimensional representations of the group $\mathrm{SO}(2)$. If we take a unit vector in each one of these one dimensional representations we get an orthonormal basis of the representation space $V_{\mathbf{m}}$. Such a basis is called a Gel'fand–Tsetlin basis. The elements of a Gel'fand–Tsetlin basis $\{v(\mu)\}$ of the representation $T_{\mathbf{m}}$ of $\mathrm{SO}(n)$ are labelled by the Gel'fand–Tsetlin patterns $\mu = (\mathbf{m}_n, \mathbf{m}_{n-1}, \dots, \mathbf{m}_3, \mathbf{m}_2)$, where the betweenness conditions are depicted in the following diagrams.

If $n = 2\ell+1$

$$\mu = \left[\begin{array}{ccccccccc} m_{1n} & & m_{2n} & \cdots\cdots & m_{\ell n} & & -m_{\ell n} \\ & m_{1,n-1} & \cdots\cdots\cdots\cdots & & & m_{\ell,n-1} & \\ & & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & & \\ & & & m_{15} & m_{25} & & -m_{25} \\ & & & & m_{14} & m_{24} & \\ & & & & m_{13} & & -m_{13} \\ & & & & & m_{12} & \end{array}\right].$$

If $n = 2\ell$

$$\mu = \left[\begin{array}{ccccccccc} m_{1n} & & m_{2n} & \cdots\cdots & & m_{\ell n} & \\ & m_{1,n-1} & \cdots\cdots & m_{\ell-1,n-1} & & & -m_{\ell-1,n-1} \\ & & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & & \\ & & & m_{15} & m_{25} & & -m_{25} \\ & & & & m_{14} & m_{24} & \\ & & & & m_{13} & & -m_{13} \\ & & & & & m_{12} & \end{array}\right].$$

The chain of subgroups $\mathrm{SO}(n-1) > \mathrm{SO}(n-2) > \cdots > \mathrm{SO}(2)$ defines the orthonormal basis $\{v(\mu)\}$ uniquely up to multiplication of the basis elements by complex numbers of absolute value one.

2.6 An explicit expression for $\dot\pi(Q_n)$

Since $Q_n \in D(\mathrm{SO}(n))^{\mathrm{SO}(n)}$, given $\dot\pi \in \hat{\mathrm{SO}}(n)$ it follows that $\dot\pi(Q_n)$ commutes with $\pi(k)$ for all $k \in \mathrm{SO}(n)$. Hence, by Schur's Lemma $\dot\pi(Q_n) = \lambda I$. From expressions (2.3) and (2.4) we can give the explicit value of λ in terms of the highest weight of π, by computing $\dot\pi(Q_n)$ on a highest weight vector.

Proposition 2.7. *Let (π, V_π) be an irreducible representation of $\mathrm{SO}(2\ell)$ of highest weight $\mathbf{m} = (m_1, m_2, \dots, m_\ell)$. Then, $\dot\pi(Q_{2\ell}) = \lambda I$, with*

$$\lambda = \sum_{1\le j\le \ell} \left(-m_j^2 - 2(\ell-j)m_j\right). \tag{2.5}$$

Proposition 2.8. *Let (π, V_π) be an irreducible representation of $\mathrm{SO}(2\ell+1)$ of highest weight $\mathbf{m} = (m_1, m_2, \dots, m_\ell)$. Then, $\dot\pi(Q_{2\ell+1}) = \lambda I$, with*

$$\lambda = \sum_{1\le j\le \ell} \left(-m_j^2 - (2(\ell-j)+1)m_j\right). \tag{2.6}$$

3 The differential operator Δ

We shall look closely at the left invariant differential operator Δ of $\mathrm{SO}(n+1)$ defined by

$$\Delta = \sum_{j=1}^{n} I_{n+1,j}^2,$$

in order to study its eigenfunctions and eigenvalues. Later we will use all this to understand the irreducible spherical functions of fundamental K-types associated with the pair $(G,K) = (\mathrm{SO}(n+1), \mathrm{SO}(n))$.

Proposition 3.1. *Let $G = \mathrm{SO}(n+1)$ and $K = \mathrm{SO}(n)$. Let us consider the following left invariant differential operator of G*

$$\Delta = \sum_{j=1}^{n} I_{n+1,j}^2.$$

Then Δ is also right invariant under K.

Proof. This is a direct consequence of the identity

$$Q_{n+1} = Q_n + \Delta$$

and Proposition 2.6. ■

Let us define the one-parameter subgroup A of G as the set of all elements of the form

$$a(s) = \begin{pmatrix} I_{n-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cos s & \sin s \\ \mathbf{0} & -\sin s & \cos s \end{pmatrix}, \qquad -\pi \le s \le \pi, \tag{3.1}$$

where I_{n-1} denotes the identity matrix of size $n-1$, and let $M = \mathrm{SO}(n-1)$ be the centralizer of A in K.

Now we want to get the expression of $[\Delta\Phi](a(s))$ for any smooth function Φ on G with values in $\mathrm{End}(V_\pi)$ such that $\Phi(kgk') = \pi(k)\Phi(g)\pi(k')$ for all $g \in G$ and all $k, k' \in K$.

We have

$$\left[I_{n+1,j}^2\Phi\right](a(s)) = \frac{\partial^2}{\partial t^2}\Phi(a(s)\exp tI_{n+1,j})\Big|_{t=0}.$$

Hence, we will use the decomposition $G = KAK$ to write $a(s)\exp tI_{n+1,j} = k(s,t)a(s,t)h(s,t)$, with $k(s,t), h(s,t) \in K$ and $a(s,t) \in A$.

Let us take on $A \setminus \{a(\pi)\}$ the coordinate function $x(a(s)) = s$, with $-\pi < s < \pi$, and let

$$F(s) = F(x(a(s))) = \Phi(a(s)).$$

From now on we will assume that $-\pi < s, t, s+t < \pi$.

If $j = n$ we have $a(s)\exp tI_{n+1,n} = a(s)a(t) = a(s+t)$. Thus we may take

$$a(s,t) = a(s+t), \qquad k(s,t) = h(s,t) = e.$$

Since $x(a(s+t)) = s+t$, we obtain

$$\begin{aligned}\left[I_{n+1,n}^2\Phi\right](a(s)) &= \frac{\partial^2}{\partial t^2}\Phi(a(s)\exp tI_{n+1,n})\Big|_{t=0} = \frac{\partial^2}{\partial t^2}\Phi(a(s+t))\Big|_{t=0}\\ &= \frac{\partial^2}{\partial t^2}F(s+t)\Big|_{t=0} = F''(s).\end{aligned}$$

For $1 \le j \le n-1$, when $s \notin \mathbb{Z}\pi$, we may take

$$k(s,t) = \begin{pmatrix} I_{j-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \frac{\sin s\cos t}{\sqrt{1-\cos^2 s\cos^2 t}} & \mathbf{0} & \frac{\sin t}{\sqrt{1-\cos^2 s\cos^2 t}} & 0\\ \mathbf{0} & \mathbf{0} & I_{n-j-1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \frac{-\sin t}{\sqrt{1-\cos^2 s\cos^2 t}} & \mathbf{0} & \frac{\sin s\cos t}{\sqrt{1-\cos^2 s\cos^2 t}} & 0\\ \mathbf{0} & 0 & \mathbf{0} & 0 & 1\end{pmatrix},$$

$$h(s,t) = \begin{pmatrix} I_{j-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \frac{\sin s}{\sqrt{1-\cos^2 s\cos^2 t}} & \mathbf{0} & \frac{-\cos s\sin t}{\sqrt{1-\cos^2 s\cos^2 t}} & 0\\ \mathbf{0} & \mathbf{0} & I_{n-j-1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \frac{\cos s\sin t}{\sqrt{1-\cos^2 s\cos^2 t}} & \mathbf{0} & \frac{\sin s}{\sqrt{1-\cos^2 s\cos^2 t}} & 0\\ \mathbf{0} & 0 & \mathbf{0} & 0 & 1\end{pmatrix},$$

$$a(s,t) = \begin{pmatrix} I_{n-1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \cos s\cos t & \sqrt{1-\cos^2 s\cos^2 t}\\ \mathbf{0} & -\sqrt{1-\cos^2 s\cos^2 t} & \cos s\cos t\end{pmatrix}.$$

Then, for $0 < s < \pi$, we have $x(a(s,t)) = \arccos(\cos s\cos t)$ and

$$\frac{\partial}{\partial t}x(a(s,t)) = \frac{\cos s\sin t}{\sqrt{1-\cos^2 s\cos^2 t}}.$$

From here we get

$$\frac{\partial}{\partial t}x(a(s,t))\Big|_{t=0} = 0 \qquad \text{and} \qquad \frac{\partial^2}{\partial t^2}x(a(s,t))\Big|_{t=0} = \frac{\cos s}{\sin s}.$$

Thus

$$\frac{\partial}{\partial t}\Phi(a(s,t))\Big|_{t=0} = F'(s)\frac{\partial}{\partial t}x(a(s,t))\Big|_{t=0} = 0 \qquad \text{and} \qquad \frac{\partial^2}{\partial t^2}\Phi(a(s,t))\Big|_{t=0} = \frac{\cos s}{\sin s}F'(s).$$

We observe that $k(s,0)=h(s,0)=e$ and that $a(s,0)=a(s)$. Then

$$\begin{aligned}[I_{nj}^2\Phi](a(s)) &= \frac{\partial^2}{\partial t^2}\pi(k(s,t))\Big|_{t=0}\Phi(a(s)) + 2\frac{\partial}{\partial t}\pi(k(s,t))\Big|_{t=0}\frac{\partial}{\partial t}\Phi(a(s,t))\Big|_{t=0} \\ &\quad + 2\frac{\partial}{\partial t}\pi(k(s,t))\Big|_{t=o}\Phi(a(s))\frac{\partial}{\partial t}\pi(h(s,t))\Big|_{t=0} + \frac{\partial^2}{\partial t^2}\Phi(a(s,t))\Big|_{t=0} \\ &\quad + 2\frac{\partial}{\partial t}\Phi(a(s,t))\Big|_{t=0}\frac{\partial}{\partial t}\pi(h(s,t))\Big|_{t=0} + \Phi(a(s))\frac{\partial^2}{\partial t^2}\pi(h(s,t))\Big|_{t=0}.\end{aligned}$$

We also have

$$\frac{\partial}{\partial t}\pi\left(k(s,t)\right)\Big|_{t=0} = \dot\pi\left(\frac{\partial}{\partial t}k(s,t)\Big|_{t=0}\right) = \frac{1}{\sin s}\dot\pi(I_{n,j}),$$

and

$$\frac{\partial}{\partial t}\pi(h(s,t))\Big|_{t=0} = \dot\pi\left(\frac{\partial}{\partial t}h(s,t)\Big|_{t=0}\right) = -\frac{\cos s}{\sin s}\dot\pi\left(I_{n,j}\right).$$

We will need the following proposition, whose proof appears in the Appendix and its idea is taken from [5].

Proposition 3.2. *If $A(s,t)=k(s,t)$ or $A(s,t)=h(s,t)$, then in either case for $0<s<\pi$, we have*

$$\frac{\partial^2(\pi\circ A)}{\partial t^2}\Big|_{t=0} = \dot\pi\left(\frac{\partial A}{\partial t}\Big|_{t=0}\right)^2.$$

Moreover in each case, for $1\le j\le n-1$ and $0<s<\pi$, we have

$$\frac{\partial^2}{\partial t^2}\pi(k(s,t))\Big|_{t=0} = \frac{1}{\sin^2 s}\dot\pi(I_{n,j})^2, \qquad \frac{\partial^2}{\partial t^2}\pi(h(s,t))\Big|_{t=0} = \frac{\cos^2 s}{\sin^2 s}\dot\pi(I_{n,j})^2.$$

Now we obtain the following corollaries.

Corollary 3.3. *Let Φ be any smooth function on G with values in $\mathrm{End}(V_\pi)$ such that $\Phi(kgk')=\pi(k)\Phi(g)\pi(k')$ for all $g\in G$ and all $k,k'\in K$. Then, if $F(s)=\Phi(a(s))$, for $0<s<\pi$ we have*

$$\begin{aligned}[\Delta\Phi](a(s)) &= F''(s) + (n-1)\frac{\cos s}{\sin s}F'(s) + \frac{1}{\sin^2 s}\sum_{j=1}^{n-1}\dot\pi(I_{n,j})^2F(s) \\ &\quad - 2\frac{\cos s}{\sin^2 s}\sum_{j=1}^{n-1}\dot\pi(I_{n,j})F(s)\dot\pi(I_{n,j}) + \frac{\cos^2 s}{\sin^2 s}F(s)\sum_{j=1}^{n-1}\dot\pi(I_{n,j})^2.\end{aligned}$$

Corollary 3.4. *Let Φ be an irreducible spherical function on G of type $\pi\in\hat K$. Then, if $F(s)=\Phi(a(s))$, we have*

$$\begin{aligned}&F''(s) + (n-1)\frac{\cos s}{\sin s}F'(s) + \frac{1}{\sin^2 s}\sum_{j=1}^{n-1}\dot\pi(I_{n,j})^2F(s) \\ &\quad - 2\frac{\cos s}{\sin^2 s}\sum_{j=1}^{n-1}\dot\pi(I_{n,j})F(s)\dot\pi(I_{n,j}) + \frac{\cos^2 s}{\sin^2 s}F(s)\sum_{j=1}^{n-1}\dot\pi(I_{n,j})^2 = \lambda F(s),\end{aligned}$$

for some $\lambda\in\mathbb{C}$ and $0<s<\pi$.

Notice that the expression in Corollary 3.4 generalizes the very well known situation when the K-type is the trivial one, as we state in the following corollary (cf. [11, p. 403, equation (10)]).

Corollary 3.5. *Let Φ be an irreducible spherical function on G of the trivial K-type. Then, for $F(s) = \Phi(a(s))$ we have*

$$F''(s) + (n-1)\frac{\cos s}{\sin s}F'(s) = \lambda F(s),$$

for some $\lambda \in \mathbb{C}$ and $0 < s < \pi$.

Let us make the change of variables $y = (1+\cos s)/2$, with $0 < s < \pi$; then $0 < y < 1$. We also have $\cos s = 2y-1$, $\sin^2 s = 4y(1-y)$ and $\frac{d}{dy} = -\frac{\sin s}{2}$. If we let $H(y) = F(s)$, i.e.

$$H(y) = \Phi(a(s)), \qquad \text{with} \quad \cos s = 2y-1,$$

we obtain

$$F'(s) = -\frac{\sin s}{2}H'(s), \qquad F''(s) = \frac{\sin^2 s}{4}H''(y) - \frac{\cos s}{2}H'(y).$$

In terms of this new variable Corollary 3.4 becomes

Corollary 3.6. *Let Φ be an irreducible spherical function on G of type $\pi \in \hat{K}$. Then, if $H(y) = \Phi(a(s))$ with $y = (1+\cos s)/2$, we have*

$$y(1-y)H''(y) + \frac{1}{2}n(1-2y)H'(y) + \frac{1}{4y(1-y)}\sum_{j=1}^{n-1}\dot{\pi}(I_{n,j})^2 H(y)$$
$$+ \frac{(1-2y)}{2y(1-y)}\sum_{j=1}^{n-1}\dot{\pi}(I_{n,j})H(y)\dot{\pi}(I_{n,j}) + \frac{(1-2y)^2}{4y(1-y)}H(y)\sum_{j=1}^{n-1}\dot{\pi}(I_{n,j})^2 = \lambda H(y),$$

for some $\lambda \in \mathbb{C}$ and $0 < y < 1$.

Remark 3.7. Let us notice that, for any $y \in (0,1)$, $H(y)$ is a scalar linear transformation when restricted to any M-submodule, see Proposition 2.2. Therefore, if m is the number of M-submodules contained in (V,π), we consider the vector valued function $H : (0,1) \to \mathbb{C}^m$ whose entries are given by those scalar values that $H(y)$ takes on every M-submodule.

If the $\mathrm{End}(V)$-valued function H satisfies the differential equation given in Corollary 3.6, then the vector valued function H satisfies

$$y(1-y)H''(y) + \frac{1}{2}n(1-2y)H'(y) + \frac{1}{4y(1-y)}N_1 H(y)$$
$$+ \frac{(1-2y)}{2y(1-y)}EH(y) + \frac{(1-2y)^2}{4y(1-y)}N_2 H(y) = \lambda H(y),$$

where E, N_1 and N_2 are matrices of size $m \times m$.

Even more, since $\sum_{j=1}^{n-1} I_{n,j}^2 = Q_n - Q_{n-1}$, Proposition 2.6 implies $\sum_{j=1}^{n-1} I_{n,j}^2 \in D(\mathrm{SO}(n))^{\mathrm{SO}(n-1)}$, therefore $\sum_{j=1}^{n-1}\dot{\pi}(I_{n,j})^2$ is scalar valued when restricted to any M-submodule. Hence, $N_1 = N_2$ and the equation above is equivalent to

$$y(1-y)H''(y)+\frac{n}{2}(1-2y)H'(y) + \frac{(1-2y)}{2y(1-y)}EH(y) + \frac{1+(1-2y)^2}{4y(1-y)}NH(y)=\lambda H(y), \quad (3.2)$$

where N is a diagonal matrix of size $m \times m$. To obtain an explicit expression of E for any K-type is a very serious matter; in the following sections we shall find explicitly the expressions of E and N, for certain K-types.

Remark 3.8. It is worth to observe that from (2.5) and (2.6) we can immediately obtain every entry of the diagonal matrix N.

4 The K-types which are M-irreducible

Let $K = \mathrm{SO}(n)$, $M = \mathrm{SO}(n-1)$, with $n = 2\ell+1$, and let $\mathbf{m}_n = (m_{1n}, \dots, m_{\ell n})$ be a K-type such that $V_{\mathbf{m}}$ is irreducible as M-module. The highest weights $\mathbf{m}_{n-1}$ of the M-submodules of $V_{\mathbf{m}}$ are those that satisfies the following intertwining relations

$$\begin{matrix} m_{1n} & & m_{2n} & & \cdots & & m_{\ell,n} & & -m_{\ell n} \\ & m_{1,n-1} & & \cdots & & \cdots & & m_{\ell,n-1} & \end{matrix}.$$

Since $V_{\mathbf{m}}$ is irreducible as M-module it follows that $m_{1n} = \cdots = m_{\ell,n} = 0$. The converse is also true, therefore $V_{\mathbf{m}}$ is M-irreducible if and only if it is the trivial representation.

Let now consider the case $K = \mathrm{SO}(n)$, $M = \mathrm{SO}(n-1)$, with $n = 2\ell$ and let $\mathbf{m}_n = (m_{1n}, \dots, m_{\ell n})$ be a K-type such that $V_{\mathbf{m}}$ is irreducible as M-module. The highest weights $\mathbf{m}_{n-1}$ of the M-submodules of $V_{\mathbf{m}}$ are those that satisfies the following intertwining relations

$$\begin{matrix} m_{1n} & & m_{2n} & & \cdots & & m_{\ell-1,n} & & m_{\ell n} & \\ & m_{1,n-1} & & \cdots & & \cdots & & m_{\ell-1,n-1} & & -m_{\ell-1,n-1} \end{matrix}.$$

Since $V_{\mathbf{m}}$ is irreducible as M-module it follows that $m_{1n} = \cdots = m_{\ell-1,n} = d$ and $m_{\ell n} = d - j$ with $0 \le j \le 2d$, since $m_{\ell-1,n} \ge |m_{\ell n}|$. This implies that $m_{1,n-1} = \cdots = m_{\ell-2,n-1} = d$ and $m_{\ell-1,n-1} = q$ with $d \ge q \ge \max\{d-j, j-d\}$. Thus, if $0 \le j \le d$ we have $d \ge q \ge d-j$ and by irreducibility we must have $j = 0$. Similarly if $d \le j \le 2d$ we have $d \ge q \ge j-d$ and by irreducibility we must have $j = 2d$. Therefore $\mathbf{m}_n = d\alpha$ or $\mathbf{m}_n = d\beta$, where

$$\alpha = (1, \dots, 1), \qquad \beta = (1, \dots, 1, -1).$$

The converse is also true, therefore $V_{\mathbf{m}}$ is M-irreducible if and only if $\mathbf{m}_n = d\alpha$ or $\mathbf{m}_n = d\beta$ for any $d \in \mathbb{N}_0$.

If Φ is an irreducible spherical function on $\mathrm{SO}(n+1)$ of type π, whose highest weight is $\mathbf{m}_n = d\alpha$ or $\mathbf{m}_n = d\beta$, then from Corollary 3.6 we get that the associated function H satisfies

$$y(1-y)H''(y) + \ell(1-2y)H'(y) + \frac{1-y}{y}\sum_{j=1}^{n-1}\dot\pi(I_{nj})^2 H(y) = \lambda H(y).$$

To compute $\sum_{j=1}^{n-1}\dot\pi(I_{nj})^2$ we write $\sum_{j=1}^{n-1}\dot\pi(I_{nj})^2 = \dot\pi(Q_n - Q_{n-1})$.

Let us first consider $\mathbf{m}_n = d\alpha$. If $v \in V_{\mathbf{m}_n}$ is a highest weight vector, then

$$\dot\pi(Q_n)v = -d\ell(d+\ell-1)v \qquad \text{and} \qquad \dot\pi(Q_{n-1})v = -d(\ell-1)(d+\ell-1)v,$$

see (2.5) and (2.6). Therefore

$$\sum_{j=1}^{n-1}\dot\pi(I_{nj})^2 v = -d(d+\ell-1)v.$$

Let us now consider $\mathbf{m}_n = d\beta$. If $v \in V_{\mathbf{m}_n}$ is a highest weight vector, then $\dot\pi(Q_n)v = -2d\ell(d+\ell-1)v$ as before, and $\dot\pi(Q_{n-1})v = -2d(\ell-1)(d+\ell-1)v$ as before because in both cases $\mathbf{m}_{n-1}$ is the same.

Therefore if $\mathbf{m}_n = (d, \dots, d, \pm d)$ we have

$$\sum_{j=1}^{n-1} \dot{\pi}(I_{nj})^2 v = -d(d + \ell - 1)v.$$

Hence, if Φ is an irreducible spherical function on $\mathrm{SO}(n + 1)$, $n = 2\ell$, of type $\mathbf{m}_n = (d, \dots, d, \pm d) \in \mathbb{C}^\ell$, then the associated scalar value function $H = h$ satisfies

$$y(1 - y)h''(y) + \ell(1 - 2y)h'(y) - \frac{d(d + \ell - 1)(1 - y)}{y}h(y) = \lambda h(y). \tag{4.1}$$

Let us now compute the eigenvalue λ corresponding to the spherical function of type $\pi \in \hat{\mathrm{SO}}(2\ell)$, of highest weight $\mathbf{m}_n = d\alpha$, associated with the irreducible representation $\tau \in \mathrm{SO}(2\ell{+}1)$, of highest weight $\mathbf{m}_{n+1} = (w, d, \dots, d) \in \mathbb{C}^\ell$. If $v \in V_{\mathbf{m}_{n+1}}$ is a highest weight vector, then from (2.6) we have

$$\dot{\tau}(Q_{n+1})v = -\left(w(w + 2\ell - 1) + d(\ell - 1)(d + \ell - 1)\right) v.$$

If $v \in V_{\mathbf{m}_n}$ is a highest weight vector, then from (2.5) we have

$$\dot{\tau}(Q_n)v = \dot{\pi}(Q_n)v = -d\ell(d + \ell - 1)v.$$

Since $\Delta = Q_{n+1} - Q_n$ it follows that

$$\lambda = -w(w + 2\ell - 1) + d(d + \ell - 1).$$

To solve (4.1) we write $h = y^\alpha f$. Then we get

$$\begin{aligned} &y(1 - y)y^\alpha f'' + (2\alpha(1 - y) + \ell(1 - 2y))y^\alpha f' \\ &\qquad + (\alpha(\alpha - 1)(1 - y) + \ell\alpha(1 - 2y) - d(d + \ell - 1)(1 - y))y^{\alpha-1} f = \lambda y^\alpha f. \end{aligned}$$

Thus the indicial equation is $\alpha(\alpha - 1) + \ell\alpha - d(d + \ell - 1) = 0$ and $\alpha = d$ is one of its solutions. If we take $h = y^d f$, then we obtain

$$y(1 - y)f'' + (2d + \ell - 2(d + \ell)y)f' - d\ell f = \lambda f.$$

If we replace $\lambda = -w(w + 2\ell - 1) + d(d + \ell - 1)$ we get

$$y(1 - y)f'' + (2d + \ell - 2(d + \ell)y)f' - (d - w)(2\ell + d + w - 1)f = 0.$$

Let $a = d - w$, $b = 2\ell + d + w - 1$, $c = 2d + \ell$ then the above equation becomes

$$y(1 - y)f'' + (c - (1 + a + b)y)f' - abf = 0.$$

A fundamental system of solutions of this equation near $y = 0$ is given by the following functions

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; y\right), \qquad y^{1-c}{}_2F_1\left(\begin{matrix} a - c + 1, b - c + 1 \\ 2 - c \end{matrix}; y\right).$$

Since $h = y^d f$ is bounded near $y = 0$ it follows that

$$h(y) = uy^d{}_2F_1\left(\begin{matrix} d - w, 2\ell + d + w - 1 \\ 2d + \ell \end{matrix}; y\right),$$

where the constant u is determined by the condition $h(1) = 1$.

Remark 4.1. Let $h_w = h_w(y)$, $w \geq d$, be the function h above. Then h_w is a polynomial of degree w. Moreover observe that the function y^d used to hypergeometrize (4.1) is precisely h_d.

Let us now compute the eigenvalue λ corresponding to the spherical function of type $\mathbf{m}_n = d\beta$ associated with an irreducible representation τ of $\mathrm{SO}(n+1)$ of highest weight $\mathbf{m}_{n+1} = (w, d, \dots, d) \in \mathbb{C}^\ell$. If $v \in V_{\mathbf{m}_{n+1}}$ is a highest weight vector, we obtain $\dot\tau(Q_{n+1})v = -(w(w+2\ell-1) + d(\ell-1)(d+\ell-1))v$.

If $v \in V_{\mathbf{m}_n}$ is a highest weight vector, then $\dot\pi(Q_n)v = -d\ell(d+\ell-1)v$ as above, because $Q_n v$ does not depend on the sign of the last coordinate of $\mathbf{m}_n$. Since $\Delta = Q_{n+1} - Q_n$ we also have

$$\lambda = -w(w+2\ell-1) + d(d+\ell-1).$$

Therefore we have proved the following result.

Theorem 4.2. *The scalar valued functions $H = h$ associated with the irreducible spherical functions on $\mathrm{SO}(n+1)$, $n = 2\ell$, of $\mathrm{SO}(n)$-type $\mathbf{m}_n = (d, \dots, d, \pm d) \in \mathbb{C}^\ell$, are parameterized by the integers $w \geq d$ and are given by*

$$h_w(y) = u y^d {}_2F_1\left(\begin{matrix} d-w, 2\ell+d+w-1 \\ 2d+\ell \end{matrix}; y\right)$$

where the constant u is determined by the condition $h_w(1) = 1$.

5 The operator Δ for fundamental K-types

We are interested in finding a more explicit expression of the differential equation given in Corollary 3.6:

$$y(1-y)H''(y) + \frac{1}{2}n(1-2y)H'(y) + \frac{1}{4y(1-y)}\sum_{j=1}^{n-1}\dot\pi(I_{n,j})^2 H(y) + \frac{(1-2y)}{2y(1-y)}\sum_{j=1}^{n-1}\dot\pi(I_{n,j})H(y)\dot\pi(I_{n,j}) + \frac{(1-2y)^2}{4y(1-y)}H(y)\sum_{j=1}^{n-1}\dot\pi(I_{n,j})^2 = \lambda H(y),$$

for certain representations $\pi \in \hat{\mathrm{SO}}(n)$, including those that are fundamental.

The obvious place to start to look for irreducible representations of $\mathrm{SO}(n)$ is among the exterior powers of the standard representation of $\mathrm{SO}(n)$. It is known that $\Lambda^p(\mathbb{C}^{2\ell})$ are irreducible $\mathrm{SO}(2\ell)$-modules for $p = 1, \dots, \ell-1$, and that $\Lambda^\ell(\mathbb{C}^{2\ell})$ splits into the direct sum of two irreducible submodules. While in the odd case $\Lambda^p(\mathbb{C}^{2\ell+1})$ are irreducible $\mathrm{SO}(2\ell+1)$-modules for $p = 1, \dots, \ell$. See Theorems 19.2 and 19.14 in [3].

Moreover, $\Lambda^p(\mathbb{C}^n)$ and $\Lambda^{n-p}(\mathbb{C}^n)$ are isomorphic $\mathrm{SO}(n)$-modules. In fact, if $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the canonical basis of $\mathbb{C}^n$, then the linear map $\xi : \Lambda^p(\mathbb{C}^n) \to \Lambda^{n-p}(\mathbb{C}^n)$ defined by

$$\xi(\mathbf{e}_{u_1} \wedge \cdots \wedge \mathbf{e}_{u_p}) = (-1)^{u_1+\cdots+u_p}\mathbf{e}_{v_1} \wedge \cdots \wedge \mathbf{e}_{v_{n-p}},$$

where $u_1 < \cdots < u_p$ and $v_1 < \cdots < v_{n-p}$ are complementary ordered set of indices, is an $\mathrm{SO}(n)$-isomorphism.

All these statements can be established directly upon observing that the elements $I_{ki} = E_{ki} - E_{ik}$ with $1 \leq i < k \leq n$ form a basis of the Lie algebra $\mathfrak{so}(n)$, and that

$$I_{ki}\mathbf{e}_k = \mathbf{e}_i, \qquad I_{ki}\mathbf{e}_i = -\mathbf{e}_k \qquad \text{and} \qquad I_{ki}\mathbf{e}_j = 0 \qquad \text{if} \quad j \neq k, i.$$

We will refer to the irreducible $\mathrm{SO}(2\ell)$-modules $\Lambda^p(\mathbb{C}^{2\ell})$ for $p = 1, \dots, \ell-1$, respectively, the irreducible $\mathrm{SO}(2\ell+1)$-modules $\Lambda^p(\mathbb{C}^{2\ell+1})$ for $p = 1, \dots, \ell$, as the fundamental $\mathrm{SO}(2\ell)$-modules, respectively, as the fundamental $\mathrm{SO}(2\ell+1)$-modules, for reasons that will be clarified in the following Sections 5.1 and 5.2.

5.1 The even case: $K = \mathrm{SO}(2\ell)$

First we will study the case $n = 2\ell$, with $\ell > 2$. The fundamental weights of $\mathfrak{so}(2\ell, \mathbb{C})$ are

$$\lambda_p = \epsilon_1 + \cdots + \epsilon_p, \qquad 1 \le p \le \ell - 2,$$
$$\lambda_{\ell-1} = \tfrac{1}{2}(\epsilon_1 + \cdots + \epsilon_{\ell-1} - \epsilon_\ell), \qquad \lambda_\ell = \tfrac{1}{2}(\epsilon_1 + \cdots + \epsilon_{\ell-1} + \epsilon_\ell).$$

Here we will consider the fundamental K-modules

$$\Lambda^1(\mathbb{C}^n),\ \Lambda^2(\mathbb{C}^n),\ \ldots,\ \Lambda^{\ell-1}(\mathbb{C}^n).$$

We will show that the highest weight of $\Lambda^p(\mathbb{C}^n)$ is $\epsilon_1 + \cdots + \epsilon_p$ for $1 \le p \le \ell - 1$. Observe that $\lambda_{\ell-1}$ and λ_ℓ are not analytically integral and therefore they will not be considered, although we will also consider the K-module with highest weight $\lambda_{\ell-1} + \lambda_\ell = \epsilon_1 + \cdots + \epsilon_{\ell-1}$. Notice that we have already considered the cases $2\lambda_{\ell-1}$ and $2\lambda_\ell$ in Section 4, which are M-irreducible. We will also show that the fundamental K-modules are direct sum of two irreducible M-submodules.

In order to obtain the explicit expression of E in (3.2) for a given irreducible representation π of $K = \mathrm{SO}(n)$, of highest weight $\varepsilon_1 + \cdots + \varepsilon_p$, we are interested to compute

$$\sum_{j=1}^{n-1} \dot\pi(I_{nj}) P_s \dot\pi(I_{nj})\big|_{V_r} = \lambda(r,s) I_{V_r},$$

with $r, s = 0, 1$ corresponding to the two M-submodules V_0 and V_1 of the representation π, associated with $\mathbf{m}_{n-1} = (1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^{\ell-1}$ with $p-1$ and p ones, respectively (see the betweenness conditions in Section 2.5); being P_0 and P_1 the respective projections.

Let us consider the standard action of $K = \mathrm{SO}(n)$ on $V = \mathbb{C}^n$, and take the canonical basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. Then we have the irreducible K-module $\Lambda^p(V)$ for $1 \le p \le \ell - 1$. The vector $(\mathbf{e}_1 - i\mathbf{e}_2) \wedge (\mathbf{e}_3 - i\mathbf{e}_4) \wedge \cdots \wedge (\mathbf{e}_{2p-1} - i\mathbf{e}_{2p})$ is the unique, up to a scalar, dominant vector and its weight is $(1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^\ell$ with p ones. Then, if V' is the subspace generated by $\{\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}\}$, $\Lambda^p(V)$ is the direct sum of two M-submodules, namely

$$\Lambda^p(V) = V_0 \oplus V_1 = \Lambda^{p-1}(V') \wedge \mathbf{e}_n \oplus \Lambda^p(V') \tag{5.1}$$

whose highest weights are $(1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^{\ell-1}$ with $p-1$ ones and $(1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^{\ell-1}$ with p ones, respectively. It is easy to see that $(\mathbf{e}_1 - i\mathbf{e}_2) \wedge (\mathbf{e}_3 - i\mathbf{e}_4) \wedge \cdots \wedge (\mathbf{e}_{2p-3} - i\mathbf{e}_{2p-2}) \wedge \mathbf{e}_n$ is an M-highest weight vector in $\Lambda^{p-1}(V') \wedge \mathbf{e}_n$ and that $(\mathbf{e}_1 - i\mathbf{e}_2) \wedge (\mathbf{e}_3 - i\mathbf{e}_4) \wedge \cdots \wedge (\mathbf{e}_{2p-1} - i\mathbf{e}_{2p})$ is an M highest weight vector in $\Lambda^p(V')$.

To get $\lambda(0,0)$ it is enough to compute

$$\sum_{j=1}^{n-1} \dot\pi(I_{nj}) P_0 \dot\pi(I_{nj})(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n).$$

Since we have that $\dot\pi(I_{nj})(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n) = \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_j$ we obtain $P_0 \dot\pi(I_{nj})(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n) = 0$ and $\lambda(0,0) = 0$.

To get $\lambda(0,1)$ it is enough to compute

$$\sum_{j=1}^{n-1} \dot\pi(I_{nj}) P_1 \dot\pi(I_{nj})(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n).$$

We have

$$P_1 \dot\pi(I_{nj})(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n) = \begin{cases} 0 & \text{if } \ 1 \le j \le p-1, \\ \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_j & \text{if } \ p \le j \le n-1. \end{cases}$$

Therefore we have

$$\dot\pi(I_{nj})P_1\dot\pi(I_{nj})(\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_{p-1}\wedge\mathbf{e}_n)=\begin{cases}0 & \text{if}\quad 1\le j\le p-1,\\ -\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_{p-1}\wedge\mathbf{e}_n & \text{if}\quad p\le j\le n-1.\end{cases}$$

Hence $\lambda(0,1)=-(n-p)$.

Similarly, to get $\lambda(1,0)$ it is enough to compute

$$\sum_{j=1}^{n-1}\dot\pi(I_{nj})P_0\dot\pi(I_{nj})(\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_p).$$

We have

$$\dot\pi(I_{nj})(\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_p)=\begin{cases}-\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_n\wedge\cdots\wedge\mathbf{e}_p & \text{if}\quad 1\le j\le p,\\ 0 & \text{if}\quad p+1\le j\le n-1,\end{cases}$$

where $\mathbf{e}_n$ appears in the j-place. Therefore

$$\dot\pi(I_{nj})P_0\dot\pi(I_{nj})(\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_p)=\begin{cases}-\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_p & \text{if}\quad 1\le j\le p,\\ 0 & \text{if}\quad p+1\le j\le n-1.\end{cases}$$

Hence $\lambda(1,0)=-p$.

Also it is clear now that $\sum_{j=1}^{n-1}\dot\pi(I_{nj})P_1\dot\pi(I_{nj})(\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_p)=0$, hence $\lambda(1,1)=0$.

Therefore, when π is the standard representation of K in $\Lambda^p(V)$, $1\le p\le\ell-1$, we have

$$(\lambda(r,s))_{0\le r,s\le 1}=\begin{pmatrix}0 & p-n\\ -p & 0\end{pmatrix}.$$

Therefore, we obtain a more explicit version of Corollary 3.6 using (3.2) and Remark 3.8.

Corollary 5.1. *Let Φ be an irreducible spherical function on G of type $\pi\in\hat{\mathrm{SO}}(n)$, $n=2\ell$. If the highest weight of π is of the form $(1,\dots,1,0,\dots,0)\in\mathbb{C}^\ell$, with p ones, $1\le p\le\ell-1$, then the function $H:(0,1)\to\mathrm{End}(\mathbb{C}^2)$ associated with Φ satisfies*

$$\begin{aligned}y(1-y)H''(y)+\frac12 n(1-2y)H'(y)+\frac{1+(1-2y)^2}{4y(1-y)}\begin{pmatrix}p-n & 0\\ 0 & -p\end{pmatrix}H(y)\\ +\frac{(1-2y)}{2y(1-y)}\begin{pmatrix}0 & p-n\\ -p & 0\end{pmatrix}H(y)=\lambda H(y),\end{aligned}$$

for some $\lambda\in\mathbb{C}$.

5.2 The odd case: $K=\mathrm{SO}(2\ell+1)$

We now study the case $n=2\ell+1$, with $\ell\ge 1$. The fundamental weights of $\mathfrak{so}(2\ell+1,\mathbb{C})$ are

$$\begin{aligned}&\lambda_p=\epsilon_1+\cdots+\epsilon_p, \qquad 1\le p\le\ell-1,\\ &\lambda_\ell=\tfrac12(\epsilon_1+\cdots+\epsilon_\ell).\end{aligned}$$

Here we will consider the fundamental K-modules

$$\Lambda^1(\mathbb{C}^n),\ \Lambda^2(\mathbb{C}^n),\ \dots,\ \Lambda^\ell(\mathbb{C}^n).$$

We will show that the highest weight of $\Lambda^p(\mathbb{C}^n)$ is $\epsilon_1 + \cdots + \epsilon_p$ for $1 \le p \le \ell$. Also we will establish that $\Lambda^p(\mathbb{C}^n)$ splits into the direct sum of two M-submodules for $1 \le p \le \ell - 1$, while $\Lambda^\ell(\mathbb{C}^n)$ splits into the sum of three M-submodules; for this reason it will be treated separately in Section 8.

Observe that λ_ℓ is not analytically integral and therefore it will not be considered, although we will consider the K-module with highest weight $2\lambda_\ell$.

As in the even case we are interested in computing

$$\sum_{j=1}^{n-1} \dot\pi(I_{nj}) P_s \dot\pi(I_{nj})\Big|_{V_r} = \lambda(r,s) I_{V_r},$$

with $r, s = 0, 1$ corresponding to the two M-submodules V_0 and V_1 of the representation π, corresponding to $\mathbf{m}_{n-1} = (1, \dots, 1, 0, \dots, 0) \in \mathbb{C}^\ell$ with $p-1$ and p ones respectively (see the betweenness conditions in Section 2.5). Being P_0 and P_1 the respective projections.

Let us consider the standard action of $K = \mathrm{SO}(n)$ on $V = \mathbb{C}^n$, and take the canonical basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. Then we have the irreducible K-module $\Lambda^p(V)$ for $1 \le p \le \ell - 1$. The vector $(\mathbf{e}_1 - i\mathbf{e}_2) \wedge (\mathbf{e}_3 - i\mathbf{e}_4) \wedge \cdots \wedge (\mathbf{e}_{2p-1} - i\mathbf{e}_{2p})$ is the unique, up to a scalar, dominant vector and its weight is $(1, \dots, 1, 0, \dots, 0) \in \mathbb{C}^\ell$ with p ones. Then, if V' is the subspace generated by $\{\mathbf{e}_1, \dots, \mathbf{e}_{n-1}\}$, $\Lambda^p(V)$ is the direct sum of two irreducible M-submodules, namely

$$\Lambda^p(V) = V_0 \oplus V_1 = \Lambda^{p-1}(V') \wedge \mathbf{e}_n \oplus \Lambda^p(V') \tag{5.2}$$

of highest weights $(1, \dots, 1, 0, \dots, 0) \in \mathbb{C}^\ell$ with $p-1$ ones, and $(1, \dots, 1, 0, \dots, 0) \in \mathbb{C}^\ell$ with p ones, respectively. It is easy to see that $(\mathbf{e}_1 - i\mathbf{e}_2) \wedge (\mathbf{e}_3 - i\mathbf{e}_4) \wedge \cdots \wedge (\mathbf{e}_{2p-3} - i\mathbf{e}_{2p-2}) \wedge \mathbf{e}_n$ is an M-highest weight vector in $\Lambda^{p-1}(V') \wedge \mathbf{e}_n$ and that $(\mathbf{e}_1 - i\mathbf{e}_2) \wedge (\mathbf{e}_3 - i\mathbf{e}_4) \wedge \cdots \wedge (\mathbf{e}_{2p-1} - i\mathbf{e}_{2p})$ is an M highest weight vector in $\Lambda^p(V')$.

To get $\lambda(0,0)$ it is enough to compute

$$\sum_{j=1}^{n-1} \dot\pi(I_{nj}) P_0 \dot\pi(I_{nj}) (\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n).$$

Since we have that $\dot\pi(I_{nj})(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n) = \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_j$, we obtain $P_0 \dot\pi(I_{nj})(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n) = 0$ and $\lambda(0,0) = 0$.

To get $\lambda(0,1)$ it is enough to compute

$$\sum_{j=1}^{n-1} \dot\pi(I_{nj}) P_1 \dot\pi(I_{nj}) (\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n).$$

We have

$$P_1 \dot\pi(I_{nj})(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n) = \begin{cases} 0 & \text{if} \quad 1 \le j \le p-1, \\ \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_j & \text{if} \quad p \le j \le n-1. \end{cases}$$

Therefore

$$\dot\pi(I_{nj}) P_1 \dot\pi(I_{nj})(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n) = \begin{cases} 0 & \text{if} \quad 1 \le j \le p-1, \\ -\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n & \text{if} \quad p \le j \le n-1. \end{cases}$$

Hence $\lambda(0,1) = -(n-p)$.

Similarly, to get $\lambda(1,0)$ it is enough to compute

$$\sum_{j=1}^{n-1} \dot\pi(I_{nj}) P_0 \dot\pi(I_{nj}) (\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_p).$$

We have that

$$\dot\pi(I_{nj})(\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_p)=\begin{cases}-\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_n\wedge\cdots\wedge\mathbf{e}_p & \text{if}\quad 1\le j\le p,\\ 0 & \text{if}\quad p+1\le j\le n-1,\end{cases}$$

where $\mathbf{e}_n$ appears in the j-place. Therefore

$$\dot\pi(I_{nj})P_0\dot\pi(I_{nj})(\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_p)=\begin{cases}-\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_p & \text{if}\quad 1\le j\le p,\\ 0 & \text{if}\quad p+1\le j\le n-1.\end{cases}$$

Hence $\lambda(1,0)=-p$.

Also it is clear now that $\sum_{j=1}^{n-1}\dot\pi(I_{nj})P_1\dot\pi(I_{nj})(\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_p)=0$, hence $\lambda(1,1)=0$.

Therefore, when π is the standard representation of K in $\Lambda^p(V)$, $1\le p\le \ell-1$, we have

$$(\lambda(r,s))_{0\le r,s\le 1}=\begin{pmatrix}0 & p-n\\ -p & 0\end{pmatrix}.$$

Therefore, we obtain a more explicit version of Corollary 3.6 using (3.2) and Remark 3.8.

Corollary 5.2. *Let Φ be an irreducible spherical function on G of type $\pi\in\hat{\mathrm{SO}}(n)$, $n=2\ell+1$. If the highest weight of π is of the form $(1,\dots,1,0,\dots,0)\in\mathbb{C}^\ell$, with p ones, $1\le p\le\ell-1$, then the function $H:(0,1)\to\mathrm{End}(\mathbb{C}^2)$ associated with Φ satisfies*

$$y(1-y)H''(y)+\frac12 n(1-2y)H'(y)+\frac{1+(1-2y)^2}{4y(1-y)}\begin{pmatrix}p-n & 0\\ 0 & -p\end{pmatrix}H(y)$$
$$+\frac{(1-2y)}{2y(1-y)}\begin{pmatrix}0 & p-n\\ -p & 0\end{pmatrix}H(y)=\lambda H(y),$$

for some $\lambda\in\mathbb{C}$.

6 The spherical functions of fundamental K-types

Let $n=2\ell$, the irreducible spherical functions of K-type

$$\mathbf{m}_n=(1,\dots,1,0,\dots,0)\in\mathbb{C}^\ell,$$

with p ones, $1\le p\le\ell-1$, are those associated with the irreducible representations of G of highest weights of the form $\mathbf{m}_{n+1}=(w+1,1,\dots,1,\delta,0,\dots,0)\in\mathbb{C}^\ell$ that interlaces $\mathbf{m}_n$,

$$\begin{matrix}w+1 & & 1 & \dots & 1 & & \delta & & 0 & \dots & 0\\ & 1 & \dots & & \dots & 1 & & 0 & \dots & & \dots & 0\end{matrix}.$$

We now consider the K-module $\Lambda^p(\mathbb{C}^n)$ which has highest weight $\mathbf{m}_n$.

For $w=0$ and $\delta=0$ we consider the G-module $\Lambda^p(\mathbb{C}^{n+1})$ whose highest weight is $\mathbf{m}_{n+1}$, and we have the following K-module decomposition

$$\Lambda^p\big(\mathbb{C}^{n+1}\big)=\Lambda^p(\mathbb{C}^n)\oplus\Lambda^{p-1}\big(\mathbb{C}^n\big)\wedge\mathbf{e}_{n+1},$$

where $\Lambda^p(\mathbb{C}^n)$ is the sum of two $\mathrm{SO}(n-1)$-modules:

$$\Lambda^p\big(\mathbb{C}^n\big)=\Lambda^p\big(\mathbb{C}^{n-1}\big)\oplus\Lambda^{p-1}\big(\mathbb{C}^{n-1}\big)\wedge\mathbf{e}_n.$$

We observe that

$$a(s)(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n) = \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge (\cos s\mathbf{e}_n - \sin s\mathbf{e}_{n+1})$$
$$= \cos s(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n) - \sin s(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_{n+1}).$$

Hence, if Φ_0 is the spherical function associated with the irreducible representation of G of highest weight $\mathbf{m}_{n+1} = (1, 1, \dots, 1, \delta, 0, \dots, 0) \in \mathbb{C}^\ell$ with $\delta = 0$, we have that

$$\Phi_0(a(s))(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n) = \cos s(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n).$$

Also we have that $a(s)(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_p) = \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_p$. Thus the vector valued function $F_0(s)$ given by the irreducible spherical function $\Phi_0(a(s))$ is

$$F_0(s) = \begin{pmatrix} \cos s \\ 1 \end{pmatrix}.$$

For $w = 0$ and $\delta = 1$ we consider the G-module $\Lambda^{p+1}(\mathbb{C}^{n+1})$ whose highest weight $\mathbf{m}_{n+1}$, and for $1 \le p \le \ell - 1$ we have the following K-module decomposition

$$\Lambda^{p+1}\big(\mathbb{C}^{n+1}\big) = \Lambda^{p+1}\big(\mathbb{C}^{n}\big) \oplus \Lambda^{p}\big(\mathbb{C}^{n}\big) \wedge \mathbf{e}_{n+1},$$

where $\Lambda^p(\mathbb{C}^n) \wedge \mathbf{e}_{n+1}$ is the sum of two $\mathrm{SO}(n-1)$-modules:

$$\Lambda^{p}\big(\mathbb{C}^{n}\big) \wedge \mathbf{e}_{n+1} = \Lambda^{p}\big(\mathbb{C}^{n-1}\big) \wedge \mathbf{e}_{n+1} \oplus \Lambda^{p-1}\big(\mathbb{C}^{n-1}\big) \wedge \mathbf{e}_n \wedge \mathbf{e}_{n+1}.$$

We observe that

$$a(s)(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n \wedge \mathbf{e}_{n+1}) = \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge (\sin s\mathbf{e}_n + \cos s\mathbf{e}_{n+1})$$
$$= \sin s(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n) + \cos s(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_{n+1}).$$

Hence, if Φ_1 is the spherical function associated with the irreducible representation of G of highest weight $\mathbf{m}_{n+1} = (1, 1, \dots, 1, \delta, 0, \dots, 0) \in \mathbb{C}^\ell$ with $\delta = 1$, we have that $\Phi_1(a(s))(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n \wedge \mathbf{e}_{n+1}) = \cos s(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n \wedge \mathbf{e}_{n+1})$. Also we have that

$$a(s)(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_p \wedge \mathbf{e}_{n+1}) = \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n \wedge \mathbf{e}_{n+1}.$$

Thus the vector valued function $F_1(s)$ given by the irreducible spherical function $\Phi_1(a(s))$ is

$$F_1(s) = \begin{pmatrix} 1 \\ \cos s \end{pmatrix}.$$

Definition 6.1. We shall consider the 2×2 matrix-valued function $\Psi = \Psi(y)$, for $0 < y < 1$, whose columns are given by the functions $H_0(y) = F_0(s)$ and $H_1(y) = F_1(s)$, with $\cos s = 2y - 1$:

$$\Psi(y) = \begin{pmatrix} 2y-1 & 1 \\ 1 & 2y-1 \end{pmatrix}. \tag{6.1}$$

Since the functions $H_0(y)$ and $H_1(y)$ are associated with irreducible spherical functions, they satisfy the differential equation given in Corollary 5.1; moreover, the respective eigenvalues are $\lambda = -p$ and $\lambda = p - n$. Therefore, we have

$$y(1-y)\Psi'' + \frac{1}{2}n(1-2y)\Psi' + \frac{1+(1-2y)^2}{4y(1-y)} \begin{pmatrix} p-n & 0 \\ 0 & -p \end{pmatrix} \Psi$$
$$+ \frac{(1-2y)}{2y(1-y)} \begin{pmatrix} 0 & p-n \\ -p & 0 \end{pmatrix} \Psi = \Psi \begin{pmatrix} -p & 0 \\ 0 & p-n \end{pmatrix}.$$

Furthermore, it is easy to check that the function $\Psi(y)$ also satisfy the equation above even when n is odd.

Theorem 6.2. *The function Ψ can be used to obtain a hypergeometric differential equation from the one given in Corollaries 5.1 and 5.2. Precisely, if H is a vector-valued solution of the differential equation in Corollaries 5.1 or 5.2, with eigenvalue λ, then $P=\Psi^{-1}H$ is a solution of $DP=\lambda P$, where D is the hypergeometric differential operator given by*

$$DP=y(1-y)P''-\begin{pmatrix}(\frac{n}{2}+1)(2y-1) & -1\\ -1 & (\frac{n}{2}+1)(2y-1)\end{pmatrix}P'-\begin{pmatrix}p & 0\\ 0 & n-p\end{pmatrix}P.$$

Proof. By hypothesis we have that

$$y(1-y)H''(y)+\frac{1}{2}n(1-2y)H'(y)+\frac{1+(1-2y)^2}{4y(1-y)}\begin{pmatrix}p-n & 0\\ 0 & -p\end{pmatrix}H(y)$$
$$+\frac{(1-2y)}{2y(1-y)}\begin{pmatrix}0 & p-n\\ -p & 0\end{pmatrix}H(y)=\lambda H(y),$$

Then, writing $H=\Psi P$, we have

$$y(1-y)P''+\big(2y(1-y)\Psi^{-1}\Psi'+\frac{n}{2}(1-2y)I\big)P'$$
$$+\Psi^{-1}\left(y(1-y)\Psi''+\frac{n}{2}(1-2y)\Psi'+\frac{1+(1-2y)^2}{4y(1-y)}\begin{pmatrix}p-n & 0\\ 0 & -p\end{pmatrix}\Psi\right.$$
$$\left.+\frac{(1-2y)}{2y(1-y)}\begin{pmatrix}0 & p-n\\ -p & 0\end{pmatrix}\Psi\right)P=\lambda P.$$

Now we compute

$$2y(1-y)\Psi^{-1}\Psi'=\frac{4y(1-y)}{4y(y-1)}\begin{pmatrix}2y-1 & -1\\ -1 & 2y-1\end{pmatrix}=-\begin{pmatrix}2y-1 & -1\\ -1 & 2y-1\end{pmatrix}.$$

Therefore

$$y(1-y)P''-\begin{pmatrix}(\frac{n}{2}+1)(2y-1) & -1\\ -1 & (\frac{n}{2}+1)(2y-1)\end{pmatrix}P'-\begin{pmatrix}\lambda+p & 0\\ 0 & \lambda+n-p\end{pmatrix}P=0.$$

This completes the proof of the theorem. ■

6.1 Δ-eigenvalues of spherical functions

As we said, when $n=2\ell$ the irreducible spherical functions of the pair $(\mathrm{SO}(n+1),\mathrm{SO}(n))$, of type $\mathbf{m}_n=(1,\dots,1,0\dots,0)\in\mathbb{C}^\ell$ with p ones, $1\le p\le \ell-1$ are those associated with the irreducible representations τ of G of highest weights of the form $\mathbf{m}_{n+1}=(w+1,1,\dots,1,\delta,0,\dots,0)\in\mathbb{C}^\ell$ with $p-1$ ones, such that the following pattern holds

$$\begin{matrix} w+1 & & 1 & \dots & 1 & & \delta & & 0 & \dots & 0 & \\ & 1 & \dots & & \dots & 1 & & 0 & \dots & & \dots & 0\end{matrix}.$$

Let $\Phi_{w,\delta}$ be the corresponding spherical function. Then $\Delta\Phi_{w,\delta}=\lambda\Phi_{w,\delta}$, where the eigenvalue $\lambda=\lambda_n(w,\delta)$ can be computed from the expression $\Delta=Q_{n+1}-Q_n$. If $v\in V_{\mathbf{m}_{n+1}}$ is a highest weight vector from (2.6) we have

$$\dot\tau(Q_{2\ell+1})v=-\big((w+1)^2+(2\ell-1)(w+1)+(2\ell-p)(p-1)+2\delta(\ell-p)\big)v.$$

If $v\in V_{\mathbf{m}_{2\ell}}$ is a highest weight vector, then from (2.5) we have

$$\dot\pi(Q_n)v=-p(2\ell-p)v.$$

Since $\Delta = Q_{n+1} - Q_n$ it follows that

$$\lambda_{2\ell}(w,\delta) = -(w+1)^2 - (2\ell-1)(w+1) + (2\ell-p) - 2\delta(\ell-p)$$

Analogously, we obtain that the eigenvalues of the spherical functions $\Phi_{w,\delta}$ of the pair $(\mathrm{SO}(2\ell+2), \mathrm{SO}(2\ell+1))$ are of the form

$$\lambda_{2\ell+1}(w,\delta) = -(w+1)(w+2\ell+1) + 2\ell - p + 1 - \delta 2(\ell-p) - \delta^2,$$

here δ is 0 or 1 when we are in the cases $1 \le p < \ell$ but δ could also be -1 in the particular case $p = \ell$.

Therefore, we have that the eigenvalues of the spherical functions $\Phi_{w,\delta}$ of the pair $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ are of the form

$$\lambda_n(w,\delta) = \begin{cases} -w(w+n+1) - p & \text{if} \quad \delta = 0, \\ -w(w+n+1) - n + p & \text{if} \quad \delta = \pm 1. \end{cases} \tag{6.2}$$

6.2 Polynomial eigenfunctions of the hypergeometric operator D

Let D be the differential operator on the real line introduced in Theorem 6.2:

$$DP = y(1-y)P'' + (C - yU)P' - VP, \tag{6.3}$$

with

$$C = \begin{pmatrix} (n/2+1) & 1 \\ 1 & (n/2+1) \end{pmatrix}, \qquad U = (n+2)I, \qquad V = \begin{pmatrix} p & 0 \\ 0 & n-p \end{pmatrix},$$

where n is of the form 2ℓ or $2\ell+1$ for $\ell \in \mathbb{N}$ and $1 \le p < \ell$.

We will study the $\mathbb{C}^2$-vector valued polynomial eigenfunctions of D.

The equation $DP = \lambda P$ is an instance of a matrix hypergeometric differential equation studied in [22]. Since the eigenvalues of C, $n/2$ and $n/2+2$, are not in $-\mathbb{N}_0$ the function P is determined by $P_0 = P(0)$. For $|y| < 1$ it is given by

$$P(y) = {}_2H_1\left(\begin{matrix} U, V+\lambda \\ C \end{matrix} ; y \right) P_0 = \sum_{j=0}^{\infty} \frac{y^j}{j!} [C; U; V+\lambda]_j P_0, \qquad P_0 \in \mathbb{C}^2,$$

where the symbol $[C; U; V+\lambda]_j$ is inductively defined by

$$[C; U; V+\lambda]_0 = 1,$$
$$[C; U; V+\lambda]_{j+1} = (C+j)^{-1} \left(j(U+j-1) + V + \lambda \right) [C; U; V+\lambda]_j,$$

for all $j \ge 0$.

Therefore, we have that there exists a polynomial solution if and only if the coefficient $[C; U; V+\lambda]_{j+1}$ is a singular matrix for some $j \in \mathbb{Z}$. Since the matrix $C+j$ is invertible for all $j \in \mathbb{N}_0$, we have that there is a polynomial solution of degree j for $DP = \lambda P$ if and only if there exists $P_0 \in \mathbb{C}^2$ such that $[C; U; V+\lambda]_j P_0 \neq 0$ and $(j(U+j-1) + V + \lambda)[C; U; V+\lambda]_j P_0 = 0$.

Now we easily observe that the only possible values for λ such that $j(U+j-1) + V + \lambda$ has non trivial kernel are those given in (6.2). Then, if $\lambda = -w(w+n+1) - p$, it is easy to check that the first and only j for which $j(U+j-1) + V + \lambda$ is singular is $j = w$, and its kernel (of dimension 1) is the subspace generated by $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Analogously, if $\lambda = -w(w+n+1) - n + p$, it is easy to check that the first and only j for which $j(U+j-1) + V + \lambda$ is singular is $j = w$, and its kernel (of dimension 1) is the subspace generated by $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ respectively. Therefore we have the following result.

Theorem 6.3. *For a given $\ell \in \mathbb{N}$ take $n = 2\ell$ or $2\ell + 1$ and $1 \le p \le \ell - 1$, then the polynomial eigenfunctions of*

$$DP = y(1-y)P'' + (C - yU)P' - VP,$$

with

$$C = \begin{pmatrix} (n/2+1) & 1 \\ 1 & (n/2+1) \end{pmatrix}, \qquad U = (n+2)I, \qquad V = \begin{pmatrix} p & 0 \\ 0 & n-p \end{pmatrix}$$

have eigenvalues $-w(w+n+1) - p$ or $-w(w+n+1) - n + p$, with $w \in \mathbb{N}_0$; in both cases the degree of the polynomial is w with leading coefficient a multiple of $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, respectively.

7 The inner product

Given a finite dimensional irreducible representation π of K in the vector space V_π let $(C(G) \otimes \operatorname{End}(V_\pi))^{K\times K}$ be the space of all continuous functions $\Phi : G \longrightarrow \operatorname{End}(V_\pi)$ such that $\Phi(k_1 g k_2) = \pi(k_1)\Phi(g)\pi(k_2)$ for all $g \in G$, $k_1, k_2 \in K$. Let us equip V_π with an inner product such that $\pi(k)$ becomes unitary for all $k \in K$. Then we introduce an inner product in the vector space $(C(G) \otimes \operatorname{End}(V_\pi))^{K\times K}$ by defining

$$\langle \Phi_1, \Phi_2 \rangle = \int_G \operatorname{tr}(\Phi_1(g)\Phi_2(g)^*)dg,$$

where dg denote the Haar measure on G normalized by $\int_G dg = 1$, and where $\Phi_2(g)^*$ denotes the adjoint of $\Phi_2(g)$ with respect to the inner product in V_π.

By using Schur's orthogonality relations for the unitary irreducible representations of G, it follows that if Φ_1 and Φ_2 are non equivalent irreducible spherical functions, then they are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle$, i.e.

$$\langle \Phi_1, \Phi_2 \rangle = 0.$$

Recall that, given an irreducible spherical function Φ of type π of the pair (G, K), the function $\Phi(a(s))$ is scalar valued when restricted to any $\mathrm{SO}(n-1)$-module (see (3.1) for $a(s)$). We shall denote by m the number of $\mathrm{SO}(n-1)$-submodules of π, and by $d_1, d_2, \dots, d_m$ the respective dimensions of each one of those submodules.

In particular, if Φ_1 and Φ_2 are two irreducible spherical functions of type $\pi \in \hat{K}$, we consider the vector valued functions $H_1(y)$ and $H_2(y)$ given by the diagonal matrix valued functions $\Phi_1(a(s))$ and $\Phi_2(a(s))$ (see Remark 3.7), with $y = (\cos s + 1)/2$, respectively, denoting

$$H_1(y) = (h_1(y), \dots, h_m(y))^t, \qquad H_2(y) = (f_1(y), \dots, f_m(y))^t.$$

Proposition 7.1. *If Φ_1, Φ_2 are two irreducible spherical functions of type $\pi \in \hat{K}$ then*

$$\langle \Phi_1, \Phi_2 \rangle = \frac{(n-1)!!}{(n-2)!!}\frac{2}{\omega_*}\sum_{i=1}^m d_i \int_0^1 (y(1-y))^{n/2-1} h_i(y)\overline{f_i(y)}dy,$$

with $\omega_* = \pi$ if n is even and $\omega_* = 2$ if n is odd.

Proof. Let $A = \exp \mathbb{R} I_{n+1,n}$ be the Lie subgroup of G of all elements of the form

$$a(s) = \exp sI_{n+1,n} = \begin{pmatrix} I_{n-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cos s & \sin s \\ \mathbf{0} & -\sin s & \cos s \end{pmatrix}, \qquad s \in \mathbb{R},$$

where I_{n-1} denotes the identity matrix of size $n-1$.

Now [12, Theorem 5.10, p. 190] establishes that for every $f \in C(G/K)$ and a suitable constant c_*

$$\int_{G/K} f(gK)dg_K = c_* \int_{K/M} \left(\int_{-\pi}^{\pi} \delta_*(a(s)) f(ka(s)K) ds \right) dk_M,$$

where dg_K and dk_M are respectively the invariant measures on G/K and K/M normalized by $\int_{G/K} dg_K = \int_{K/M} dk_M = 1$ and the function $\delta_* : A \longrightarrow \mathbb{R}$ is defined by

$$\delta_*(a(s)) = \prod_{\nu\in\Sigma^+} |\sin is\nu(I_{n+1,n})|,$$

with Σ^+ the set of those positive roots whose restrictions to $\mathfrak{a}$, the Lie algebra of A, are not zero. In our case we have $\delta_*(a(s)) = |\sin^{n-1} s|$.

To find the value of c_* we consider the function $f \equiv 1$, having then

$$1 = 2c_* \int_0^{\pi} \sin^{n-1} s ds.$$

Since

$$\int \sin^{n-1} s ds = -\frac{1}{n-1} \sin^{n-2} s \cos s + \frac{n-2}{n-1} \int \sin^{n-3} ds,$$

we obtain that, for $n = 2\ell$ or $2\ell + 1$,

$$\int_0^{\pi} \sin^{n-1} s ds = \frac{n-2}{n-1}\frac{n-4}{n-3} \cdots \frac{n-2\ell+1}{n-2\ell+2} \int_0^{\pi} \sin^{n-2\ell} s ds.$$

Therefore

$$c_* = \frac{(n-1)!!}{(n-2)!!}\frac{1}{2\omega_*},$$

with $\omega_* = \pi$ for $n = 2\ell$ and $\omega_* = 2$ for $2\ell + 1$.

Since the function $g \mapsto \operatorname{tr}(\Phi_1(g)\Phi_2(g)^*)$ is invariant under left and right multiplication by elements in K, we have

$$\langle \Phi_1, \Phi_2 \rangle = \int_G \operatorname{tr}(\Phi_1(g)\Phi_2(g)^*)dg = 2c_* \int_0^{\pi} \sin^{n-1} s \operatorname{tr}\left(\Phi_1(a(s)\Phi_2(a(s))^*)\right) ds.$$

If we put $y = \frac{1}{2}(\cos s + 1)$ for $0 < s < \pi$ we have

$$\operatorname{tr}\left(\Phi_1(a(s)\Phi_2(a(s))^*)\right) = \sum_{i=1}^{m} d_i h_i(y)\overline{f_i(y)}.$$

Then

$$\langle \Phi_1, \Phi_2 \rangle = 4c_* \sum_{i=1}^{m} d_i \int_0^1 (4y(1-y))^{(n-2)/2} h_i(y)\overline{f_i(y)} dy,$$

and the proposition follows. ■

Proposition 7.2. *If* $\Phi_1, \Phi_2 \in (C^\infty(G) \otimes \operatorname{End}(V_\pi))^{K\times K}$ *then*

$$\langle \Delta\Phi_1, \Phi_2 \rangle = \langle \Phi_1, \Delta\Phi_2 \rangle.$$

Proof. If we apply a left invariant vector field $X \in \mathfrak{g}$, to the function on G given by $g \mapsto \operatorname{tr}(\Phi_1(g)\Phi_2(g)^*)$, and then we integrate over G we obtain

$$0 = \int_G \operatorname{tr}\left((X\Phi_1)(g)\Phi_2(g)^*\right) dg + \int_G \operatorname{tr}\left(\Phi_1(g)(X\Phi_2)(g)^*\right) dg.$$

Therefore $\langle X\Phi_1, \Phi_2\rangle = -\langle \Phi_1, X\Phi_2\rangle$. Now let $\tau : \mathfrak{g}_{\mathbb{C}} \longrightarrow \mathfrak{g}_{\mathbb{C}}$ be the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to the real linear form $\mathfrak{g}$. Then $-\tau$ extends to a unique antilinear involutive * operator on $D(G)$ such that $(D_1D_2)^* = D_2^*D_1^*$ for all $D_1, D_2 \in D(G)$. This follows easily from the fact that the universal enveloping algebra over $\mathbb{C}$ of $\mathfrak{g}$ is canonically isomorphic to $D(G)$. Then it follows that $\langle D\Phi_1, \Phi_2\rangle = \langle \Phi_1, D^*\Phi_2\rangle$.

Finally, it is easy to verify that $\Delta^* = \Delta$. ■

7.1 Spherical functions as polynomial solutions of $DP = \lambda P$

Let us consider $\widetilde{D}$, the differential operator on $(0,1)$ introduced in Corollaries 5.1 and 5.2:

$$\begin{aligned} &y(1-y)H''(y) + \frac{1}{2}n(1-2y)H'(y) + \frac{1+(1-2y)^2}{4y(1-y)}\begin{pmatrix} p-n & 0 \\ 0 & -p\end{pmatrix} H(y) \\ &\quad + \frac{(1-2y)}{2y(1-y)}\begin{pmatrix} 0 & p-n \\ -p & 0\end{pmatrix} H(y) = \lambda H(y), \end{aligned} \tag{7.1}$$

Recall that the operator D that appears in (6.3) extends the differential operator $D = \Psi\widetilde{D}\Psi^{-1}$ to the whole real line, where

$$\Psi(y) = \begin{pmatrix} 2y-1 & 1 \\ 1 & 2y-1\end{pmatrix}$$

is the matrix function given in (6.1) and used in Theorem 6.2.

We want to focus our attention on the following vector spaces of $\mathbb{C}^2$-valued analytic functions on $(0,1)$:

$$\begin{aligned} S_\lambda &= \left\{H = H(y) : \widetilde{D}H = \lambda H,\ H\left(\tfrac{\cos s+1}{2}\right) \text{ analytic at } s=0\right\}, \\ W_\lambda &= \left\{P = P(y) : DP = \lambda P,\ \text{analytic on } [0,1]\right\}. \end{aligned}$$

From Theorem 6.2 we know that the correspondence $P \mapsto \Psi P$ is an injective linear map from W_λ into S_λ. Now we want to prove that this map is bijective.

Theorem 7.3. *The linear map $P \mapsto \Psi P$ is an isomorphism from W_λ onto S_λ.*

Proof. A vector valued function $P \in W_\lambda$ is an eigenfunction of the hypergeometric operator D. Since it is analytic at $y = 1$ it is determined by $P(1)$, therefore $\dim(W_\lambda) = 2$.

On the other hand, if $H \in S_\lambda$ then there is a function $F(s)$ analytic at $s = 0$, such that it extends the function $H(\frac{\cos s+1}{2})$ defined on $(0,\pi)$. Then, F satisfies the following differential equation

$$\begin{aligned} &F''(s) + (n-1)\frac{\cos s}{\sin s}F'(s) + \frac{1+\cos^2 s}{\sin^2 s}\begin{pmatrix} p-n & 0 \\ 0 & -p\end{pmatrix} F(s) \\ &\quad - 2\frac{\cos s}{\sin^2 s}\begin{pmatrix} 0 & p-n \\ -p & 0\end{pmatrix} F(s) = \lambda F(s), \end{aligned}$$

or equivalently

$$\sin^2 sF''(s) + \frac{n-1}{2}\sin(2s)F'(s) + (2-\sin^2 s)\begin{pmatrix} p-n & 0 \\ 0 & -p\end{pmatrix} F(s)$$

$$- 2\cos s \begin{pmatrix} 0 & p-n \\ -p & 0 \end{pmatrix} F(s) = \lambda \sin^2 s F(s), \tag{7.2}$$

Let $a_j \in \mathbb{C}^2$ and $\alpha_j, \beta_j, \gamma_j \in \mathbb{C}$, for $j \geq 0$, be the Taylor coefficients of F, sin, $\sin^2$ and cos at $s = 0$:

$$\begin{aligned}
F(s) &= \sum_{j\geq 0} a_j s^j, & \sin s &= \sum_{j\geq 1} \alpha_j s^j, \\
F'(s) &= \sum_{j\geq 0} a_{j+1}(j+1)s^j, & \sin^2 s &= \sum_{j\geq 2} \beta_j s^j, \\
F''(s) &= \sum_{j\geq 0} a_{j+2}(j+2)(j+1)s^j, & \cos s &= \sum_{j\geq 0} \gamma_j s^j.
\end{aligned}$$

Therefore, from (7.2) we have

$$\begin{aligned}
&\sum_{j\geq 0} \Bigg[\sum_{k=0}^{j-2} \beta_{j-k} a_{k+2}(k+2)(k+1) + \frac{n-1}{2} \sum_{k=0}^{j-1} 2^{j-k} \alpha_{j-k} a_{k+1}(k+1) + \begin{pmatrix} p-n & 0 \\ 0 & -p \end{pmatrix} \\
&\quad \times \left(2a_j - \sum_{k=0}^{j-2} \beta_{j-k} a_k \right) - 2 \begin{pmatrix} 0 & p-n \\ -p & 0 \end{pmatrix} \sum_{k=0}^{j} \gamma_{j-k} a_k \Bigg] s^j = \lambda \sum_{j\geq 0} \left[\sum_{k=0}^{j-2} \beta_{j-k} a_k \right] s^j.
\end{aligned}$$

Hence, since $\beta_2 = \alpha_1 = \gamma_0 = 1$, we have that

$$\left[j(j-1) + (n-1)j + 2 \begin{pmatrix} p-n & -p+n \\ p & -p \end{pmatrix} \right] a_j$$

is a linear combination with matrix coefficients of $\{a_0, a_1, \dots, a_{j-1}\}$; it is clear that for $j = 1$ and $j > 2$ the matrix above is non singular, therefore $\{a_0, a_2\}$ determine completely the sequence $\{a_j\}_{j\geq 0}$. Also it is clear that when $j = 0$ or 2, that matrix has nullity 1. Therefore we can conclude that $\dim(S_\lambda) = 2$. The theorem follows. ■

Theorem 7.4. *Let H be the $\mathbb{C}^2$-valued analytic function on $(0,1)$ given by an irreducible spherical function Φ on G of fundamental K-type $(1,\dots,1,0,\dots,0) \in \mathbb{C}^\ell$, with p ones, $0 < p < \ell$. If $P = \Psi^{-1}H$, then P is polynomial.*

Proof. We know that the function H is analytic in $(0,1)$, and from Corollary 5.1 we know that it is an eigenfunction of the operator $\widetilde{D}$ (see (7.1)). Also we know that the function $H(\frac{1+\cos s}{2})$ is analytic at $s = 0$, since $\Phi(a(s))$ it is. Therefore from Theorem 7.3 the function $P = \Psi^{-1}H$ is an analytic eigenfunction of D on the closed interval $[0,1]$.

If we introduce the following matrix-weight function $V = V(y)$ supported on the interval $[0,1]$

$$V(y) = \frac{(n-1)!!}{(n-2)!!} \frac{2}{\omega_*} (y(1-y))^{n/2-1} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},$$

with $\omega_* = \pi$ if n is even and $\omega_* = 2$ if n is odd, then from Proposition 7.1 we have

$$\langle \Phi_0, \Phi_1 \rangle = \int_0^1 H_2^*(y) V(y) H_1^*(y) dy.$$

It follows from Propositions 7.1 and 7.2 that $\widetilde{D}$ is a symmetric operator with respect to the inner product defined among continuous vector-valued functions on $[0,1]$ by

$$\langle H_1, H_2 \rangle_V = \int_0^1 H_2^*(y) V(y) H_1(y) dy.$$

Then, since $D = \Psi^{-1}\widetilde{D}\Psi$, we have that D is a symmetric operator with respect to the inner product defined among continuous vector-valued functions on $[0,1]$ by

$$\langle P_1, P_2\rangle_W = \int_0^1 P_2^*(y)W(y)P_1(y)dy,$$

where

$$W = \Psi^* V \Psi.$$

Actually, we have that (W, D) is a classical pair in the sense of [7], see also [2]. As the weight W has finite moments there exists a sequence $\{Q_r\}_{r\geq 0}$ of 2×2 matrix-valued orthonormal polynomials, such that $DQ_r = Q_r\Lambda_r$ where Λ_r is a real diagonal matrix (for precise definitions and general facts on matrix-valued orthogonal polynomials see [5] and [2]).

Let $\{e_1, e_2\}$ be the canonical basis of $\mathbb{C}^2$. Then

$$\langle Q_r e_j, Q_s e_i\rangle_W = e_i^*\left(\int_0^1 Q_s^*(y)W(y)Q_r^*(y)dy\right)e_j = e_i^*\delta_{si}Ie_j = \delta_{r,s}\delta_{i,j}.$$

Therefore, for $r\geq 0$, $j=1,2$, $\{Q_r e_j\}$ is a family of $\mathbb{C}^2$-valued orthonormal polynomials such that

$$D(Q_r e_j) = (DQ_r)e_j = (Q_r\Lambda_r)e_j = Q_r(\Lambda_r e_j) = \lambda_r^j(Q_r e_j),$$

where $\Lambda_r = \operatorname{diag}(\lambda_r^1, \lambda_r^2)$.

Now we write our function $P = \Psi^{-1}H$ as $P = \sum_{r,j} a_{r,j}Q_r e_j$, where $a_{r,j} = \langle P, Q_r e_j\rangle_W$. Since P is analytic on $[0,1]$ the sum converges not only in the L^2-norm but also in the topology based on uniform convergence of sequences of functions and their successive derivatives.

Therefore,

$$\lambda P = DP = \sum_{r,j} a_{r,j}\lambda_r^j Q_r e_j.$$

Then $a_{r,j} = 0$ if $\lambda_r^j \neq \lambda$. Since $\dim W_\lambda = 2$ it follows that P is a polynomial. ■

Remark 7.5. It is easy to see from (5.1) and (5.2) that the dimensions of the M-submodules of the fundamental representation of K with highest weight of the form $(1,\dots,1,0\dots,0)$, with p ones, are given by

$$d_1 = \frac{(n-1)!}{(p-1)!(n-p)!}, \qquad d_2 = \frac{(n-1)!}{p!(n-1-p)!},$$

therefore the weight W is given by

$$W = \frac{(n-1)!!}{(n-2)!!}\frac{2}{\omega_*}\frac{(n-1)!}{p!(n-p)!}(y(1-y))^{n/2-1}\Psi^*\begin{pmatrix} p & 0\\ 0 & n-p\end{pmatrix}\Psi,$$

with $\omega_* = \pi$ if n is even and $\omega_* = 2$ if n is odd. Then, W is a scalar multiple of

$$\begin{pmatrix} p(2y-1)^2 + n - p & n(2y-1)\\ n(2y-1) & (n-p)(2y-1)^2 + p\end{pmatrix}.$$

Even more, since $0 < p < \ell$ and $n = 2\ell, 2\ell+1$ it follows that $p \neq n-p$. Then it can be proved that the weight W does not reduce to a smaller size, i.e., there is not any invertible matrix M such that $M^*W(y)M$ is diagonal for all $y\in[0,1]$.

For a given fundamental K type $\pi \in \hat{\mathrm{SO}}(n)$, $n = 2\ell$ or $2\ell+1$, with highest weight of the form $(1,\dots,1,0,\dots,0) \in \mathbb{C}^\ell$ with p ones $(0 < p < \ell)$, let $\Phi_{w,\delta}$ denote the irreducible spherical function of the pair $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ given by $\tau \in \hat{\mathrm{SO}}(n+1)$ with highest weight of the form $(w+1, 1, \dots, 1, \delta, 0 \dots, 0)$ with $p-1$ ones.

Therefore, combining (6.2), Theorems 6.3 and 7.4 we have the following statement.

Theorem 7.6. *Given $w \in \mathbb{N}_0$, every irreducible spherical function $\Phi_{w,\delta}$ of the pair $(\mathrm{SO}(n+1), \mathrm{SO}(n))$, with $n = 2\ell$ or $2\ell+1$, of type $\mathbf{m}_n = (1,\dots,1,0,\dots,0) \in \mathbb{C}^\ell$ with p ones $(0 < p < \ell)$, corresponds to a vector valued function $P_{w,\delta}$ $(\delta = 0,1)$, which is a polynomial of degree w; and the leading coefficients of $P_{w,0}$ and $P_{w,1}$ are multiples of $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ respectively. Precisely*

$$P_{w,\delta}(y) = \sum_{j=0}^{w} \frac{y^j}{j!} [C; U; V+\lambda]_j P_{w,\delta}(0),$$

with

$$C = \begin{pmatrix} (n/2+1) & 1 \\ 1 & (n/2+1) \end{pmatrix}, \qquad U = (n+2)I, \qquad V = \begin{pmatrix} p & 0 \\ 0 & n-p \end{pmatrix},$$
$$\lambda = \lambda_n(w,\delta) = \begin{cases} -w(w+n+1) - p & \text{if} \quad \delta = 0, \\ -w(w+n+1) - n + p & \text{if} \quad \delta = 1. \end{cases}$$

Even more, the value of $P_{w,\delta}(0)$ can be computed.

Proof. It only remains to prove that $P_{w,\delta}(0)$ can be computed.

Let us consider the case $\delta = 0$. We know from (6.2) and Theorem 6.3 that there is some $c \in \mathbb{C}$ such that

$$[C; U; V+\lambda]_w P_{w,0}(0) = c \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Since $[C; U; V+\lambda]_w$ is invertible, this c is univocally determined by the condition $\Phi(e) = I$, which implies

$$\Psi(1) \sum_{j=0}^{w} \frac{1}{j!} [C; U; V+\lambda]_j P_{w,0}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Similarly, we can prove the same for $P_{w,1}(0)$. ■

Remark 7.7. It is worth to observe that for w, $w' \geq 0$ and $\delta, \delta' = 0, 1$, since $\langle P_{w,\delta}, P_{w',\delta'} \rangle_W = \langle \Phi_{w,\delta}, \Phi_{w',\delta'} \rangle$, we have that if $(w,\delta) \neq (w',\delta')$ then

$$\langle P_{w,\delta}, P_{w',\delta'} \rangle_W = 0.$$

Therefore, our construction encodes all equivalent classes of irreducible spherical functions of a fundamental K-type of highest weight λ_p, $0 < p < \ell$, in the orthogonal set of $\mathbb{C}^2$-valued polynomials $\{P_{w,0}, P_{w,1}\}$. The degree of $P_{w,0}$ and $P_{w,1}$ is w, and the leading coefficient is a multiple of $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, respectively.

8 Matrix valued orthogonal polynomials

8.1 Matrix valued orthogonal polynomials

In this subsection, given n of the form 2ℓ or $2\ell+1$ with $\ell \in \mathbb{N}$, for a fixed $0 < p < \ell$ we shall construct a sequence of matrix-valued polynomials $\{P_w\}_{w\geq 0}$ directly related to irreducible spherical functions of type $\pi \in \hat{\mathrm{SO}}(n)$ of highest weight $\mathbf{m}_\pi = (1,\dots,1,0\dots,0) \in \mathbb{C}^\ell$, with p ones.

Given a nonnegative integer w and $\delta = 0,1$, we can consider $\Phi_{w,\delta}$, the irreducible spherical function of type π associated with the irreducible representation $\tau \in \hat{\mathrm{SO}}(n+1)$ of highest weight of the form $\mathbf{m}_\tau = (w+1,1,\dots,1,\delta,0,\dots,0)$ with $p-1$ ones.

We insist on recalling that, since π has only two $\mathrm{SO}(n-1)$-submodules, we can interpret the diagonal matrix-valued function $\Phi_{w,\delta}(a(s))$, $s \in (0,\pi)$, as a 2 column vector function.

Now we consider the vector-valued function

$$P_{w,\delta}: \ (0,1) \to \mathbb{C}^2$$

given by the vector function $P_{w,\delta}(y) = \Psi^{-1}(y)\Phi_{w,\delta}(a(s))$, with $\cos(s) = 2y-1$. Then, we define the matrix-valued function

$$P_w = P_w(y),$$

whose δ-th column ($\delta = 0,1$) is given by the $\mathbb{C}^2$-valued polynomial $P_{w,\delta}(y)$.

Let consider the matrix-valued skew symmetric bilinear form defined among C^∞ 2×2 matrix-valued functions on $[0,1]$ by

$$\langle P, Q\rangle_W = \int_0^1 Q^*(y)W(y)P(y)dy,$$

where

$$W = \frac{(n-1)!!}{(n-2)!!}\frac{2}{\omega_*}\frac{(n-1)!}{p!(n-p)!}(y(1-y))^{n/2-1}\begin{pmatrix} p(2y-1)^2+n-p & n(2y-1) \\ n(2y-1) & (n-p)(2y-1)^2+p \end{pmatrix}.$$

See Remark 7.5. Then we state the following theorem.

Theorem 8.1. *The matrix-valued polynomial functions P_w, $w \geq 0$, form a sequence of orthogonal polynomials with respect to W, which are eigenfunctions of the symmetric differential operator D in (6.3). Moreover,*

$$DP_w = P_w\begin{pmatrix} \lambda(w,0) & 0 \\ 0 & \lambda(w,1) \end{pmatrix},$$

where

$$\lambda(w,\delta) = \begin{cases} -w(w+n+1)-p & \text{if} \quad \delta = 0, \\ -w(w+n+1)-n+p & \text{if} \quad \delta = 1. \end{cases}$$

Proof. From Theorem 6.2 we have that the δ-th column of P_w is an eigenfunction of the operator D with eigenvalue $\lambda(w,\delta)$, see (6.2) and (6.3). Therefore we have

$$DP_w = P_w\Lambda_w,$$

with

$$\Lambda_w = \begin{pmatrix} \lambda(w,0) & 0 \\ 0 & \lambda(w,1) \end{pmatrix}.$$

From Theorem 7.6 we know that each column of P_w is a polynomial function of degree w and, even more, that P_w is a polynomial whose leading coefficient is a nonsingular diagonal matrix.

Given w and w', non negative integers, by using Remark 7.7 we have

$$\begin{aligned}\langle P_{w'}, P_w\rangle_W &= \int_0^1 P_w(y)^* W(y) P_{w'}(y) du = \sum_{\delta,\delta'=0}^{1} \int_0^1 \big(P_{w,\delta}(y)^* W(y) P_{w',\delta'}(y) du\big) E_{\delta,\delta'} \\ &= \sum_{\delta,\delta'=0}^{1} \delta_{w,w'}\delta_{\delta,\delta'} \left(\int_0^1 P_{w,\delta}(y)^* W(y) P_{w',\delta'}(y) du\right) E_{\delta,\delta'} \\ &= \delta_{w,w'} \sum_{\delta=0}^{1} \int_0^1 \big(P_{w,\delta}(y)^* W(y) P_{w'\delta}(y) du,\big) E_{\delta,\delta},\end{aligned}$$

which proves the orthogonality. Even more, it also shows us that $\langle P_w, P_w\rangle_W$ is a diagonal matrix. Also, making a few simple computations we have that

$$\langle DP_w, P_{w'}\rangle = \delta_{w,w'}\langle P_w, P_{w'}\rangle \Lambda_w = \delta_{w,w'}\Lambda_w^*\langle P_w, P_{w'}\rangle = \langle P_w, DP_{w'}\rangle,$$

for every w, $w' \in \mathbb{N}_0$, since Λ_w is real and diagonal. This concludes the proof of the theorem. ■

9 The SO$(2\ell+1)$-type with highest weight $2\lambda_\ell$

In this section $K = \mathrm{SO}(2\ell+1)$. We will focus on the particular case when the K-type is given by an irreducible representation π with highest weight $2\lambda_\ell = (1, 1, \dots, 1)$. We will first see that such K-module is the direct sum of three M-submodules, and we will find similar results to those obtained for the fundamental K-types $\lambda_1, \dots, \lambda_{\ell-1}$ that are direct sum of two M-submodules.

Let us consider the irreducible K-module $\Lambda^\ell(V)$, with $V = \mathbb{C}^n$, $n = 2\ell+1$. The vector $v = (\mathbf{e}_1 - i\mathbf{e}_2) \wedge (\mathbf{e}_3 - i\mathbf{e}_4) \wedge \cdots \wedge (\mathbf{e}_{2\ell-1} - i\mathbf{e}_{2\ell})$ is the unique, up to a scalar, dominant vector and its weight is $2\lambda_\ell = (1, 1, \dots, 1)$.

It is not difficult to see that $\Lambda^\ell(V)$ is the sum of three M-irreducible submodules, namely

$$\Lambda^\ell(V) = V_1 \oplus V_0 \oplus V_{-1} \tag{9.1}$$

with respective highest weights $(1, \dots, 1), (1, \dots, 1, 0), (1, \dots, 1, -1) \in \mathbb{C}^\ell$ and having $V_0 = \Lambda^{\ell-1}(V) \wedge \mathbf{e}_n$ and $V_1 \oplus V_{-1} \simeq \Lambda^\ell(\mathbb{C}^{n-1})$.

The vectors

$$\begin{aligned} v_1 &= (\mathbf{e}_1 - i\mathbf{e}_2) \wedge (\mathbf{e}_3 - i\mathbf{e}_4) \wedge \cdots \wedge (\mathbf{e}_{2\ell-1} - i\mathbf{e}_{2\ell}), \\ v_0 &= -(\mathbf{e}_1 - i\mathbf{e}_2) \wedge (\mathbf{e}_3 - i\mathbf{e}_4) \wedge \cdots \wedge (\mathbf{e}_{2\ell-3} - i\mathbf{e}_{2\ell-2}) \wedge \mathbf{e}_n, \\ v_{-1} &= (\mathbf{e}_1 - i\mathbf{e}_2) \wedge (\mathbf{e}_3 - i\mathbf{e}_4) \wedge \cdots \wedge (\mathbf{e}_{2\ell-1} + i\mathbf{e}_{2\ell}) \end{aligned}$$

are M-highest weight vectors in V_1, V_0 and V_{-1}, respectively. Also let us call P_1, P_0 and P_{-1} the respective projections on V_1, V_0 and V_{-1}, according to the decomposition (9.1).

In order to obtain the explicit expression of E in (3.2) we are interested to compute

$$\sum_{j=1}^{n-1} \dot\pi(I_{nj}) P_s \dot\pi(I_{nj})\big|_{V_r} = \lambda(r,s) I_{V_r},$$

with $r, s = 1, 0, -1$ corresponding to the three M-submodules V_1, V_0 and V_{-1} of the representation π.

If $1 \le j \le \ell$, then

$$\dot\pi(I_{n,2j-1})(\mathbf{e}_{2k-1} - i\mathbf{e}_{2k}) = \begin{cases} 0 & \text{if} \quad k \neq j, \\ -\mathbf{e}_n & \text{if} \quad k = j, \end{cases}$$
$$\dot\pi(I_{n,2j})(\mathbf{e}_{2k-1} - i\mathbf{e}_{2k}) = \begin{cases} 0 & \text{if} \quad k \neq j, \\ i\mathbf{e}_n & \text{if} \quad k = j, \end{cases}$$

therefore, it is easy to see that $P_0\dot\pi(I_{n,2j-1})v_0 = P_0\dot\pi(I_{n,2j})v_0 = 0$ and that $P_r\dot\pi(I_{n,2j-1})v_s = P_r\dot\pi(I_{n,2j})v_s = 0$ when $s \pm 1$ and $r \pm 1$; i.e.

$$\lambda(0,0) = \lambda(-1,-1) = \lambda(1,-1) = \lambda(-1,1) = \lambda(1,1) = 0.$$

Furthermore, it is easy to see that, for $1 \le j \le \ell$ and r equal to 1 or -1, we have

$$\dot\pi(I_{n,2j-1})P_0\dot\pi(I_{n,2j-1})v_r + \dot\pi(I_{n,2j})P_0\dot\pi(I_{n,2j})v_r = -v_r,$$

then $\lambda(-1,0) = \lambda(1,0) = -\ell$. Therefore, it only remains to compute

$$\sum_{j=1}^{\ell} \left(\dot\pi(I_{n,2j-1})P_s\dot\pi(I_{n,2j-1})v_0 + \dot\pi(I_{n,2j})P_s\dot\pi(I_{n,2j})v_0\right),$$

for $s = \pm 1$.

To obtain $P_s\dot\pi(I_{n,k})v_0$ it is necessary to decompose $\dot\pi(I_{n,k})v_0$ according to the direct sum (9.1). We know that $\dot\pi(X_{-\varepsilon_j-\varepsilon_\ell})v_1 \in V_1$ and $\dot\pi(X_{-\varepsilon_j+\varepsilon_\ell})v_{-1} \in V_{-1}$; recall that

$$X_{-\epsilon_j-\epsilon_\ell} = I_{2\ell-1,2j-1} - I_{2\ell,2j} + i(I_{2\ell-1,2j} + I_{2\ell,2j-1}),$$
$$X_{-\epsilon_j+\epsilon_\ell} = I_{2\ell-1,2j-1} + I_{2\ell,2j} + i(I_{2\ell-1,2j} - I_{2\ell,2j-1}),$$

see (2.2). We have

$$\dot\pi\left(X_{-\varepsilon_j-\varepsilon_\ell}\right)(\mathbf{e}_{2j-1} - i\mathbf{e}_{2j}) = -2(\mathbf{e}_{2\ell-1} + i\mathbf{e}_{2\ell}),$$
$$\dot\pi\left(X_{-\varepsilon_j-\varepsilon_\ell}\right)(\mathbf{e}_{2\ell-1} - i\mathbf{e}_{2\ell}) = 2\left(\mathbf{e}_{2j-1} + i\mathbf{e}_{2j}\right),$$
$$\dot\pi\left(X_{-\varepsilon_j-\varepsilon_\ell}\right)(\mathbf{e}_{2k-1} - i\mathbf{e}_{2k}) = 0, \qquad \text{for} \quad k \neq s, \ell.$$

Therefore, for $1 \le j < \ell$,

$$\dot\pi\left(X_{-\varepsilon_j-\varepsilon_\ell}\right)v_1 = 2(\mathbf{e}_1 - i\mathbf{e}_2) \wedge \cdots \wedge (\mathbf{e}_{2(\ell-1)-1} - i\mathbf{e}_{2(\ell-1)}) \wedge (\mathbf{e}_{2j-1} + i\mathbf{e}_{2j})$$
$$- 2(\mathbf{e}_1 - i\mathbf{e}_2) \wedge \cdots \wedge (\mathbf{e}_{2j-3} - i\mathbf{e}_{2j-2}) \wedge (\mathbf{e}_{2\ell-1} + i\mathbf{e}_{2\ell}) \wedge (\mathbf{e}_{2j+1} - i\mathbf{e}_{2j+2}) \wedge$$
$$\cdots \wedge (\mathbf{e}_{2\ell-1} - i\mathbf{e}_{2\ell}).$$

Similarly, for $1 \le j < \ell$,

$$\dot\pi\left(X_{-\varepsilon_j+\varepsilon_\ell}\right)v_1 = 2(\mathbf{e}_1 - i\mathbf{e}_2) \wedge \cdots \wedge (\mathbf{e}_{2(\ell-1)-1} - i\mathbf{e}_{2(\ell-1)}) \wedge (\mathbf{e}_{2j-1} + i\mathbf{e}_{2j})$$
$$+ 2(\mathbf{e}_1 - i\mathbf{e}_2) \wedge \cdots \wedge (\mathbf{e}_{2j-3} - i\mathbf{e}_{2j-2}) \wedge (\mathbf{e}_{2\ell-1} + i\mathbf{e}_{2\ell}) \wedge (\mathbf{e}_{2j+1} - i\mathbf{e}_{2j+2}) \wedge$$
$$\cdots \wedge (\mathbf{e}_{2\ell-1} - i\mathbf{e}_{2\ell}).$$

Hence, for $1 \le j < \ell$, we have

$$\tfrac{-i}{8}\left(\dot\pi\left(X_{-\varepsilon_j-\varepsilon_\ell}\right)v_1 + \dot\pi\left(X_{-\varepsilon_j+\varepsilon_\ell}\right)v_{-1}\right)$$
$$= (\mathbf{e}_1 - i\mathbf{e}_2) \wedge \cdots \wedge (\mathbf{e}_{2(\ell-1)-1} - i\mathbf{e}_{2(\ell-1)}) \wedge \mathbf{e}_{2j} = \dot\pi(I_{n,2j})v_0,$$
$$\tfrac{1}{8}\left(\dot\pi\left(X_{-\varepsilon_j-\varepsilon_\ell}\right)v_1 + \dot\pi\left(X_{-\varepsilon_j+\varepsilon_\ell}\right)v_{-1}\right)$$

$$= (\mathbf{e}_1 - i\mathbf{e}_2) \wedge \cdots \wedge (\mathbf{e}_{2(\ell-1)-1} - i\mathbf{e}_{2(\ell-1)}) \wedge \mathbf{e}_{2j-1} = \dot\pi(I_{n,2j-1})v_0.$$

Then, for $1 \leq j < \ell$,

$$\begin{aligned}\dot\pi\left(I_{n,2j-1}\right) P_1 \dot\pi\left(I_{n,2j-1}\right) v_0 &= \tfrac{1}{8}\dot\pi\left(I_{n,2j-1}\right)\dot\pi\left(X_{-\varepsilon_j-\varepsilon_\ell}\right) v_1\\ &= \tfrac{i}{2}\big(\mathbf{e}_1 - i\mathbf{e}_2\big) \wedge \cdots \wedge \mathbf{e}_{2j} \wedge \cdots \wedge \big(\mathbf{e}_{2(\ell-1)-1} - i\mathbf{e}_{2(\ell-1)}\big) \wedge \mathbf{e}_n,\\ \dot\pi\left(I_{n,2j}\right) P_1 \dot\pi\left(I_{n,2j}\right) v_0 &= \tfrac{-i}{8}\dot\pi\left(I_{n,2j}\right)\dot\pi\left(X_{-\varepsilon_j-\varepsilon_\ell}\right) v_1\\ &= -\tfrac{1}{2}\big(\mathbf{e}_1 - i\mathbf{e}_2\big) \wedge \cdots \wedge \mathbf{e}_{2j-1} \wedge \cdots \wedge \big(\mathbf{e}_{2(\ell-1)-1} - i\mathbf{e}_{2(\ell-1)}\big) \wedge \mathbf{e}_n.\end{aligned}$$

Therefore, for $1 \leq j < \ell$,

$$\dot\pi\left(I_{n,2j-1}\right) P_1 \dot\pi\left(I_{n,2j-1}\right) v_0 + \dot\pi\left(I_{n,2j}\right) v_0 P_1 \dot\pi\left(I_{n,2j}\right) v_0 = -\tfrac{1}{2} v_0.$$

Besides, for $j = \ell$ we have

$$\dot\pi(I_{n,2\ell})v_0 = \tfrac{1}{2i}(-v_1 + v_{-1}) \qquad \text{and} \qquad \dot\pi(I_{n,2\ell-1})v_0 = \tfrac{1}{2}(v_1 + v_{-1}).$$

Therefore, since

$$\begin{aligned}\dot\pi\left(I_{n,2\ell}\right) P_1 \dot\pi(I_{n,2\ell})v_0 &= -\tfrac{1}{2i}\dot\pi\left(I_{n,2\ell}\right) v_1 = -\tfrac{1}{2}v_0,\\ \dot\pi\left(I_{n,2\ell-1}\right) P_1 \dot\pi(I_{n,2\ell-1})v_0 &= \tfrac{1}{2}\dot\pi\left(I_{n,2\ell-1}\right) v_1 = -\tfrac{1}{2}v_0,\end{aligned}$$

we have that

$$\sum_{j=0}^{n-1} \dot\pi\left(I_{n,j}\right) P_1 \dot\pi(I_{n,j})v_0 = -\frac{\ell+1}{2} v_0,$$

i.e.

$$\lambda(0,1) = -\frac{\ell+1}{2}.$$

Analogously we obtain

$$\lambda(0,-1) = -\frac{\ell+1}{2}.$$

Hence

$$(\lambda(r,s))_{-1\leq r,s\leq 1} = \begin{pmatrix} 0 & -\ell & 0 \\ -\frac{\ell+1}{2} & 0 & -\frac{\ell+1}{2} \\ 0 & -\ell & 0 \end{pmatrix}.$$

Therefore, we obtain a more explicit version of Corollary 3.6 using (3.2) and Remark 3.8. Confront Corollary 5.2.

Corollary 9.1. *Let Φ be an irreducible spherical function on G of type $\pi \in \hat{\mathrm{SO}}(n)$, $n = 2\ell + 1$. If the highest weight of π is of the form $(1, \dots, 1) \in \mathbb{C}^\ell$, then the function $H : (0,1) \to \mathbb{C}^3$ associated with Φ satisfies $\widetilde{D}H = \lambda H$, for some $\lambda \in \mathbb{C}$ with*

$$\begin{aligned}\widetilde{D}H = y(1-y)H''(y) + \frac{1}{2}n(1-2y)H'(y) + \frac{(1-2y)^2+1}{4y(1-y)}\begin{pmatrix} -\ell & 0 & 0 \\ 0 & -\ell-1 & 0 \\ 0 & 0 & -\ell \end{pmatrix} H(y)\\ + \frac{(1-2y)}{2y(1-y)}\begin{pmatrix} 0 & -\ell & 0 \\ -\frac{\ell+1}{2} & 0 & -\frac{\ell+1}{2} \\ 0 & -\ell & 0 \end{pmatrix} H(y).\end{aligned}$$

9.1 Spherical functions of SO$(2\ell+1)$-type $2\lambda_\ell$

Let $n = 2\ell + 1$, we now focus on the spherical functions $\Phi_{w,\delta}$ of type $\mathbf{m}_n = (1, \dots, 1) \in \mathbb{C}^\ell$, which are associated with the irreducible representations of $\mathrm{SO}(n+1)$ of highest weights of the form $\mathbf{m}_{n+1} = (w+1, 1, \dots, 1, \delta) \in \mathbb{C}^{\ell+1}$ such that the following pattern holds

$$\begin{matrix} w+1 & & 1 & \dots & 1 & & \delta \\ & 1 & \dots & & \dots & 1 & & -1 \end{matrix}.$$

As before we make the function Ψ whose columns are given by the spherical functions $\Phi_{0,\delta}$, $\delta = -1, 0, 1$. When $w = 0$, this is calculable using [24, p. 364, equation (8)] or alternatively by considering the G-modules $\Lambda^{\ell+1}(\mathbb{C}^{n+1}) = V_1 \oplus V_{-1}$ and $\Lambda^\ell(\mathbb{C}^{n+1}) = V_0$ and working in the same way that we already did in the beginning of Section 6 for the 2×2 cases (here V_t, for $t = 1, 0, -1$, are the irreducible G-modules with highest weights $(1, \dots, 1, t) \in \mathbb{C}^{\ell+1}$).

Therefore, if $\cos s = 2y - 1$ we have

$$\Psi(y) = \begin{pmatrix} e^{is} & 1 & e^{-is} \\ 1 & \frac{1}{2}(e^{is} + e^{-is}) & 1 \\ e^{-is} & 1 & e^{is} \end{pmatrix}$$
$$= \begin{pmatrix} 2y-1+2i\sqrt{y-y^2} & 1 & 2y-1-2i\sqrt{y-y^2} \\ 1 & 2y-1 & 1 \\ 2y-1-2i\sqrt{y-y^2} & 1 & 2y-1+2i\sqrt{y-y^2} \end{pmatrix}.$$

Each column of Ψ satisfies the differential equation given in Corollary 9.1. And it is easy to check that we have

$$y(1-y)H''(y) + \frac{1}{2}n(1-2y)H'(y) + \frac{(1-2y)^2+1}{4y(1-y)} \begin{pmatrix} -\ell & 0 & 0 \\ 0 & -\ell-1 & 0 \\ 0 & 0 & -\ell \end{pmatrix} \Psi(y)$$
$$+ \frac{(1-2y)}{2y(1-y)} \begin{pmatrix} 0 & -\ell & 0 \\ -\frac{\ell+1}{2} & 0 & -\frac{\ell+1}{2} \\ 0 & -\ell & 0 \end{pmatrix} \Psi(y) = \Psi(y) \begin{pmatrix} -\ell-1 & 0 & 0 \\ 0 & -\ell & 0 \\ 0 & 0 & -\ell-1 \end{pmatrix}.$$

Theorem 9.2. *The function Ψ can be used to obtain a hypergeometric differential equation from the one given in Corollary 9.1. Precisely, if H is a vector-valued solution of the differential equation in Corollary 9.1, with eigenvalue λ, then $P = \Psi^{-1}H$ is a solution of $DP = \lambda P$, where D is the hypergeometric differential operator given by*

$$DP = y(1-y)P'' + (C - yU)P' - VP,$$

with

$$C = \begin{pmatrix} (n+2)/2 & 1/2 & 0 \\ 1 & (n+2)/2 & 1 \\ 0 & 1/2 & (n+2)/2 \end{pmatrix}, \qquad U = (n+2)I,$$
$$V = \begin{pmatrix} -\ell-1 & 0 & 0 \\ 0 & -\ell & 0 \\ 0 & 0 & -\ell-1 \end{pmatrix}.$$

Proof. Let us write $H = \Psi P$. Then

$$y(1-y)P'' + \Big(2y(1-y)\Psi^{-1}\Psi' + \frac{n}{2}(1-2y)I\Big) P'$$

$$+ \Psi^{-1}\left(y(1-y)\Psi'' + \frac{n}{2}(1-2y)\Psi' + \frac{1+(1-2y)^2}{4y(1-y)}\begin{pmatrix} -\ell & 0 & 0 \\ 0 & -\ell-1 & 0 \\ 0 & 0 & -\ell \end{pmatrix}\Psi \right.$$
$$\left. + \frac{(1-2y)}{2y(1-y)}\begin{pmatrix} 0 & -\ell & 0 \\ -\frac{\ell+1}{2} & 0 & -\frac{\ell+1}{2} \\ 0 & -\ell & 0 \end{pmatrix}\Psi \right) P = \lambda P.$$

Now we compute

$$2y(1-y)\Psi^{-1}\Psi' + \frac{n}{2}(1-2y)I = -(n+2)yI + \begin{pmatrix} (n+2)/2 & 1/2 & 0 \\ 1 & (n+2)/2 & 1 \\ 0 & 1/2 & (n+2)/2 \end{pmatrix}.$$

Therefore

$$y(1-y)P'' + \left(-(n+2)yI + \begin{pmatrix} (n+2)/2 & 1/2 & 0 \\ 1 & (n+2)/2 & 1 \\ 0 & 1/2 & (n+2)/2 \end{pmatrix} \right) P'$$
$$+ \left(\begin{pmatrix} -\ell-1 & 0 & 0 \\ 0 & -\ell & 0 \\ 0 & 0 & -\ell-1 \end{pmatrix} - \lambda I \right) P = 0.$$

This completes the proof of the theorem. ∎

We obtain a similar result to Theorem 6.3, with an analogous proof:

Theorem 9.3. *For a given $\ell \in \mathbb{N}$ let $n = 2\ell+1$, then the nonzero polynomial eigenfunctions of*

$$DP = y(1-y)P'' + (C - yU)P' - VP,$$

with

$$C = \begin{pmatrix} (n+2)/2 & 1/2 & 0 \\ 1 & (n+2)/2 & 1 \\ 0 & 1/2 & (n+2)/2 \end{pmatrix}, \qquad U = (n+2)I,$$
$$V = \begin{pmatrix} -\ell-1 & 0 & 0 \\ 0 & -\ell & 0 \\ 0 & 0 & -\ell-1 \end{pmatrix},$$

have eigenvalues $-w(w+n+1)-\ell$ or $-w(w+n+1)-\ell-1$, with $w \in \mathbb{N}_0$. In both cases the degree of the polynomial is w and the leading coefficient can be any multiple of $\left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right)$ or any linear combination of $\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$, respectively.

Let us consider $\widetilde{D}$, the differential operator on $(0,1)$ introduced in Corollary 9.1:

$$\widetilde{D}H = y(1-y)H''(y) + \frac{1}{2}n(1-2y)H'(y)$$
$$+ \frac{(1-2y)^2+1}{4y(1-y)}\begin{pmatrix} -\ell & 0 & 0 \\ 0 & -\ell-1 & 0 \\ 0 & 0 & -\ell \end{pmatrix}H(y) + \frac{(1-2y)}{2y(1-y)}\begin{pmatrix} 0 & -\ell & 0 \\ -\frac{\ell+1}{2} & 0 & -\frac{\ell+1}{2} \\ 0 & -\ell & 0 \end{pmatrix}H(y).$$

Recall that the operator D that appears in Theorem 9.3 extends the differential operator $D = \Psi\widetilde{D}\Psi^{-1}$ to the whole real line.

We want to focus our attention on the following vector spaces of $\mathbb{C}^3$-valued analytic functions on $(0,1)$:

$$S_\lambda = \big\{H = H(y) : \widetilde{D}H = \lambda H,\ H(\tfrac{\cos s+1}{2}) \text{ analytic at } s=0\big\},$$
$$W_\lambda = \big\{P = P(y) : DP = \lambda P,\ \text{analytic on } [0,1]\big\}.$$

From Theorem 9.2 we know that the correspondence $P \mapsto \Psi P$ is an injective linear map from W_λ into S_λ. In fact, $\Psi((\cos s+1)/2)$ is analytic as a function of s and P is analytic at $y=1$, hence $H((\cos s+1)/2) = (\Psi P)((\cos s+1)/2)$ is analytic at $s=0$.

Then, we have an analogous result to Theorem 7.3, whose proof is quite similar and therefore we will omit it.

Theorem 9.4. *The linear map $P \mapsto \Psi P$ is an isomorphism from W_λ onto S_λ.*

Now, we can easily make a proof similar to that one of Theorem 7.4 in order to obtain next theorem.

Theorem 9.5. *Let H be the $\mathbb{C}^3$-valued analytic function on $(0,1)$ given by an irreducible spherical function Φ on $\mathrm{SO}(2\ell+2)$ of fundamental $\mathrm{SO}(2\ell+1)$-type $(1,\dots,1)\in\mathbb{C}^\ell$. If $P=\Psi^{-1}H$, then P is polynomial.*

For a given fundamental K-type $\pi \in \hat{\mathrm{SO}}(n)$, $n=2\ell+1$, with highest weight $(1,\dots,1)\in\mathbb{C}^\ell$, let $\Phi_{w,\delta}$ denote the irreducible spherical function of the pair $(\mathrm{SO}(n+1),\mathrm{SO}(n))$ given by $\tau\in\hat{\mathrm{SO}}(n+1)$ with highest weight of the form $(w+1,1,\dots,1,\delta)\in\mathbb{C}^{\ell+1}$, $\delta=-1,0,1$.

Now, combining Theorems 9.3, 9.5 and the expression of the eigenvalue $\lambda_n(w,\delta)$ given in (6.2) we have the following statement.

Theorem 9.6. *Given $w\in\mathbb{N}$, every irreducible spherical function $\Phi_{w,\delta}$ of the pair $(\mathrm{SO}(n+1),\mathrm{SO}(n))$ with $n=2\ell+1$, of type $\mathbf{m}_n=(1,\dots,1)\in\mathbb{C}^\ell$, corresponds to a vector-valued function $P_{w,\delta}$ ($w\ge 0$, $\delta=-1,0,1$), which is a polynomial of degree w. The leading coefficients of $P_{w,0}$ is a multiple of $\left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right)$ and the leading coefficients of $P_{w,-1}$ and $P_{w,1}$ are both linear combinations of $\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)$. Precisely*

$$P_{w,\delta}(y) = \sum_{j=0}^{w} \frac{y^j}{j!}[C;U;V+\lambda]_j P_{w,\delta}(0),$$

with

$$C = \begin{pmatrix} (n+2)/2 & 1/2 & 0 \\ 1 & (n+2)/2 & 1 \\ 0 & 1/2 & (n+2)/2 \end{pmatrix},$$
$$U=(n+2)I, \qquad V = \begin{pmatrix} -\ell-1 & 0 & 0 \\ 0 & -\ell & 0 \\ 0 & 0 & -\ell-1 \end{pmatrix},$$
$$\lambda = \lambda_n(w,\delta) = \begin{cases} -w(w+n+1)-\ell & \text{if} \quad \delta=0, \\ -w(w+n+1)-\ell-1 & \text{if} \quad \delta=\pm1. \end{cases}$$

Even more, the value of $P_{w,\delta}(0)$ can be computed.

Proof. It only remains to prove that $P_{w,\delta}(0)$ can be computed.

Let us consider the case $\delta = 0$. We know from (6.2) and Theorem 9.3 that there is some $c \in \mathbb{C}$ such that

$$[C;U;V+\lambda]_w P_{w,0}(0) = c \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Since $[C;U;V+\lambda]_w$ is invertible this c is univocally determined by the condition $\Phi(e) = I$ which implies

$$\Psi(1) \sum_{j=0}^{w} \frac{1}{j!}[C;U;V+\lambda]_j P_{w,0}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Now let us consider the cases $\delta = \pm 1$. We know from (6.2) and Theorem 9.3 that

$$[C;U;V+\lambda]_w P_{w,\delta}(0) \in \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle;$$

since $[C;U;V+\lambda]_w$ is invertible, this condition tells us that $P_{w,\delta}(0)$ belongs to a plane which contains the origin and does not depend on δ.

Besides, the condition $\Phi_{w,\delta}(e) = I$, for $\delta = \pm 1$, tells us

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \sum_{j=0}^{w} \frac{1}{j!}[C;U;V+\lambda]_j P_{w,\delta}(0).$$

Then, $P_{w,\delta}(0)$ belongs to a plane, parallel to the kernel of

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \sum_{j=0}^{w} \frac{1}{j!}[C;U;V+\lambda]_j,$$

which does not contain the origin and does not depend on δ. Therefore we know that both $P_{w,1}(0)$ and $P_{w,-1}(0)$ are in the same straight line.

On the other hand, recall that we have

$$\Phi_{w,\delta}(a(s)) = \Psi\left(\frac{\cos s + 1}{2}\right) P_{w,\delta}\left(\frac{\cos s + 1}{2}\right),$$

where

$$a(s) = \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & \cos s & \sin s \\ 0 & -\sin s & \cos s \end{pmatrix},$$

then

$$\left.\frac{d}{ds}\right|_{s=0} \Phi(a(s)) = \begin{pmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ -i & 0 & i \end{pmatrix} P_{w,\delta}(1).$$

From [24, p. 364, equation (8)] we can easily compute $\frac{d}{ds}\Phi_{w,\delta}(a(s))$ at $s=0$, which is obtained by looking at the action of $\dot\tau(I_{n+1,n})$ and considering the corresponding projection, see (2.1); having then

$$\delta\frac{i(w+\ell+1)}{1+\ell}\begin{pmatrix}-1\\0\\1\end{pmatrix}=\begin{pmatrix}i&0&-i\\0&0&0\\-i&0&i\end{pmatrix}\sum_{j=0}^{w}\frac{1}{j!}[C;U;V+\lambda]_j P_{w,\delta}(0).$$

This last condition establishes that $P_{w,1}(0)$ and $P_{w,-1}(0)$ are in two different and parallel planes, and the line mentioned above does not belong to any of them since each plane has to intersect it. Therefore the values of $P_{w,1}(0)$ and $P_{w,-1}(0)$ are univocally determined. ■

9.2 Matrix-valued orthogonal polynomials of size 3

In this subsection, given n of the form $2\ell+1$ with $\ell\in\mathbb{N}$, we shall construct a sequence of matrix-valued polynomials $\{P_w\}_{w\geq 0}$ directly related to irreducible spherical functions of type $\pi\in\hat{\mathrm{SO}}(n)$ of highest weight $\mathbf{m}_\pi=(1,\dots,1)\in\mathbb{C}^\ell$.

Given a nonnegative integer w and $\delta=-1,0,1$, we can consider $\Phi_{w,\delta}$, the irreducible spherical function of type π associated with the irreducible representation $\tau\in\hat{\mathrm{SO}}(n+1)$ of highest weight of the form $\mathbf{m}_\tau=(w+1,1,\dots,1,\delta)$.

We insist on recalling that, since π has only three $\mathrm{SO}(2\ell)$-submodules, we can interpret the diagonal matrix-valued function $\Phi_{w,\delta}(a(s))$, $s\in(0,\pi)$, as a 3 column vector function.

Now we consider the vector-valued function

$$P_{w,\delta}:\ (0,1)\to\mathbb{C}^3$$

given by the vector function $P_{w,\delta}(y)=\Psi^{-1}(y)\Phi_{w,\delta}(a(s))$, with $\cos(s)=2y-1$. Then, we define the matrix-valued function

$$P_w=P_w(y),$$

whose δ-th column ($\delta=-1,0,1$) is given by the $\mathbb{C}^3$-valued polynomial $P_{w,\delta}(y)$.

Let consider the matrix-valued skew symmetric bilinear form defined among continuous 3×3 matrix-valued functions on $[0,1]$ by

$$\langle P,Q\rangle_W=\int_0^1 Q^*(y)W(y)P(y)dy,$$

where the 3×3 weight-matrix W is given by

$$W(y)=\frac{(n-1)!!}{(n-2)!!}(y(1-y))^{n/2-1}\Psi^*(y)\begin{pmatrix}d_1&0&0\\0&d_2&0\\0&0&d_3\end{pmatrix}\Psi(y)$$

with

$$d_1=d_3=\frac{(2\ell+1)!}{\ell!(\ell+2)!},\qquad d_2=\frac{(2\ell+1)!}{\ell!(\ell+1)!},$$

and

$$\Psi(y)=\begin{pmatrix}2y-1+2i\sqrt{y-y^2}&1&2y-1-2i\sqrt{y-y^2}\\1&2y-1&1\\2y-1-2i\sqrt{y-y^2}&1&2y-1+2i\sqrt{y-y^2}\end{pmatrix}$$

Let us recall that, from Proposition 7.1, we have

$$\langle\Phi_{w,\delta},\Phi_{w',\delta'}\rangle=\int_0^1 P_{w,\delta}^*W(y)P_{w',\delta'}dy.$$

Remark 9.7. Notice that W reduces to a smaller size: if $M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix}$ we have

$$MW(y)M^* = \frac{(n-1)!!}{(n-2)!!}(y(1-y))^{n/2-1}4 \\ \times \begin{pmatrix} 2d_1(2y-1)^2 + d_2 & d_1(2y-1) + d_2(2y-1)/\sqrt{2} & 0 \\ d_1(2y-1)\sqrt{2} + d_2(2y-1)/\sqrt{2} & d_1 + d_2(2y-1)^2/2 & 0 \\ 0 & 0 & d_1 8(y-y^2) \end{pmatrix}.$$

Then we state the following theorem.

Theorem 9.8. *The matrix-valued polynomial functions P_w, $w \geq 0$, form a sequence of orthogonal polynomials with respect to W, which are eigenfunctions of the symmetric differential operator D from Theorem 9.2. Moreover,*

$$DP_w = P_w \begin{pmatrix} \lambda(w,-1) & 0 & 0 \\ 0 & \lambda(w,0) & 0 \\ 0 & 0 & \lambda(w,1) \end{pmatrix},$$

where

$$\lambda(w,\delta) = \begin{cases} -w(w+n+1) - p & \text{if} \quad \delta = 0, \\ -w(w+n+1) - n + p & \text{if} \quad \delta = \pm 1. \end{cases}$$

Proof. The proof is completely analogous to the proof of Theorem 8.1 ■

Appendix

Proof of Proposition 3.2. For $|t|$ sufficiently small $A(s,t)$ is close to the identity of K, i.e. to the identity matrix I_n. So we can consider the function

$$X(s,t) = \log(A(s,t)) = B(s,t) - \frac{B(s,t)^2}{2} + \frac{B(s,t)^3}{3} - \cdots, \tag{9.2}$$

where $B(s,t) = A(s,t) - I_n$. Then

$$\pi(A(s,t)) = \pi(\exp X(s,t)) = \exp \dot{\pi}(X(s,t)) = \sum_{j \geq 0} \frac{\dot{\pi}(X(s,t))^j}{j!}.$$

Now we differentiate with respect to t to obtain

$$\frac{\partial(\pi \circ A)}{\partial t} = \dot{\pi}\left(\frac{\partial X}{\partial t}\right) + \tfrac{1}{2!}\dot{\pi}\left(\frac{\partial X}{\partial t}\right)\dot{\pi}(X) + \tfrac{1}{2!}\dot{\pi}(X)\dot{\pi}\left(\frac{\partial X}{\partial t}\right) \\ + \tfrac{1}{3!}\dot{\pi}\left(\frac{\partial X}{\partial t}\right)\dot{\pi}(X)^2 + \tfrac{1}{3!}\dot{\pi}(X)\dot{\pi}\left(\frac{\partial X}{\partial t}\right)\dot{\pi}(X) + \tfrac{1}{3!}\dot{\pi}(X)^2\dot{\pi}\left(\frac{\partial X}{\partial t}\right) + \cdots. \tag{9.3}$$

Since $X(s,0) = 0$, if we differentiate (9.2) with respect to t and evaluate at $(s,0)$ we obtain

$$\frac{\partial^2(\pi \circ A)}{\partial t^2}\Big|_{t=0} = \dot{\pi}\left(\frac{\partial^2 X}{\partial t^2}\Big|_{t=0}\right) + \dot{\pi}\left(\frac{\partial X}{\partial t}\Big|_{t=0}\right)^2.$$

To compute $\frac{\partial X}{\partial t}\big|_{t=0}$ and $\frac{\partial^2 X}{\partial t^2}\big|_{t=0}$ we differentiate (9.2) and we get

$$\frac{\partial X}{\partial t} = \frac{\partial B}{\partial t} - \tfrac{1}{2}\left(\frac{\partial B}{\partial t}\right)B - \tfrac{1}{2}B\left(\frac{\partial B}{\partial t}\right) + \tfrac{1}{3}\left(\frac{\partial B}{\partial t}\right)B^2 + \tfrac{1}{3}B\left(\frac{\partial B}{\partial t}\right)B + \tfrac{1}{3}B^2\left(\frac{\partial B}{\partial t}\right) + \cdots.$$

Since $B(s,0)=0$ we have

$$\frac{\partial X}{\partial t}\Big|_{t=0} = \frac{\partial B}{\partial t}\Big|_{t=0} = \frac{\partial A}{\partial t}\Big|_{t=0}.$$

We also get

$$\frac{\partial^2 X}{\partial t^2}\Big|_{t=0} = \frac{\partial^2 A}{\partial t^2}\Big|_{t=0} - \left(\frac{\partial A}{\partial t}\Big|_{t=0}\right)^2.$$

Now we will first consider the case $A(s,t)=k(s,t)$. A direct computation yields to

$$\frac{\partial k}{\partial t} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-\sin s \sin t}{(1-\cos^2 s\cos^2 t)^{3/2}} & \mathbf{0} & \frac{\sin^2 s \cos t}{(1-\cos^2 s\cos^2 t)^{3/2}} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-\sin^2 s \cos t}{(1-\cos^2 s\cos^2 t)^{3/2}} & \mathbf{0} & \frac{-\sin s \sin t}{(1-\cos^2 s\cos^2 t)^{3/2}} & 0 \\ \mathbf{0} & 0 & \mathbf{0} & 0 & 0 \end{pmatrix},$$

in particular $\frac{\partial k}{\partial t}\big|_{t=0} = \frac{1}{\sin s} I_{n,j}$. Differentiating once more with respect to t and evaluating at $t=0$ we obtain $\frac{\partial^2 k}{\partial t^2}\big|_{t=0} = -\frac{1}{\sin^2 s}(E_{jj}+E_{n,n})$. Then we get

$$\frac{\partial^2 A}{\partial t^2}\Big|_{t=0} - \left(\frac{\partial A}{\partial t}\Big|_{t=0}\right)^2 = -\frac{1}{\sin^2 s}(E_{jj}+E_{n,n}) - \frac{1}{\sin^2 s} I_{n,j}^2 = 0.$$

Similarly when $A(s,t)=h(s,t)$ we obtain

$$\frac{\partial h}{\partial t} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{-\sin s \cos^2 s \cos t \sin t}{(1-\cos^2 s\cos^2 t)^{3/2}} & \mathbf{0} & \frac{-\cos s \cos t \sin^2 s}{(1-\cos^2 s\cos^2 t)^{3/2}} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\cos s \cos t \sin^2 s}{(1-\cos^2 s\cos^2 t)^{3/2}} & \mathbf{0} & \frac{-\sin s \cos^2 s \cos t \sin t}{(1-\cos^2 s\cos^2 t)^{3/2}} & 0 \\ \mathbf{0} & 0 & \mathbf{0} & 0 & 0 \end{pmatrix},$$

in particular $\frac{\partial h}{\partial t}\big|_{t=0} = -\frac{\cos s}{\sin s} I_{n,j}$. Differentiating once more with respect to t and evaluating at $t=0$ we obtain $\frac{\partial^2 h}{\partial t^2}\big|_{t=0} = -\frac{\cos^2 s}{\sin^2 s}(E_{jj}+E_{n,n})$. Then we get

$$\frac{\partial^2 A}{\partial t^2}\Big|_{t=0} - \left(\frac{\partial A}{\partial t}\Big|_{t=0}\right)^2 = -\frac{\cos^2 s}{\sin^2 s}(E_{jj}+E_{n,n}) - \frac{\cos^2 s}{\sin^2 s} I_{n,j}^2 = 0.$$

Proposition follows. ■

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