Gravitation, Relativistic Astrophysics and Cosmology
Second Argentinian-Brazilian Meeting, 2014
G. S. Vila, F. L. Vieyro and J. Fabris, eds.

Averages associated to the energy momentum tensor and study of a two scale system in general relativity

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Abstract. We present a brief study of compound systems of different scales. It is shown that the detailed dynamical studies of massless and massive particles can hardly be associated to standard averaging techniques.

1. Introduction

Recently new expressions for the deviation angle and for the optical scalars in the study of weak lensing have been derived in terms of curvature scalar of the lens geometry [GM11]. This formulas, in contrast to standard treatments found in the literature, take into account the spacelike components of the energy-momentum tensor of the lens. One of the advantage of the new expression is that they allow us to model the lens with a broader kind of objects.

In particular, it have been shown the case of a peculiar geometry without mass but with non-vanishing spacelike components[GM12]. This geometry is an exact solution of the Einstein's equations with a non-conventional energy-momentum tensor; despite its bizarre nature it has remarkable features, as we now mention.

It was shown that this geometry can fit the shear profile in studies of weak lensing in Coma's cluster.

A dynamical study of the rotation curves in this geometry shows that if the observations of the tangential velocity, v_t, of the rotation curves in this geometry are interpreted with the usual Newtonian relations

$$M_N(r) = \frac{rc^2}{G} v_t^2; \tag{1}$$

it gives the expected linear growth of the deduced Newtonian mass $M_N(r)$ with the radial coordinate. Here G is the gravitational constant and c denotes the speed of light in vacuum.

The radial mass profile of a matter distribution deduced by means of the estimation in the scape velocity of the system can be fitted using the radial scape velocity, v_e, in this geometry together with a Newtonian interpretation, this is associating a Newtonian mass $M_N(r)$ given by

$$M_N(r) = \frac{rc^2}{2G} v_e^2. \tag{2}$$

This is pertinent to observations in systems where the issue of the missing mass, or dark matter problem is manifest. This geometry, although a toy model since it has not mass density, gives account in a very acceptable way some of the features found it in the observations of “dark matter” in astrophysical systems.

This fact rises the question about the possible nature of this peculiar solution and its relation to the dark matter problem and also to the way in which observations are carried out.

The study of phenomenology of dark matter in the new peculiar solutions presented in [GM12] involves the use of two tools: The geodesic equation for massive and massless particles, and the deviation geodesic equation for a congruence of null geodesics.

The former only contains the information that comes from the connection associated to the geometry while the last has information of the curvature (second derivatives) of the geometry.

We want to consider a system composed of small point-like subsystems that contribute to a big complete system.

Each subsystem is considered to have very small velocity with respect to each other so that all of them can be considered as geometric linear stationary contribution over a common flat background.

We employ a generalization of the optical scalars for the case of a such distribution in the approximation of thin lens.

2. The system

The distribution of the small constituents of the big system can be described in terms of the stationary distribution function $\mathscr{P}(x^i)$ with $i = 1, 2, 3$ denoting the spacelike coordinates of the flat background. This is a continuous distribution that models the density of the small subsystems.

We work with several pictures in mind: Each subsystem is considered as a vacuum gravitating central object; which therefore is associated to a Schwarzschild geometry. We also consider the case of spherically symmetric geometries with halos which contribute only to their respective P_r component of the microscopic energy-momentum tensor. We consider astrophysical useful distributions to the energy density ϱ, as is the isothermal mass distribution.

We are assuming that the nature of the observations is such that one can consider each subsystem and the compound system as stationary; so that we can assume the existence of a global timelike Killing vector field, $\tilde{t}^a$.

2.1. Decomposition of the geometry

Let us express the metric g_{ab} of the spacetime in terms of a reference metric η_{ab}, such that

$$g_{ab} = \eta_{ab} + h_{ab}. \tag{3}$$

Let ∂_a denote the torsion free metric connection of η_{ab} and ∇_a the torsion free metric connection of g_{ab}; then one can express the covariant derivative of an arbitrary vector v^a by

$$\nabla_a v^b = \partial_a v^b + \Gamma_{a\ c}^{\ b} v^c; \tag{4}$$

and one can prove that

$$\Gamma_{a\,b}^{\,c} = \frac{1}{2} g^{cd} \left(\partial_a h_{bd} + \partial_b h_{ad} - \partial_d h_{ab}\right) = \Gamma_{b\,a}^{\,c}. \tag{5}$$

Since we are considering contributions of many subsystems A's, the tensor h_{ab} must be the sum of all the contributions, namely

$$h_{ab} = \sum_{(A)} h_{ab}^{(A)}. \tag{6}$$

3. Testing a spherically symmetric system with massless particles

In reference [GM11] we have deduced the general equations for the description of weak lensing. We will use here those that are appropriate for the study of spherically symmetric systems.

Let us recall that the lens scalars in the thin lens approximation, in terms of the curvature invariants Ψ_0 and Φ_{00} associated to the Weyl's and Ricci's tensor respectively, are given by

$$\begin{aligned} \kappa =& \frac{d_l d_{ls}}{d_s} \int_{-d_l}^{d_{ls}} \Phi_{00}\, dy, \\ \gamma =& \frac{d_l d_{ls}}{d_s} \int_{-d_l}^{d_{ls}} |\Psi_0|\, dy; \end{aligned} \tag{7}$$

where here γ refers to the modulus of the shear.

For thin lenses the bending angle is given by[GM11]

$$\alpha(J) = J\left(\hat{\Phi}_{00}(J) + \hat{\Psi}_0(J)\right); \tag{8}$$

$$\hat{\Phi}_{00} = \int \Phi_{00} d\lambda, \qquad \hat{\Psi}_0 = \int \Psi_0 d\lambda. \tag{9}$$

This expression are valid for each subsystem.

We use J to denote the impact parameter of the null geodesic to center of the lens; y is the Cartesian coordinate along which the photons path, λ the affine parameter along the null geodesics and the coordinate r is satisfies $r^2 = J^2 + y^2$.

The above expressions can be put in terms of the total mass, $M(r)$ and the components of the energy-momentum tensor of the lens, using the following relations;

$$\Psi_0 = -3\frac{J^2}{r^2}\tilde{\Psi}_2 e^{2i\vartheta} = -3\frac{J^2}{r^2}\left[\frac{4\pi}{3}\left(\varrho - P_r - P_t\right) - \frac{M}{r^3}\right] e^{2i\vartheta}; \tag{10}$$

where here ϑ is the angle of polar coordinates in the plane $y = 0$ in a Cartesian coordinate system; and

$$\Phi_{00} = 2\frac{J^2}{r^2}\left(\tilde{\Phi}_{11} - \frac{1}{4}\tilde{\Phi}_{00}\right) + \tilde{\Phi}_{00} = 4\pi\frac{J^2}{r^2}\Big(P_t - P_r\Big) + 4\pi\left(\varrho + P_r\right). \tag{11}$$

In particular, the well known results for a lens with the Schwarzschild geometry

$$\kappa(J) = 0, \tag{12}$$

$$\gamma(J) = \frac{d_l d_{ls}}{d_s}\frac{4M}{J^2}, \tag{13}$$

$$\alpha(J) = \frac{4M}{J}. \tag{14}$$

are obtained by taking $\rho = P_t = P_r = 0$ and M = constant with the integration limits going to infinity.

3.1. Treatment of a compound system

When dealing with a compound system, the standard treatment find in textbooks is to consider the situation of a monopole mass (Schwarzschild) and to generalize eq. (14) to a vector equation in the plane of the thin lens.

Here, we generalize the bending angle equation for a an axially symmetric distribution; but before let us recall that given the scalar expression $\alpha(J)$ for bending angle, one can write[GM11] the 2-dimensional equation in terms of the components of $\alpha^i = (\alpha^1, \alpha^2)$ as

$$(\alpha^i) = \alpha(J)(\frac{z_0}{J}, \frac{x_0}{J}); \tag{15}$$

taking into account the appropriate orientation in the two dimensional space of the images.

3.2. Generalization for a distribution of spherically symmetric deflectors:

It could be convenient to change the notation to a most common one when we consider a distribution of sources. Then, let us denote by $\boldsymbol{\xi}'$ the vector in the plane of the thin lens joining an arbitrary location in the distribution with a given point in the plane of the lens. Then we rewrite equation (15) in the form

$$\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \alpha(|\boldsymbol{\xi} - \boldsymbol{\xi}'|)\frac{(\boldsymbol{\xi} - \boldsymbol{\xi}')}{|\boldsymbol{\xi} - \boldsymbol{\xi}'|}; \tag{16}$$

which for a macroscopic distribution $\mathscr{P}(\boldsymbol{\xi}', y')$ results in

$$\boldsymbol{\alpha}(\boldsymbol{\xi}) = \int_{\mathbb{R}^2}\int_{-\infty}^{\infty} \mathscr{P}(\boldsymbol{\xi}', y')\alpha(|\boldsymbol{\xi} - \boldsymbol{\xi}'|)\frac{(\boldsymbol{\xi} - \boldsymbol{\xi}')}{|\boldsymbol{\xi} - \boldsymbol{\xi}'|} d^2\xi' dy'; \tag{17}$$

or put it in a compact form we have

$$\boldsymbol{\alpha}(\boldsymbol{\xi}) = \int_{\mathbb{R}^2} \Sigma(\boldsymbol{\xi}, \boldsymbol{\xi}')\frac{(\boldsymbol{\xi} - \boldsymbol{\xi}')}{|\boldsymbol{\xi} - \boldsymbol{\xi}'|^2} d^2\xi'; \tag{18}$$

with

$$\Sigma(\boldsymbol{\xi}, \boldsymbol{\xi}') = \int_{-\infty}^{\infty} \mathscr{P}(\boldsymbol{\xi}', y')\alpha(|\boldsymbol{\xi} - \boldsymbol{\xi}'|)|\boldsymbol{\xi} - \boldsymbol{\xi}'| dy'. \tag{19}$$

In the case of small deflectors of Schwarzschild type, $\alpha(|\boldsymbol{\xi}-\boldsymbol{\xi}'|)=\frac{4m}{|\boldsymbol{\xi}-\boldsymbol{\xi}'|}$; the quantity $\Sigma(\boldsymbol{\xi},\boldsymbol{\xi}')$ represent the total mass of the distribution projected in the plane of the thin lens. For analogy with this case we will refer to $\Sigma(\boldsymbol{\xi},\boldsymbol{\xi}')$ in the most general case as the *generalized projected mass.*

4. Testing the system with massive particles

4.1. The equation of motion

The dynamics of massive particles is determined by the geodesic equation. Let the vector u^a be the four velocity of the particle, then one can express

$$u^a \nabla_a u^b = u^a \partial_a u^b + u^a \Gamma_{a\,c}^{\,b} u^c = 0. \tag{20}$$

Since we are assuming small velocities we can express, in first order in the velocities

$$u^a = (1+\frac{u^2}{2})t^a + \mathrm{v}^a + \mathscr{O}(u^3); \tag{21}$$

where

$$\eta_{ab} t^a \mathrm{v}^b = 0, \tag{22}$$

$$\eta_{ab} \mathrm{v}^a \mathrm{v}^b = -u^2. \tag{23}$$

In this way, we see that the equation of motion can be expressed as

$$t^a \partial_a \mathrm{v}^b + \Gamma_{a\,c}^{\,b} t^a t^c + 2\Gamma_{a\,c}^{\,b} t^a \mathrm{v}^c = 0; \tag{24}$$

4.2. The case of spherical symmetry

A stationary spherically symmetric geometry can be expressed by

$$ds^2 = e^{2\Phi(r)}dt^2 - \frac{dr^2}{1-\frac{2M(r)}{r}} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right); \tag{25}$$

in terms of a standard spherical coordinate system (t, r, θ, ϕ).

For this case one has that the non-vanishing components of the connection tensor are

$$\Gamma_{\theta\,\theta}^{\,r} = 2M(r), \tag{26}$$

$$\Gamma_{\phi\,\phi}^{\,r} = 2M(r)\sin^2\theta, \tag{27}$$

$$\Gamma_{t\,r}^{\,t} = \frac{d\Phi(r)}{dr}, \tag{28}$$

$$\Gamma_{t\,t}^{\,r} = \frac{d\Phi(r)}{dr}, \tag{29}$$

$$\Gamma_{r\,r}^{\,r} = \frac{d}{dr}\left(\frac{M(r)}{r}\right); \tag{30}$$

where we are considering only linear terms.

Then, considering the non-zero contributions from the connection, one has

$$\frac{dv^r}{dt} = -\frac{d\Phi(r)}{dr}. \tag{31}$$

Therefore, we have only one equation dynamically interesting.

It is important to emphasize that the notion of the r direction is dependent on the system we are considering the interaction with. To carry out the sum over all subsystems A it would be better to introduce a Cartesian description with respect to the background.

When the sum it is carried one obtains the Newton's equation for a particle in a gravitational field. Where the effective potential is given by the sum of the individual contributions of the distribution of the form (31). In particular, when the big scale distribution is spherically symmetric one can use the Newton's theorem on spherical systems to evaluate the effective potential inside of a central sphere of radius r with respect to the origin.

5. Summary and perspectives

We have just seen that when studying the dynamics of massive and massless particles, while the first reduces to the simple application of Newtonian techniques, the later is much more complicated, specially when the spacelike components of the energy momentum tensor can not be neglected; as is the case for the geometries that we have presented elsewhere. Therefore, the physical smoothing procedures are not necessarily associated with standard averaging of geometrical quantities as tensors; but come from a detailed study of the particular observation.

Acknowledgments. We acknowledge support from CONICET, SeCyT-UNC and Foncyt.

References

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