Research Article

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One-dimensional singular problems involving the p-Laplacian and nonlinearities indefinite in sign

Abstract: Let Ω be a bounded open interval, let $p > 1$ and $\gamma > 0$, and let $m : \Omega \to \mathbb{R}$ be a function that may change sign in Ω. In this article we study the existence and nonexistence of positive solutions for one-dimensional singular problems of the form $-(|u'|^{p-2}u')' = m(x)u^{-\gamma}$ in Ω, $u = 0$ on $\partial\Omega$. As a consequence we also derive existence results for other related nonlinearities.

Keywords: One-dimensional singular problems, indefinite nonlinearities, p-Laplacian, positive solutions

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1 Introduction

For $a < b$, let $\Omega := (a, b)$, and let $\gamma > 0$. Let $p \in (1, \infty)$ and $m \in L^{p'}(\Omega)$ (where as usual we define p' by $1/p + 1/p' = 1$) be a possibly sign changing function, and consider the problem

$$\begin{cases} -(|u'|^{p-2}u')' = m(x)u^{-\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

One-dimensional singular problems involving the p-Laplacian like problem (1.1) arise in applications such as non-Newtonian fluid theory or the turbulent flow of a gas in a porous medium (cf. [11, 22]), and they have been widely studied over the years if m is nonnegative. We cite, among many others, the papers [1, 2, 17–19, 24, 25]. However, to the best of our knowledge, there are no results available in the literature when m is allowed to change sign in Ω. Let us note that if m has an indefinite sign, (1.1) becomes a much more involved problem. In fact, (when m changes sign) these problems are quite intriguing even when (1.1) is sublinear (i.e., $\gamma \in (1 - p, 0)$), and only lately existence of positive solutions have been obtained in this case (see [14] for $p \in (1, \infty)$), and [13] and its references for the special case $p = 2$).

On the other side, for the Laplace operator (that is, $p = 2$) problem (1.1) has recently been considered in [12] for sign changing functions m. Our aim in this article is to establish similar results in the general situation $1 < p < \infty$, adapting and extending the approach developed in [12] combined also with some of the ideas in [14]. Let us mention that this is far from being trivial due to the nonlinearity of the p-Laplacian and its corresponding solution operator. Moreover, we remark that some of the conditions presented in this paper improve the ones found in [12] for the Laplacian operator.

In order to derive our results we shall mainly rely on the well-known sub- and supersolution method. The major difficulty here (as with various nonlinear problems with indefinite nonlinearities) is to find a (strictly) positive subsolution. We shall provide such subsolution by means of Schauder's fixed point theorem applied

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to some related nonlinear problems. More precisely, in Theorem 3.1 (i) we shall give a sufficient condition on m that assures the existence of solutions of (1.1) for all $\gamma > 0$ small enough, and further conditions are stated in Theorem 3.1 (ii) without the smallness restriction on γ (see also Remark 3.2 below).

On the other hand, two necessary conditions on m are exhibited in Theorem 3.3 (see also Remark 3.4). Let us point out that the first of the aforementioned sufficient conditions on m turns out to be also “almost” necessary (compare (3.1) with (3.7), and see the last paragraph in Remark 3.4). Finally, as a consequence of the above theorems, we shall prove in Corollary 3.5 an existence result for singular nonlinearities of the form $m(x)f(u)$ with no monotonicity nor convexity assumptions on f.

We conclude this introduction with some few comments on some related open interesting problems. Based on the results in [14] for the analogous sublinear problem, we think that similar theorems to the ones proved here should still be true replacing the p-Laplacian by operators of the form

$$\mathcal{L}u = -(|u'|^{p-2}u')' + c(x)|u|^{p-2}u,$$

where $c \geq 0$ in Ω. We note however that, for instance, the proof of the key Lemma 2.4 does not work in this case and it is not clear how to adapt it. Also somehow similar results should be valid for the analogous n-dimensional problem (in fact, this occurs when $p = 2$ (and $c \equiv 0$), see [12, Section 4]; and also [8–10] for related elliptic problems), and in our opinion proving this if $p \neq 2$ is not a trivial task. Let us finally mention that in the one-dimensional case one could also consider (1.1) with the so-called ϕ-Laplacian in place of the p-Laplacian, that is, taking $(\phi(u'))'$ instead of the p-Laplacian, where ϕ is an increasing odd homeomorphism with $\phi(\mathbb{R}) = \mathbb{R}$ (for singular problems with the ϕ-Laplacian we refer to the book [23, Part II]).

2 Preliminaries

For $1 < p < \infty$, let $\mathcal{L}$ be the differential operator given by

$$\mathcal{L}v := -(|v'|^{p-2}v')'.$$

We start collecting some necessary facts concerning the problem

$$\begin{cases} \mathcal{L}v = h(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Remark 2.1. Let $h \in L^q(\Omega)$, $q > 1$. It is well known that (2.1) admits a unique solution $v \in C^1(\overline{\Omega})$ such that $|v'|^{p-2}v'$ is absolutely continuous and that the equation holds in the pointwise sense. In fact, if

$$\varphi_p(t) := |t|^{p-2}t \quad \text{for } t \neq 0, \quad \varphi_p(0) := 0,$$

and φ_p^{-1} denotes its inverse, it can be seen that

$$v(x) = \int_a^x \varphi_p^{-1}\left(c_h - \int_a^y h(t)\,dt\right)dy, \tag{2.2}$$

where c_h is the unique constant such that $v(b) = 0$ (see e.g. [5, Section 2]). Furthermore, the solution operator $\mathcal{S}$ satisfies that $\mathcal{S} : L^q(\Omega) \to C^1(\overline{\Omega})$ is continuous (e.g. [20, Lemma 2.1] or [21, Lemma 4.2]) and $\mathcal{S} : L^q(\Omega) \to C(\overline{\Omega})$ is compact (cf. [5, Corollary 2.3]).

The so-called weak comparison principle shall be repeatedly used along the paper, and so we state it here for the reader’s convenience (for a proof, see for instance [7, Corollary 6.5.3]).

Lemma 2.2. *Let $u, v \in W_0^{1,p}(\Omega)$ be such that $u \leq v$ on $\partial\Omega$ and $\mathcal{L}u \leq \mathcal{L}v$ in weak sense in Ω, that is,*

$$\int_a^b |u'|^{p-2}u'\varphi' \leq \int_a^b |v'|^{p-2}v'\varphi' \quad \textit{for all } 0 \leq \varphi \in W_0^{1,p}(\Omega).$$

Then $u \leq v$ in Ω.

The next remark compiles some properties concerning the first eigenvalue of the p-Laplacian and its corresponding eigenfunctions.

Remark 2.3. There exists a first eigenvalue $\lambda_1(\Omega) > 0$ and $\Phi \in W_0^{1,p}(\Omega)$, $\|\Phi\|_{L^\infty(\Omega)} = 1$, satisfying

$$\begin{cases} \mathcal{L}\Phi = \lambda_1(\Omega)\Phi^{p-1} & \text{in } \Omega, \\ \Phi > 0 & \text{in } \Omega, \\ \Phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Moreover,

$$\lambda_1(\Omega) = \Big(\frac{\pi_p}{b-a}\Big)^p, \quad \text{where} \quad \pi_p := \frac{2\pi(p-1)^{\frac{1}{p}}}{p\sin(\frac{\pi}{p})},$$

and Φ is a multiple of the function $\sin_p(\pi_p(x-a)/(b-a))$ which is strictly positive and symmetric in Ω and increasing in $(a,(a+b)/2)$ (see e.g. [7, Section 6.3]; and for the precise definition and further properties of $\sin_p$, see e.g. [15] and [3, Section 2]).

In the following lemma we establish some useful upper and lower bounds for $\mathcal{S}(h)$. We write as usual $h = h^+ - h^-$ with $h^+ := \max(h,0)$ and $h^- := \max(-h,0)$. We also set

$$\delta_\Omega(x) := \operatorname{dist}(x,\partial\Omega) = \min(x-a, b-x).$$

Lemma 2.4. *Let $p \in (1,\infty)$ and $h \in L^q(\Omega)$ for some $q > 1$.*

(i) *If $h \geq 0$, then in $\overline{\Omega}$ it holds that*

$$\mathcal{S}(h) \leq \Big(\int_a^b h\Big)^{\frac{1}{p-1}} \delta_\Omega. \tag{2.4}$$

(ii) *Let $I := (x_0,x_1) \subseteq \Omega$ and let $x_I := (x_0+x_1)/2$. If*

$$\inf_I h > \lambda_1(I)\max\Big((x_I-a)^{p-1}\int_a^{x_0} h^-, (b-x_I)^{p-1}\int_{x_1}^b h^-\Big), \tag{2.5}$$

then in $\overline{\Omega}$ it holds that

$$\mathcal{S}(h) \geq \min(\mathcal{H}_a,\mathcal{H}_b)^{\frac{1}{p-1}}\delta_\Omega, \tag{2.6}$$

where

$$\mathcal{H}_a := \frac{\inf_I h}{\lambda_1(I)(x_I-a)^{p-1}} - \int_a^{x_0} h^-, \quad \mathcal{H}_b := \frac{\inf_I h}{\lambda_1(I)(b-x_I)^{p-1}} - \int_{x_1}^b h^-.$$

Proof. Let us prove (i). We assume here without loss of generality that $h \not\equiv 0$. Then by the strong maximum principle (e.g. [6, Theorem 2]), $\mathcal{S}(h) > 0$ in Ω. We observe now that $\varphi_p^{-1} = t^{\frac{1}{p-1}}$ for $t \geq 0$ and $\varphi_p^{-1} = -|t|^{\frac{1}{p-1}}$ if $t < 0$, and so using (2.2) we discover that

$$\mathcal{S}(h)'(x) = \varphi_p^{-1}\Big(c_h - \int_a^x h(t)\,dt\Big)$$

is nonincreasing because $h \geq 0$. Hence, $\mathcal{S}(h)$ is concave in Ω and thus it must hold that $\mathcal{S}(h)'(b) < 0 < \mathcal{S}(h)'(a)$ and therefore

$$0 < c_h < \int_a^b h(t)\,dt. \tag{2.7}$$

Noticing that φ_p^{-1} is increasing and (2.7) we get that

$$\mathcal{S}(h)'(a), |\mathcal{S}(h)'(b)| \leq \Big(\int_a^b h\Big)^{\frac{1}{p-1}}$$

and then from the concavity of $\mathcal{S}(h)$ we derive (2.4).

On the other side, let $I := (x_0, x_1) \subseteq \Omega$, and let $\lambda_1(I) > 0$ and $\Phi > 0$ with $\|\Phi\|_{L^\infty(I)} = 1$ be the corresponding normalized positive eigenfunction for the p-Laplacian in I, that is, satisfying (2.3) with I in place of Ω. Suppose that (2.5) holds (in particular, $\inf_I h > 0$) and fix $\lambda^* := \lambda_1(I)/\inf_I h$. In order to prove (ii) we start building some $0 < u \in W_0^{1,p}(\Omega)$ such that $\mathcal{L}u \leq \lambda^* h(x)$ in weak sense in Ω. Its construction is inspired in some of the computations made in the proofs of Theorems 3.1 and 3.5 in [14] and [13] respectively. Let us first point out that since $0 < \Phi \leq 1$,

$$\mathcal{L}\Phi = \lambda_1(I)\Phi^{p-1} \leq \lambda^* h(x) \quad \text{in } I. \tag{2.8}$$

On the other hand, define

$$c_a := \frac{1}{(x_I - a)^{p-1}} - \lambda^* \int_a^{x_0} h^-,$$

$$v(x) := \int_a^x \left(c_a + \lambda^* \int_a^y h^- \right)^{\frac{1}{p-1}} dy, \quad x \in [a, x_I].$$

(Recall that $x_I := (x_0 + x_1)/2$, and note that $c_a > 0$ due to (2.5).) It is easy to check that v is increasing and convex, $v(a) = 0$ and $\mathcal{L}v = -\lambda^* h^-(x) \leq \lambda^* h(x)$ in (a, x_I). Also, (2.5) implies that $h > 0$ in I and thus

$$\|v\|_{L^\infty(a,x_I)} \leq \int_a^{x_I} \left(c_a + \lambda^* \int_a^{x_0} h^- \right)^{\frac{1}{p-1}} dy = 1.$$

Similarly, if for $x \in [x_I, b]$ we set

$$c_b := \frac{1}{(b - x_I)^{p-1}} - \lambda^* \int_{x_1}^{b} h^-,$$

$$w(x) := \int_x^b \left(c_b + \lambda^* \int_y^b h^- \right)^{\frac{1}{p-1}} dy,$$

then w is decreasing and convex, $w(b) = 0$, $\mathcal{L}w \leq \lambda^* h(x)$ in (x_I, b) and $\|w\|_{L^\infty(x_I,b)} \leq 1$.

Now, since $v(a) = w(b) = \Phi(x_0) = \Phi(x_1) = 0$ and $\|v\|_\infty, \|w\|_\infty \leq 1 = \|\Phi\|_\infty$, and since Φ is increasing in $[x_0, x_I]$ and decreasing in $[x_I, x_1]$ (see Remark 2.3), reasoning as in [13, proof of Theorem 3.1 (i)] we find some $\underline{x}_0 \in (x_0, x_I)$ and $\overline{x}_1 \in (x_I, x_1)$ such that

$$\begin{aligned} v(\underline{x}_0) &= \Phi(\underline{x}_0), & \Phi(\overline{x}_1) &= w(\overline{x}_1), \\ v'(\underline{x}_0) &\leq \Phi'(\underline{x}_0), & \Phi'(\overline{x}_1) &\leq w'(\overline{x}_1). \end{aligned} \tag{2.9}$$

Let us define a function u by $u := v$ in $[a, \underline{x}_0]$, $u := \Phi$ in $[\underline{x}_0, \overline{x}_1]$ and $u := w$ in $[\overline{x}_1, b]$. (We mention that if $x_0 = a$, in order to build u we only use Φ and w, if $x_1 = b$ then we do not need w, and if $I = \Omega$ we simply put $u = \Phi$.) Taking into account the above paragraph, (2.8) and (2.9), a simple integration by parts gives that $\mathcal{L}u \leq \lambda^* h(x)$ in weak sense in Ω. Moreover, since

$$v'(a) = c_a^{\frac{1}{p-1}} \quad \text{and} \quad -w'(b) = c_b^{\frac{1}{p-1}},$$

by the convexity of v and w and the aforementioned monotonicity properties of Φ it follows that

$$u \geq \min(c_a, c_b)^{\frac{1}{p-1}} \delta_\Omega \quad \text{in } \overline{\Omega},$$

and from the weak comparison principle (see Lemma 2.2) the same estimate is also true for $\mathcal{S}(\lambda^* h)$. Furthermore, by the homogeneity of the differential operator $\mathcal{L}$ we get that

$$\mathcal{S}(h) \geq \left(\frac{\min(c_a, c_b)}{\lambda^*} \right)^{\frac{1}{p-1}} \delta_\Omega \quad \text{in } \overline{\Omega}$$

which in turn yields (2.6), and this ends the proof of the lemma. □

Remark 2.5. Let us note that in particular (ii) establishes the strong maximum principle and Hopf's lemma for the operator $\mathcal{L}$, even if h changes sign in Ω. Moreover, it provides explicit lower and upper bounds for $\mathcal{S}(h)'(a)$ and $\mathcal{S}(h)'(b)$ respectively, in terms of Ω, p and h.

Let $f : \Omega \times (0, \infty) \to \mathbb{R}$ be a Carathéodory function (that is, $f(\cdot, \xi)$ is measurable for all $\xi \in (0, \infty)$ and $f(x, \cdot)$ is continuous for a.e. $x \in \Omega$). We consider next singular problems of the form

$$\begin{cases} \mathcal{L}u = f(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.10}$$

in a suitable sense. We say that $v \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ is a *subsolution* (in the sense of distributions) of (2.10) if $v > 0$ in Ω, $v = 0$ on $\partial\Omega$, and

$$\int_a^b |v'|^{p-2} v' \phi' \le \int_a^b f(x, v)\phi \quad \text{for all } 0 \le \phi \in C_c^\infty(\Omega).$$

Analogously, $w \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ is a *supersolution* of (2.10) if $w > 0$ in Ω, $w = 0$ on $\partial\Omega$, and

$$\int_a^b |w'|^{p-2} w' \phi' \ge \int_a^b f(x, w)\phi \quad \text{for all } 0 \le \phi \in C_c^\infty(\Omega).$$

For the sake of completeness we state the following existence theorem in the presence of well-ordered sub- and supersolutions (for the proof, see [16, Theorem 4.1]).

Theorem 2.6. *Assume there exist $v, w \in C^1(\Omega)$ sub- and supersolutions respectively of* (2.10)*, satisfying $v \le w$ in Ω. Suppose also that there exists $g \in L^{p'}_{\mathrm{loc}}(\Omega)$ such that $|f(x, \xi)| \le g(x)$ for a.e. $x \in \Omega$ and all $\xi \in [v(x), w(x)]$. Then there exists $u \in C^1(\Omega) \cap C(\overline{\Omega})$ solution (in the sense of distributions) of* (2.10) *with $v \le u \le w$, that is,*

$$\int_a^b |u'|^{p-2} u' \phi' = \int_a^b f(x, u)\phi \quad \textit{for all } \phi \in C_c^\infty(\Omega).$$

Remark 2.7. If $m \in L^q(\Omega)$ with $q > 1$ and $m^+ \not\equiv 0$, one can quickly verify that (1.1) possesses arbitrarily big supersolutions. Indeed, let $\psi := \mathcal{S}(m^+)$ and let us choose $\beta \in (0, 1)$ and $\sigma > 0$ satisfying

$$\beta := \frac{p-1}{p-1+\gamma}, \qquad \sigma \ge \frac{1}{\beta^\beta}.$$

Notice that $\psi^\beta \in C^1(\Omega) \cap C(\overline{\Omega})$, $\psi^\beta = 0$ on $\partial\Omega$ and $\psi^\beta > 0$ in Ω by the strong maximum principle. Also, a simple computation shows that

$$\begin{aligned} \mathcal{L}(\sigma\psi^\beta) &= -(\sigma\beta)^{p-1}\big(|\psi'|^{p-2}\psi'\psi^{(\beta-1)(p-1)}\big)' \\ &= (\sigma\beta)^{p-1}\big(m^+(x)\psi^{(\beta-1)(p-1)} - (\beta-1)(p-1)|\psi'|^p\psi^{(\beta-1)(p-1)-1}\big) \\ &\ge (\sigma\beta)^{p-1} m^+(x)\psi^{(\beta-1)(p-1)} \\ &\ge m^+(x)(\sigma\psi^\beta)^{-\gamma} \\ &\ge m(x)(\sigma\psi^\beta)^{-\gamma} \quad \text{in } \Omega' \end{aligned}$$

for all $\Omega' \Subset \Omega$, and hence $\sigma\psi^\beta$ is a supersolution of (1.1).

3 Main results

We denote

$$P^\circ := \text{interior of the positive cone of } C_0^1(\overline{\Omega})$$

(that is, the functions $v \in C^1(\overline{\Omega})$ with $v(a) = v(b) = 0$, $v > 0$ in Ω, $v'(a) > 0$ and $v'(b) < 0$), and for any $I = (x_0, x_1) \subseteq \Omega$ we shall write

$$x_I := \frac{x_0 + x_1}{2}, \qquad c_I := \max(x_I - a, b - x_I).$$

Theorem 3.1. *Let $m \in L^{p'}(\Omega)$ and $\gamma > 0$.*

(i) *Suppose*
$$\mathcal{S}(m) \in P^{\circ}. \tag{3.1}$$
Then there exists $\gamma_0 > 0$ such that problem (1.1) has a solution $u \in P^{\circ}$ for all $\gamma \in (0, \gamma_0]$.

(ii) *Suppose $m^- \delta_\Omega^{-\gamma} \in L^q(\Omega)$ with $q > 1$. If for some $I = (x_0, x_1) \subseteq \Omega$ it holds that*
$$\frac{(\inf_I m^+)^{p-1+\gamma}}{(\int_a^b m^+)^\gamma} \geq c_{\gamma,p,\Omega,I} \max\Bigg(\int_a^{x_0} m^-\delta_\Omega^{-\gamma}, \int_{x_1}^b m^-\delta_\Omega^{-\gamma}\Bigg)^{p-1}, \tag{3.2}$$
where
$$c_{\gamma,p,\Omega,I} := \Big(\frac{p-1}{\gamma}\Big)^\gamma \Big(\frac{p-1+\gamma}{p-1}\Big)^{p-1+\gamma} \Big(\frac{b-a}{2}\Big)^{\gamma(p-1)} (c_I^{p-1}\lambda_1(I))^{p-1+\gamma},$$
then problem (1.1) has a solution $u \in C^1(\Omega) \cap C(\overline{\Omega})$, and $u \in P^{\circ}$ whenever $m^+\delta_\Omega^{-\gamma} \in L^r(\Omega)$ with $r > 1$.

Proof. Since Remark 2.7 provides arbitrarily large supersolutions of (1.1), it suffices to find a subsolution. Let us start proving (i). We first observe that (1.1) admits a solution for m if and only if it has one for τm for any constant $\tau > 0$, and therefore we shall also assume without loss of generality that $\mathcal{S}(m^+) \leq 1$ in Ω.

Due to (3.1), we can fix $\varepsilon > 0$ such that $\mathcal{S}(m) \geq 2\varepsilon\delta_\Omega$ in Ω. We also pick $\gamma_0 > 0$ such that for every $\gamma \in (0, \gamma_0]$ it holds that $m^-\delta_\Omega^{-\gamma} \in L^r(\Omega)$ with $r > 1$. Since $\mathcal{S} : L^r(\Omega) \to C^1(\overline{\Omega})$ is a continuous operator for any $r > 1$ (see Remark 2.1), making γ_0 smaller if necessary, we obtain that for all such γ it holds that
$$\mathcal{S}(m^+ - m^-(\varepsilon\delta_\Omega)^{-\gamma}) \geq \varepsilon\delta_\Omega \quad \text{in } \Omega. \tag{3.3}$$
Define now the set
$$\mathcal{C} := \{v \in C(\overline{\Omega}) : \varepsilon\delta_\Omega \leq v \leq \mathcal{S}(m^+) \text{ in } \Omega\},$$
and for $v \in \mathcal{C}$ let $u := \mathcal{S}(m^+ - m^- v^{-\gamma}) := \mathcal{T}(v)$. Utilizing (3.3) and the weak comparison principle we see that
$$\mathcal{S}(m^+) \geq \mathcal{S}(m^+ - m^- v^{-\gamma}) = u \geq \mathcal{S}(m^+ - m^-(\varepsilon\delta_\Omega)^{-\gamma}) \geq \varepsilon\delta_\Omega \quad \text{in } \Omega$$
and hence $u \in \mathcal{C}$. Furthermore, one can verify that $v \to m^+ - m^- v^{-\gamma}$ is continuous from $\mathcal{C}$ into $L^r(\Omega)$ for some $r > 1$, and thus employing the compactness of the solution operator $\mathcal{S}$ (cf. Remark 2.1) we deduce that $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is continuous and compact. It follows from Schauder's fixed point theorem that there exists some $v \in \mathcal{C}$ solution of
$$\begin{cases} \mathcal{L}v = m^+(x) - m^-(x)v^{-\gamma} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$
Moreover, $v \in C^1(\overline{\Omega})$ and, since $v \leq 1$ (due to $v \leq \mathcal{S}(m^+) \leq 1$), it follows from (3.4) that v is a subsolution of (1.1). Therefore, recalling Remark 2.7 and Theorem 2.6 we obtain some $u \in C^1(\Omega) \cap C(\overline{\Omega})$ solution of (1.1). Finally, decreasing γ_0 if necessary so that $m^+\delta_\Omega^{-\gamma} \in L^r(\Omega)$ with $r > 1$, by standard regularity arguments we get that $u \in C^1(\overline{\Omega})$, and also $u \in P^{\circ}$ in view of the fact that $u \geq c\delta_\Omega$ for some $c > 0$. This concludes the proof of (i).

In order to prove (ii) we proceed similarly. We shall prove (ii) for τm, where
$$\tau := \Big(\frac{2}{b-a}\Big)^{p-1} \Bigg(\int_a^b m^+\Bigg)^{-1}.$$
Since $\delta_\Omega \leq (b-a)/2$ in Ω, employing (2.4) one can check that $\mathcal{S}(\tau m^+) \leq 1$ in Ω. We shall also assume that
$$\max\Bigg(\int_a^{x_0} m^-\delta_\Omega^{-\gamma}, \int_{x_1}^b m^-\delta_\Omega^{-\gamma}\Bigg) = \int_a^{x_0} m^-\delta_\Omega^{-\gamma} \tag{3.5}$$
because the other case is completely analogous. We define next
$$c_1 := \frac{\inf_I m^+}{\lambda_1(I) c_I^{p-1}}, \quad c_2 := \int_a^{x_0} m^-\delta_\Omega^{-\gamma}, \quad r := \Big(\frac{\tau c_2 \gamma}{p-1}\Big)^{\frac{1}{p-1+\gamma}}, \quad \mathcal{C} := \{v \in C(\overline{\Omega}) : r\delta_\Omega \leq v \leq \mathcal{S}(\tau m^+) \text{ in } \Omega\}.$$
(Let us mention that if (3.5) is not valid, then we set $c_2 := \int_{x_1}^b m^-\delta_\Omega^{-\gamma}$.) One can readily verify that (3.2) implies

that

$$c_1^{p-1+\gamma} \geq \Big(\frac{p-1}{\tau\gamma}\Big)^{\gamma}\Big(\frac{p-1+\gamma}{p-1}\Big)^{p-1+\gamma} c_2^{p-1}. \tag{3.6}$$

Taking into account this fact and the definition of c_I we now observe that

$$\begin{aligned}\lambda_1(I)(x_I-a)^{p-1}\int_a^{x_0} m^-(r\delta_\Omega)^{-\gamma} &\leq \lambda_1(I)c_I^{p-1}r^{-\gamma}\int_a^{x_0} m^-\delta_\Omega^{-\gamma}\\ &= \lambda_1(I)c_I^{p-1}\Big(\Big(\frac{p-1}{\tau\gamma}\Big)^{\gamma}c_2^{p-1}\Big)^{\frac{1}{p-1+\gamma}}\\ &\leq \lambda_1(I)c_I^{p-1}c_1\frac{p-1}{p-1+\gamma}\\ &< \inf_I m^+\end{aligned}$$

and thus we may apply Lemma 2.4 (ii) with $m^+-m^-(r\delta_\Omega)^{-\gamma}$ in place of h (and so also with $\tau(m^+-m^-(r\delta_\Omega)^{-\gamma})$).

Given any $v \in \mathcal{C}$, we next define $u := \mathcal{S}(\tau(m^+-m^-v^{-\gamma}))$. Recalling the above paragraph, from Lemma 2.4 (ii) and again making use of (3.5) and (3.6), after some computations we deduce that

$$\mathcal{S}(\tau m^+) \geq u \geq \mathcal{S}(\tau(m^+-m^-(r\delta_\Omega)^{-\gamma})) \geq (\tau(c_1-c_2r^{-\gamma}))^{\frac{1}{p-1}}\delta_\Omega \geq r\delta_\Omega \quad \text{in } \Omega$$

and therefore $v \in \mathcal{C}$. Now the proof of (ii) can be finished as in (i), and this concludes the proof. □

Remark 3.2. (i) Let us notice that by Lemma 2.4 (ii), (3.1) is true if for instance

$$\inf_I m^+ > \lambda_1(I)\max\Big((x_I-a)^{p-1}\int_a^{x_0} m^-, (b-x_I)^{p-1}\int_{x_1}^{b} m^-\Big)$$

for some $I = (x_0, x_1) \subset \Omega$.

(ii) We also remark that several distinct conditions guarantee that $m^-\delta_\Omega^{-\gamma} \in L^q(\Omega)$ for some $q > 1$. Indeed, for example, this occurs for all $\gamma \in (0, 1/p)$, or more generally if $m^- \in L^q(\Omega)$ with $q \geq p'$ and $\gamma \in (0, (q-1)/q)$. Also, the same is valid for every $\gamma > 0$ when $m \geq 0$ in the set $\{x \in \Omega : \delta_\Omega(x) < \varepsilon\}$ for some $\varepsilon > 0$. Of course, analogous statements hold for $m^+\delta_\Omega^{-\gamma}$.

Theorem 3.3. *Suppose* (1.1) *has a solution* $u \in C^1(\overline{\Omega})$ *such that* $\varphi_p(u')$ *is absolutely continuous. Then*

$$\mathcal{S}(m) > 0 \quad \text{in } \Omega \tag{3.7}$$

and

$$\int_a^b m > 0. \tag{3.8}$$

Proof. Let $u > 0$ be a solution of (1.1) and fix

$$\beta := \frac{p-1+\gamma}{p-1}.$$

Let $0 \leq \phi \in C_c^\infty(\Omega)$, and let Ω' be an open set such that $\operatorname{supp}\phi \subset \Omega' \Subset \Omega$. We have that

$$\begin{aligned}\mathcal{L}(u^\beta) &= -\beta^{p-1}(|u'|^{p-2}u'u^{(\beta-1)(p-1)})'\\ &= \beta^{p-1}(m(x)u^{-\gamma}u^{(\beta-1)(p-1)} - |u'|^p(\beta-1)(p-1)u^{(\beta-1)(p-1)-1})\\ &\leq \beta^{p-1}m(x)u^{-\gamma}u^{(\beta-1)(p-1)}\\ &= \beta^{p-1}m(x) \quad \text{in } \Omega'\end{aligned}$$

and hence, multiplying the above inequality by ϕ, integrating over Ω' and using the integration by parts formula, we see that

$$\int_a^b |(u^\beta)'|^{p-2}(u^\beta)'\phi' \leq \beta^{p-1}\int_a^b m(x)\phi.$$

On the other hand, let $0 \le v \in W_0^{1,p}(\Omega)$. It is easy to check that there exists $\{\phi_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\Omega)$ with $\phi_j \ge 0$ in Ω and such that $\phi_j \to v$ in $W^{1,p}(\Omega)$ (see e.g. [4, p. 50]). Utilizing the last inequality with ϕ_j in place of ϕ and passing to the limit, we get that $\mathcal{L}(u^\beta) \le \beta^{p-1} m(x)$ in weak sense in Ω and so from the weak comparison principle we deduce that $0 < u^\beta \le \beta\mathcal{S}(m)$ in Ω and this ends the proof of (3.7).

Finally, we observe that multiplying (1.1) by u^γ and integrating by parts on $(a+\varepsilon, b-\varepsilon)$ with $\varepsilon > 0$ small, we get that

$$(\varphi_p(u')u^\gamma)(a+\varepsilon)-(\varphi_p(u')u^\gamma)(b-\varepsilon) + \gamma \int_{a+\varepsilon}^{b-\varepsilon} |u'|^p u^{\gamma-1} \le \int_{a+\varepsilon}^{b-\varepsilon} m$$

and letting $\varepsilon \to 0$ it is easy to deduce (3.8). □

Remark 3.4. Let us note that conditions (3.7) and (3.8) are not comparable. Indeed, suppose first $p = 2$, and let $\Omega := (0, 3\pi)$ and $m(x) := \sin x$. Then $m = \mathcal{S}(m)$ and $\int_0^{3\pi} m > 0$, but $\mathcal{S}(m) < 0$ in $(\pi, 2\pi)$.

On the other side, integrating (2.1) (with m in place of h) we get that

$$\varphi_p(\mathcal{S}(m)'(a)) - \varphi_p(\mathcal{S}(m)'(b)) = \int_a^b m. \tag{3.9}$$

It follows that we may have $\mathcal{S}(m) > 0$ in Ω but $\int_a^b m = 0$. (Take for instance again $p = 2$, $\Omega := (0, \pi)$, $m(x) := 2(\sin^2 x - \cos^2 x)$ and $\mathcal{S}(m)(x) = \sin^2 x$.)

What it is indeed true from (3.9) is that $\mathcal{S}(m) > 0$ in Ω implies $\int_a^b m \ge 0$. Moreover, from Theorem 3.3 and (3.9) we have that if (1.1) admits a solution, then either $\mathcal{S}(m)'(a) \ne 0$ or $\mathcal{S}(m)'(b) \ne 0$. It is an interesting open question to see if it is necessary that both derivatives are nonzero.

We conclude the paper showing an existence theorem for singular problems of the form

$$\begin{cases} \mathcal{L}u = m(x)f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.10}$$

for certain continuous functions $f : (0, \infty) \to (0, \infty)$. Let us observe that we make no monotonicity nor convexity assumptions on f.

We state the following hypothesis:

Hypothesis (H). There exist $c_f, C_f > 0$ and $\gamma > 0$ such that

$$c_f \xi^{-\gamma} \le f(\xi) \le C_f \xi^{-\gamma} \quad \text{for all } \xi > 0.$$

Corollary 3.5. *Let $m \in L^{p'}(\Omega)$, let f satisfy* (H) *and suppose* (1.1) *has a solution with $c_f m^+ - C_f m^-$ in place of m. Then there exists a solution of* (3.10).

Proof. Let u be a solution of (1.1) with $c_f m^+ - C_f m^-$ in place of m. Employing (H) we find that

$$\mathcal{L}u = (c_f m^+(x) - C_f m^-(x))u^{-\gamma} \le m(x)f(u) \quad \text{in } \Omega.$$

On the other hand, let $\psi := \mathcal{S}(m^+) > 0$ and fix $\beta \in (0, 1)$ and $\sigma > 0$ satisfying

$$\beta := \frac{p-1}{p-1+\gamma}, \qquad \sigma \ge \frac{C_f^{\frac{1}{p-1+\gamma}}}{\beta^\beta}.$$

Enlarging σ if necessary, recalling that $\beta < 1$ and that by Lemma 2.4,

$$\mathcal{S}(m^+)'(a) > 0 > \mathcal{S}(m^+)'(b),$$

we may assume that $\sigma\psi^\beta \ge u$ in Ω. Now, arguing as in Remark 2.7 and taking into account (H), we obtain that

$$\mathcal{L}(\sigma\psi^\beta) \ge (\sigma\beta)^{p-1} m^+(x)\psi^{(\beta-1)(p-1)} \ge C_f m^+(x)(\sigma\psi^\beta)^{-\gamma} \ge m^+(x)f(\sigma\psi^\beta) \ge m(x)f(\sigma\psi^\beta) \quad \text{in } \Omega'$$

for every $\Omega' \Subset \Omega$, and the corollary follows. □

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