LIFTING VIA COCYCLE DEFORMATION

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Abstract. We develop a strategy to compute all liftings of a Nichols algebra over a finite dimensional cosemisimple Hopf algebra. We produce them as cocycle deformations of the bosonization of these two. In parallel, we study the shape of any such lifting.

1. Introduction

Let A be a finite-dimensional Hopf algebra whose coradical is a Hopf subalgebra H. Then the graded algebra associated to the coradical filtration of A is again a Hopf algebra, which is given by a smash product $\operatorname{gr} A \simeq R\#H$, for $R = \bigoplus_{n\geq 0} R^n$ a graded Hopf algebra in ${}^H_H\mathcal{YD}$, the category of Yetter-Drinfeld modules over H. Let $V = R^1$, then the subalgebra of R generated by V is the *Nichols algebra* $\mathcal{B}(V)$ [AS2]; this is a braided Hopf algebra in ${}^H_H\mathcal{YD}$ which is also defined for every $V \in {}^H_H\mathcal{YD}$ by a universal quotient $T(V)/\mathcal{J}(V)$, for $\mathcal{J}(V)$ an ideal generated by homogeneous elements of degree ≥ 2.

If $\operatorname{gr} A = \mathcal{B}(V)\#H$, then A is called a *lifting* or *deformation* of $\mathcal{B}(V)$ (over H). Hence, deformations of $\mathcal{B}(V)$ give rise to new examples of Hopf algebras. Moreover, there are classes of Hopf algebras (as pointed Hopf algebras over abelian groups) in which every example arises as such a deformation.

1.1. **The problem.** In this article, we develop a strategy to compute all the liftings or deformations of a Nichols algebra. More precisely, we consider

(1.1) a Hopf algebra H which is finite-dimensional and cosemisimple;

(1.2) $V \in {}^H_H\mathcal{YD}$ such that $\dim V < \infty$ and $\mathcal{J}(V)$ is finitely generated.

The problem is to describe all Hopf algebras A such that

$$\operatorname{gr} A \simeq \mathcal{B}(V)\#H. \tag{1.3}$$

Notice that the coradical of A is isomorphic to H by (1.3), see [AS2]. This problem is one of the steps in the Lifting Method [AS1, AS2], see also the generalization proposed in [AC]. To deal with it, we split it into two parts:

(a) To detect the shape of all possible deformations.

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(b) To show that these proposed deformations actually are so.

Problem (a) is usually taken by examination of the comodule structure of the first term of the coradical filtration, what would give possible deformations by defining relations, see Section 4.

However it is not apparent that the proposed deformations have the desired property; namely, such deformation A would bear an epimorphism $\mathcal{B}(V)\#H \twoheadrightarrow \operatorname{gr} A$ but whether this is an isomorphism requires an extra reasoning. This is Problem (b) and there have been different approaches to face up to it: the Diamond Lemma [AS1, AG2, AV1]; a reduction to the first term of the coradical filtration followed by some representation theory, assuming that the Nichols algebra is quadratic [GGI]; a combination of deformation by cocycles and an examination of the PBW basis [AS3].

We briefly recall this last approach highlighting some features that are present in the strategy below; see *loc. cit.* for more details and undefined notation. There, H is assumed to be the group algebra of a finite abelian group Γ (with some restrictions on the order) and $V \in {}^H_H\mathcal{YD}$ has a finite-dimensional Nichols algebra; therefore, by the restrictions alluded to, V is of Cartan type and gives rise to a Dynkin diagram $\mathit{\Delta}$. The defining ideal $\mathcal{J}(V)$ is generated by three kind of relations:

(i) Serre relations in the same connected component of $\mathit{\Delta}$,
(ii) Serre relations between vertices in different connected components,
(iii) powers of root vectors.

It is then shown that in any deformation A the Serre relations in the same connected component still hold, and the other relations deform respectively to the so-called linking relations, controlled by a family of parameters $\boldsymbol{\lambda}$, and the so-called power of root vector relations, controlled by a second family of parameters $\boldsymbol{\mu}$. Hence the A should be of the form $u(\mathcal{D}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = T(V)\#H/\mathcal{J}$, where the ideal $\mathcal{J}$ is generated by:

(i) Serre relations (in the same connected component),
(ii) linking relations,
(iii) power of root vector relations.

To show that $u(\mathcal{D}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ has the desired dimension $\dim \mathcal{B}(V)|\Gamma|$, the procedure in [AS3] goes as follows.

(a) Let $U(\mathcal{D}, \boldsymbol{\lambda}) = T(V)\#H/\mathcal{J}_0$, where the ideal $\mathcal{J}_0$ is generated by the Serre relations (in the same connected component) and the linking relations. Then $U(\mathcal{D}, \boldsymbol{\lambda})$ has the "right" basis; it is proved by induction on the number of connected components, via cocycle deformation in the inductive step.
(b) Finally, $u(\mathcal{D}, \boldsymbol{\lambda}) = U(\mathcal{D}, \boldsymbol{\lambda})/\mathcal{J}_1$, where $\mathcal{J}_1$ is generated by the power of root vector relations, has the right dimension by a delicate argument using centrality of these last relations in $U(\mathcal{D}, \boldsymbol{\lambda})$.

1.2. **The background.** The family of Hopf algebras $u(\mathcal{D}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ contains the liftings of quantum linear spaces defined in [AS1]. It was shown in [Ma2] that these liftings of quantum linear spaces are cocycle deformations of their associated graded Hopf algebras. Further work in this direction was done in [D, BDR, GrM]; in this last paper it was stated that any Hopf algebra $u(\mathcal{D}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ is a cocycle deformation of its associated graded Hopf algebra, but the argument had a gap and a complete proof was given in [Ma4].

The result in [Ma4] is first extended to the non-abelian case in [GIM] where it is shown that every finite-dimensional pointed Hopf algebra H over $\mathbb{S}_3$ or $\mathbb{S}_4$ is again a cocycle deformation of $\operatorname{gr} H$. In [AV2] it is shown that this is also the case for finite-dimensional copointed Hopf algebras over $\mathbb{S}_3$. Also, in [GIV] this is shown for some pointed or copointed Hopf algebras associated to affine racks. In all of these papers the results are achieved by computing Hopf biGalois objects. In [GM], the authors pick up the work in [GrM] to explicitly compute cocycles as exponentials of Hochschild 2-cocycles. They show that every finite-dimensional pointed Hopf algebra H over the dihedral groups D_{4t} is a cocycle deformation of $\operatorname{gr} H$.

1.3. **The strategy.** In the present paper, we propose to reverse the order and start by computing all cocycle deformations following ideas in [Ma4]. Observe that, since a deformation by cocycle affects only the multiplication, the coradical filtration of a cocycle deformation A of $\mathcal{B}(V)\#H$ remains unchanged, hence it is isomorphic to $\mathcal{B}(V)\#H$ as coalgebras. Also, it is possible to decide when A is a lifting of $\mathcal{B}(V)$ over H.

Set $\mathcal{T}(V) = T(V)\#H$, $\mathcal{H} = \mathcal{B}(V)\#H$. Our strategy is as follows:

(a) We decompose a minimal set of generators of the ideal defining $\mathcal{B}(V)$ and recover $\mathcal{H}$ as the last link in a chain of subsequent quotients $\mathcal{T}(V) \twoheadrightarrow \mathcal{B}_1\#H \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{B}_n\#H \twoheadrightarrow \mathcal{H}$. We choose this decomposition in such a way that every intermediate quotient is achieved by dividing by primitive elements in $\mathcal{B}_i$, $i = 1, \dots, n$.

(b) At each step, we compute the Galois objects of $\mathcal{H}_{i+1}$ as quotients of the Galois objects of $\mathcal{H}_i$, following the results in [Gu]. We start with the trivial Galois object for $\mathcal{T}(V)$. In the final step, we have a set Λ of Galois objects of $\mathcal{H}$ and hence a list of cocycle deformations L, which arise as $L \simeq L(\mathcal{A}, \mathcal{H})$, for $\mathcal{A} \in \Lambda$ as in [S1].

(c) We check that any lifting is obtained as one of these deformations.

The paper is organized as follows: In Section 2 we fix the notation and introduce the preliminaries on Hopf algebras, Nichols algebras, cocycles and Hopf Galois objects. In Section 3 we recall the two theorems in [Gu] about cleft and Galois objects of quotient Hopf algebras and study the validity of the hypotheses of these results in order to apply them in our context. In Section 4 we investigate the shape of any lifting candidate of a Nichols algebra $\mathcal{B}(V)$. We also study this problem in the opposite sense, that is to say we investigate the shape of the graded algebra associated to a deformation. Finally, in Section 5 we present our strategy to compute all cocycle deformations of $\mathcal{B}(V)\#H$. As an illustration, we apply it to classify all liftings of a Nichols algebra associated to an example with diagonal braiding. We end this article with a question related to the extent of this strategy.

2. Conventions and preliminaries

2.1. **Conventions.** The base field, denoted by $\Bbbk$, is assumed to be algebraically closed. Let G be a group; then $Z(G)$, resp. $\widehat{G}$, denotes its center, resp. the group of multiplicative characters. If $\chi \in \widehat{G}$ and V is a G-module, then V^{χ} denotes the isotypic component of V of type χ. Let A be a $\Bbbk$-algebra and $S \subset A$ a subset. Then we denote by $Z(A)$ the center of A, by $\langle S\rangle$ (or $\langle S\rangle_A$ if an explicit mention to A is needed) the two-sided ideal generated by S and by $\Bbbk\langle S\rangle$ the subalgebra generated by S.

Let H be a Hopf algebra. We will use the (summation free) Sweedler's notation for the comultiplication Δ, ε will denote the counit and $\mathcal{S}$ the antipode. Where needed, we stress the connection with H by a subscript H, *e.g.* Δ_H. We denote by $H_{[0]}$ the *coradical* of H and by $(H_{[n]})_{n\in\mathbb{N}}$ the coradical filtration; $G(H)$ is the group of group-like elements. For $g, h \in G(H)$, we denote by $\mathcal{P}_{g,h}(H) = \{u \in H : \Delta(u) = u \otimes h + g \otimes u\}$ the set of (g, h) skew-primitive elements in H; $\mathcal{P}(H) = \mathcal{P}_{1,1}(H)$ for short. If A is a right (resp. left) H-comodule algebra, then $A^{\operatorname{co} H}$ (resp. ${}^{\operatorname{co} H}A$) denotes the subalgebra of right (resp. left) coinvariants. The right adjoint action of H on itself is

$$\operatorname{ad}_r(h)(b) = \mathcal{S}(h_{(1)})bh_{(2)}, \qquad b, h \in H. \tag{2.1}$$

Right, resp. left, coactions are denoted by ρ, resp. λ. We shall also use the Sweedler's notation for coactions.

Given a Hopf algebra H with bijective antipode, we denote by ${}^H_H\mathcal{YD}$, resp. $\mathcal{YD}^H_H$, the category of left, resp. right, Yetter-Drinfeld modules over H. If $K \subseteq H$ is a Hopf subalgebra, then $\mathcal{YD}^H_K$ is the category whose objects are H-comodules and K-modules, with the compatibility inherited from $\mathcal{YD}^H_H$, and H-colinear, K-linear morphisms. We refer to [Mo] for unexplained notation and notions about Hopf algebras.

We say that (g, χ), with $g \in G(H)$ and $\chi \in \operatorname{Alg}(H, \Bbbk)$, is a *YD-pair* when the following equivalent conditions hold for all $h \in H$:

$$\chi(h)\, g = \chi(h_{(2)})h_{(1)}\, g\, \mathcal{S}(h_{(3)}) \iff \chi(h_{(1)})\, g\, h_{(2)} = h_{(1)}\, g\, \chi(h_{(2)}). \tag{2.2}$$

In particular, such g should belong to $Z(G(H))$. If (g, χ) is a YD-pair, then $\Bbbk^{\chi}_g$ denotes the vector space $\Bbbk$ with coaction $x \mapsto g \otimes x$ and action $h{\cdot}x = \chi(h)x$, for $x \in \Bbbk$, $h \in H$; (2.2) guarantees that $\Bbbk^{\chi}_g \in {}^H_H\mathcal{YD}$. Conversely, any one-dimensional Yetter-Drinfeld module over H arises in this way. If $V \in {}^H_H\mathcal{YD}$, then V^{χ}_g denotes the isotypic component of V of type $\Bbbk^{\chi}_g$.

We refer to [M, EGNO, Mü] for details about braided Hopf algebras, that is Hopf algebras in braided tensor categories. Recall that a Nichols algebra $\mathcal{B}(V) = \bigoplus_{n\geq 0} \mathcal{B}^n(V)$ is a graded braided Hopf algebra in ${}^H_H\mathcal{YD}$ generated by $V = \mathcal{B}^1(V)$ that coincides with the space $\mathcal{P}(\mathcal{B}(V))$ of primitive elements in $\mathcal{B}(V)$. We denote by $\mathcal{J}(V) = \bigoplus_{n\geq 2} \mathcal{J}^n(V)$ the defining ideal of $\mathcal{B}(V)$, *i.e.* $\mathcal{B}(V) = T(V)/\mathcal{J}(V)$. See [AS2] for more details. A *pre-Nichols algebra* is an intermediate graded braided Hopf algebra between $T(V)$ and $\mathcal{B}(V)$, see [Ma4].

2.2. **Cocycles.** Let H be a Hopf algebra. A 2-cocycle $\sigma : H \otimes H \to \Bbbk$ is a convolution-invertible linear map $h \otimes k \mapsto \sigma(h,k)$ satisfying, for $x, y, z \in H$,

$$\sigma(x,1) = \sigma(1,x) = \varepsilon(x) \quad \text{and} \tag{2.3}$$

$$\sigma(x_{(1)}, y_{(1)})\sigma(x_{(2)}y_{(2)}, z) = \sigma(y_{(1)}, z_{(1)})\sigma(x, y_{(2)}z_{(2)}). \tag{2.4}$$

Let σ be a 2-cocycle. Then $\cdot_\sigma : H \otimes H \to H$, given by

$$x \cdot_\sigma y = \sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)}\sigma^{-1}(x_{(3)}, y_{(3)}), \quad x, y \in H, \tag{2.5}$$

defines an associative product on the vector space H with unit 1_H. Moreover, the collection $(H, \cdot_\sigma, 1_H, \Delta, \varepsilon, \mathcal{S}_\sigma)$ is a Hopf algebra with antipode $\mathcal{S}_\sigma = f * \mathcal{S} * f^{-1}$, for $f = \sigma \circ (\mathrm{id} \otimes \mathcal{S}) \circ \Delta$. This Hopf algebra is denoted H_σ.

The group of convolution-invertible linear functionals of H acts on the set $Z^2(H, \Bbbk)$ of 2-cocycles. If $\alpha \in \mathrm{Hom}(H, \Bbbk)$ is convolution-invertible, then

$$\sigma^\alpha(x,y) = \alpha(x_{(1)})\alpha(y_{(1)})\sigma(x_{(2)}, y_{(2)})\alpha^{-1}(x_{(3)}y_{(3)}), \quad \forall x, y \in H$$

is again a 2-cocycle and $\alpha^{-1} * \mathrm{id} * \alpha : H_{\sigma^\alpha} \longrightarrow H_\sigma$ is an isomorphism of Hopf algebras. The quotient of $Z^2(H, \Bbbk)$ under this action is denoted $H^2(H, \Bbbk)$.

Remarks 2.1. Let H be a Hopf algebra and let $\sigma : H \otimes H \to \Bbbk$ be a 2-cocycle. Since the comultiplications of H and H_σ coincide, we have

(a) The coradicals of H and H_σ coincide.
(b) The coradical filtrations of H and H_σ coincide; this is valid for any wedge filtration (*e.g.* the *standard filtration* defined in [AC]).
(c) If C and D are subcoalgebras of H, then $C \cdot D = C \cdot_\sigma D$.
(d) Let C be a subcoalgebra stable by the antipode $\mathcal{S}_H$. Let K be the subalgebra of H generated by C (a Hopf subalgebra indeed) and set $\sigma' = \sigma_{|K \otimes K}$. Then $K_{\sigma'}$ is the subalgebra of H_σ generated by C.

Given a 2-cocycle $\sigma : H \otimes H \to \Bbbk$ there is another way to define an associative product on H:

$$x \cdot_{(\sigma)} y = \sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)}, \quad x, y \in H. \tag{2.6}$$

We denote this algebra by $H_{(\sigma)}$. Then $\Delta : H_{(\sigma)} \to H_{(\sigma)} \otimes H$ is an algebra map and $H_{(\sigma)}$ becomes a right H-comodule algebra. Moreover, $\big(H_{(\sigma)}\big)^{\mathrm{co}\, H} = \Bbbk$.

2.3. **Galois objects.** Let H be a Hopf algebra and let A be a right H-comodule algebra with $\Bbbk \simeq A^{\mathrm{co}\, H}$. Then A is a (right) H-*Galois object* if the *canonical* linear map $\mathrm{can} : A \otimes A \to A \otimes H$, $a \otimes b \mapsto ab_{(0)} \otimes b_{(1)}$ is an isomorphism. Left H-Galois objects are defined analogously. We set $\mathrm{Gal}(H) = \{$isomorphism classes of (right) H-Galois objects$\}$. Let H, L be Hopf algebras. An (L, H)-bicomodule algebra is an (L, H)-*biGalois object* if it is both a left L-Galois object and a right H-Galois object.

Let A be a right H-comodule algebra, $B = A^{\operatorname{co} H}$. The extension $B \subset A$ is called *cleft* if there exists an H-colinear convolution-invertible map $\gamma : H \to A$ or, equivalently when $A \simeq B\#_\sigma H$ for some 2-cocycle $\sigma : H \otimes H \to B$, see [Mo, Section 7] and [DT2]. We may assume that $\gamma(1) = 1$, in which case γ is called a *section*. The cocycle σ is given by

$$\sigma(h,k) = \gamma(h_{(1)})\gamma(k_{(1)})\gamma^{-1}(h_{(2)}k_{(2)}), \quad h,k \in H. \tag{2.7}$$

If $A^{\operatorname{co} H} = \Bbbk$, then A is called *cleft object*. Set

$$\operatorname{Cleft}(H) := \{\text{isomorphism classes of } H\text{-cleft objects}\} \simeq H^2(H, \Bbbk).$$

Remarks 2.2. (1) $B \subset A$ is a cleft extension if and only if A is an H-Galois extension and has the *normal basis* property, *i.e.* $A \simeq B \otimes H$ as right H-comodules and left B-modules [DT2]. If $\gamma : H \to A$ is a section, then

$$\operatorname{can}^{-1}(a \otimes h) = a\gamma^{-1}(h_{(1)}) \otimes \gamma(h_{(2)}), \qquad a \in A,\ h \in H. \tag{2.8}$$

(2) If H is pointed, then any H-Galois extension is cleft [Gu, Remark 10].

Given a right H-Galois object A, there is a Hopf algebra $L = L(A,H)$ attached to the pair (A,H) in such a way that A becomes an (L,H)-biGalois object [S1, Section 3]. As an algebra, $L(A,H) = (A \otimes A^{\mathrm{op}})^{\operatorname{co} H}$; the coproduct Δ_L and the coaction $\lambda : A \to L \otimes A$ are:

$$\Delta_L\Big(\sum_i x_i \otimes y_i\Big) = \sum_i x_{i(0)} \otimes \operatorname{can}^{-1}(1 \otimes x_{i(1)}) \otimes y_i, \quad \sum_i x_i \otimes y_i \in L;$$

$$\lambda(x) = x_{(0)} \otimes \operatorname{can}^{-1}(1 \otimes x_{(1)}), \qquad x \in A. \tag{2.9}$$

Here can^{-1} is the inverse of the *right canonical map* $\operatorname{can} : A \otimes A \to A \otimes H$. In turn, the inverse of the *left canonical map* $\operatorname{can} : A \otimes A \to L(A,H) \otimes A$ is:

$$\operatorname{can}^{-1}\Big((\sum_i x_i \otimes y_i) \otimes a\Big) = \sum_i x_i \otimes y_i a, \quad \sum_i x_i \otimes y_i \in L, a \in A. \tag{2.10}$$

The Hopf algebra L is uniquely characterized by this property [S1, Theorem 3.3]: if L' is another bialgebra and λ' is a left L'-coaction on A making it an (L', H)-biGalois object, then there exists a unique isomorphism $\vartheta : L \to L'$ such that $\lambda' = (\vartheta \otimes \mathrm{id})\lambda$. Explicitly, see [S1, Lemma 3.2],

$$\vartheta\Big(\sum_i x_i \otimes y_i\Big) \otimes 1_A = \sum_i \lambda'(x_i)(1 \otimes y_i), \qquad \sum_i x_i \otimes y_i \in L. \tag{2.11}$$

Remark 2.3. If $\sigma \in Z^2(H,\Bbbk)$, then $L(H_{(\sigma)}, H) \simeq H_\sigma$ [S1, Theorem 3.9].

3. Hopf Galois objects for quotient Hopf algebras

Our argument involves a recurrence on a chain of Hopf algebra quotients. We will use [Gu, Theorems 4 & 8], which we cite next, to study cocycle deformations for a quotient Hopf algebra.

We start with some preliminaries. Let $\pi : L \to K$ be a projection of Hopf algebras with bijective antipode. Then the right coideal subalgebra

$X = {}^{\operatorname{co} K}L$ of the left coinvariants is an algebra in $\mathcal{YD}_L^L$ with the right adjoint action (2.1) and the coaction given by the restriction of Δ.

Let $A \in \operatorname{Gal}(L)$. For $h \in L$, write $\sum_i \ell_i(h) \otimes r_i(h) = \operatorname{can}^{-1}(1 \otimes h) \in A \otimes A$. Then A is an algebra in $\mathcal{YD}_L^L$ via the *Miyashita-Ulbrich action* [DT1]

$$a \leftharpoonup h = \sum_i \ell_i(h) a r_i(h), \qquad a \in A,\ h \in L. \tag{3.1}$$

If A, B are algebras in $\mathcal{YD}_L^L$, $\operatorname{Alg}_L^L(A, B)$ denotes the set of algebra morphisms in $\mathcal{YD}_L^L$ between them.

Theorem 3.1. [Gu, Theorem 4] *Let L, K, π and $X = {}^{\operatorname{co} K}L$ be as above. Assume that L is left and right K-coflat. There are bijective correspondences*

$$\operatorname{Gal}(K) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \{(A, f) : [A] \in \operatorname{Gal}(L), f \in \operatorname{Alg}_L^L(X, A)\}/\sim,$$

$$\Psi\big([(A, f)]\big) = [A/Af(X^+)], \qquad \Phi\big([B]\big) = [(B \square_K L, x \mapsto 1 \otimes x)]\,.$$

The equivalence $\sim$ is defined so that $(A, f) \sim (A', f')$ if and only if there exists an isomorphism $\alpha : A \to A'$ of L-comodule algebras such that $f' = \alpha \circ f$. The coaction on $A/Af(X^+)$ is given by $(\tau \otimes \pi)\lambda_A$, for $\tau : A \to A/Af(X^+)$ the projection. If $B \in \operatorname{Gal}(K)$, then the L-coaction on $B \square_K L$ is $\operatorname{id}_B \otimes \Delta_L$.

If there is a subcoalgebra of L that is mapped isomorphically onto the coradical of K, then this correspondence restricts to cleft objects. □

There is another approach to compute cleft objects of quotient Hopf algebras given by an expansion of [Gu, Theorem 8]. To prove this, we will use the following result of Takeuchi. Let us recall that a right coideal subalgebra B of a Hopf algebra $\mathcal{H}$ is *normal* when it is stable under the right adjoint action (2.1).

Theorem 3.2. [T2, Theorem 3.2] *Let $\mathcal{H}$ be a Hopf algebra with bijective antipode. There exist mutually inverse bijective correspondences between the set of Hopf ideals I such that $\mathcal{H}$ is $\mathcal{H}/I$-coflat and the set of normal right coideal subalgebras B such that $\mathcal{H}$ is right B-faithfully flat given by*

$$I \text{ Hopf ideal } \rightsquigarrow \mathcal{X}(I) = \mathcal{H}^{\operatorname{co} \mathcal{H}/I};$$
$$B \text{ normal right coideal subalgebra} \rightsquigarrow \mathcal{I}(B) = \mathcal{H}B^+. \quad \square$$

If A, A' are right L-comodule algebras, then $\operatorname{Alg}^L(A, A')$ is the set of comodule algebra morphisms between them. If $X \subset L$ is a right coideal subalgebra, then $N(X)$ is the subalgebra generated by $\{\mathcal{S}(h_{(1)}) x h_{(2)} : h \in L, x \in X\}$; this is the normal subalgebra generated by X.

Theorem 3.3. *Let L be a Hopf algebra with bijective antipode. Let $Y \subset L$ be a right coideal subalgebra. Set $I = LY^+L$ and $K = L/I$; then K is a* *quotient Hopf algebra of L. Assume that L is K-coflat and that L is faithfully flat over $N(Y)$. Then there are bijective correspondences*

$$\mathrm{Cleft}(K) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \left\{(A, f) : \begin{array}{l} [A] \in \mathrm{Cleft}(L), f \in \mathrm{Alg}^L(Y, A) \\ \textit{such that } Af(Y^+)A \neq A \end{array} \right\} / \sim, \tag{3.2}$$

$$\Psi\big([(A, f)]\big) = [A/Af(Y^+)A], \qquad \Phi\big([B]\big) = [(B\square_K L, x \mapsto 1 \otimes x)] .$$

The corresponding coactions and the relation $\sim$ are as in Theorem 3.1.

Proof. Set $X = N(Y)$. First, we use Theorem 3.2 to show $X = {}^{\mathrm{co}\,K}L$. Indeed, we have, on the one hand, $\mathcal{I}(X) = LN(Y)^+ = LY^+L = I$. On the other, $\mathcal{X}(I) = {}^{\mathrm{co}\,K}L$ and the statement follows.

The proof now runs as that of [Gu, Theorem 8], as the hypotheses ${}^{\mathrm{co}\,K}L \cap L_{[0]} \subseteq N(Y)$ (which we recover trivially) and that of L being pointed in *loc. cit.* are precisely used to show $N(Y) = {}^{\mathrm{co}\,K}L$. □

In order to apply Theorems 3.1 and 3.3 we need to investigate when a Hopf algebra L is coflat over a quotient Hopf algebra K. This will be the content of Subsection 3.1. For Theorem 3.3, we need to study when L is faithfully flat over a right coideal subalgebra, we will also deal with this question in the next subsection. Also, see Section 5.9 for a detailed comparison of these two theorems.

3.1. **On the coflatness of quotients.** Fix a Hopf algebra H with bijective antipode. Let R be a connected (*i.e.* the coradical of R is $\Bbbk$) Hopf algebra in ${}^H_H\mathcal{YD}$. In particular, the antipode of R, hence that of $A = R\#H$, is bijective. Clearly, see [Ma3, Section 1], we have

- If B is a right coideal subalgebra of R, then R/RB^+ is a quotient left R-module coalgebra of R.
- If T is a quotient left R-module coalgebra of R, then the left T-coinvariants ${}^{\mathrm{co}\,T}R$ form a right coideal subalgebra of R.

Recall that the structures on R arise from the obvious Hopf algebra maps $H \to A \to H$, whose composite is the identity on H, as follows: we have $R = A^{co\,H}$, so that R is a left coideal subalgebra of A, and is thus an algebra and left H-comodule, while we have $R = A/AH^+$, so that R is a quotient left A-module coalgebra of A, and is thus a coalgebra and left H-module; the last left H-module structure coincides with the adjoint action. Let P (resp. Q) denote the braided Hopf algebra in ${}^{H^{\mathrm{op}}}_{H^{\mathrm{op}}}\mathcal{YD}$ (resp. ${}^{H^{\mathrm{cop}}}_{H^{\mathrm{cop}}}\mathcal{YD}$) which arises from $H^{\mathrm{op}} \to A^{\mathrm{op}} \to H^{\mathrm{op}}$ (resp. $H^{\mathrm{cop}} \to A^{\mathrm{cop}} \to H^{\mathrm{cop}}$). Then $P = Q = R$ as vector spaces. As an algebra, P equals the opposite algebra R^{op} of R, while as a coalgebra, Q equals the co-opposite coalgebra R^{cop} of R.

Lemma 3.4. *(i) The sub-objects (resp., right coideal subalgebras) of R in ${}^H_H\mathcal{YD}$ coincide with those of P in ${}^{H^{\mathrm{op}}}_{H^{\mathrm{op}}}\mathcal{YD}$.*

(ii) The quotient objects (resp., left module coalgebras) of R in ${}^H_H\mathcal{YD}$ coincide with those of Q in ${}^{H^{\mathrm{cop}}}_{H^{\mathrm{cop}}}\mathcal{YD}$.

Proof. (i) Since the comultiplication does not change, $R = P$ as left comodules over the coalgebra $H = H^{\mathrm{op}}$. If $h = \mathcal{S}_H^{-1}(k) \in H$, $a \in A$, then the adjoint actions of H and H^{op} are related by

$$\mathrm{ad}_H\, h(a) = h_{(1)} a \mathcal{S}_H(h_{(2)}) = k_{(1)} \cdot_{\mathrm{op}} a \cdot_{\mathrm{op}} \mathcal{S}_{H^{\mathrm{op}}}(k_{(2)}) = \mathrm{ad}_{H^{\mathrm{op}}}\, k(a).$$

This settles the claim for sub-objects. Let now X be a sub-object of R, or equivalently of P. Clearly, X is a subalgebra of R if and only if it is a subalgebra of $P = R^{\mathrm{op}}$. For $x \in R$, let $x \mapsto \sum x^{(1)} \otimes x^{(2)}$ denote the coproduct on R. Then the coproduct $\Delta(x)$ on A is given by

$$\Delta(x) = (x^{(2)})_{(-1)}(\mathcal{S}_H^{-1}((x^{(2)})_{(-2)}) \rightharpoonup x^{(1)}) \otimes (x^{(2)})_{(0)}.$$

Hence Δ_P is given by $x \mapsto \mathcal{S}_H^{-1}((x^{(2)})_{(-1)}) \rightharpoonup x^{(1)} \otimes (x^{(2)})_{(0)}$. It follows that X is a right coideal of R, if and only if it is such of P. (ii) is similar. □

Let B be a right coideal subalgebra of R in ${}^H_H\mathcal{YD}$. Then one can define the category $({}^H_H\mathcal{YD})^R_B$ of right (R, B)-Hopf modules in ${}^H_H\mathcal{YD}$; a *right* (R, B)-*Hopf module* is here a right B-module and right R-comodule in ${}^H_H\mathcal{YD}$ which satisfies the compatibility condition formulated as in the ordinary situation, but involving the braiding $R \otimes B \xrightarrow{\simeq} B \otimes R$.

Lemma 3.5. *Every object M in $({}^H_H\mathcal{YD})^R_B$ includes a sub-object X in ${}^H_H\mathcal{YD}$ such that the action map $X \otimes B \to M$ is a bijection, necessarily an isomorphism in $({}^H_H\mathcal{YD})_B$.*

Proof. The lemma follows from the following claim.

Claim. If $M \neq 0$, then M includes a non-zero sub-object N in $({}^H_H\mathcal{YD})^R_B$ which includes a sub-object X in ${}^H_H\mathcal{YD}$ such that $X \otimes B \xrightarrow{\simeq} N$.

Indeed, assume that we have proven the claim. We consider all pairs (N, X), where N is a sub-object of M in $({}^H_H\mathcal{YD})^R_B$, and X is a sub-object of N in ${}^H_H\mathcal{YD}$ such that $X \otimes B \xrightarrow{\simeq} N$ naturally, and introduce the natural order given by inclusion to the pairs. By Zorn's Lemma we have a maximal pair (N, X). To see $N = M$, suppose on the contrary $N \subsetneq M$. With the assumed fact applied to M/L, we have sub-objects $\tilde{N} \subset M$ in $({}^H_H\mathcal{YD})^R_B$ and $\tilde{X} \subset \tilde{N}$ in ${}^H_H\mathcal{YD}$ such that $N \subsetneq \tilde{N}$, $X \subsetneq \tilde{X}$ and $\tilde{X}/X \otimes B \xrightarrow{\simeq} \tilde{N}/N$. A map of short exact exact sequences which is isomorphic on the kernels and the cokernels shows that $\tilde{X} \otimes B \xrightarrow{\simeq} \tilde{N}$, which contradicts the maximality of (N, X), and hence shows $N = M$. This argument is the same as the one in [Ra, Proposition 1].

We now prove the claim. Suppose $0 \neq M \in ({}^H_H\mathcal{YD})^R_B$. We wish to prove M includes a nonzero pair. Set $X = M^{\mathrm{co}\, R}$. This is a subject of M in ${}^H_H\mathcal{YD}$, and is the socle $\mathrm{soc}\, M$ of the right R-comodule M, whence $X \neq 0$. The tensor product $X \otimes B$ is naturally an object in $({}^H_H\mathcal{YD})^R_B$ whose R-comodule socle $\mathrm{soc}(X \otimes B) = X$. We see that $f : X \otimes B \to M$, $f(v \otimes b) = vb$ is a morphism in $({}^H_H\mathcal{YD})^R_B$, which is injective since it is restricted to the identity on the socles. If $L = \mathrm{Im}\, f = XB$ then (N, X) is a desired pair. □

Proposition 3.6. *Let R be a connected Hopf algebra in ${}^H_H\mathcal{YD}$.*

(a) *R is a free left and right module over every right coideal subalgebra.*

(b) *R is a cofree left and right comodule over every quotient left module coalgebra T.*

(c) *$B \mapsto R/RB^+$ and $T \mapsto {}^{\operatorname{co} T}R$ give a bijection between the set of right coideal subalgebras B of R and the set of quotient left R-module coalgebras T of R.*

(d) *If B and T correspond to each other via this bijection, then there exists a left T-colinear and right B-linear isomorphism $T \otimes B \xrightarrow{\simeq} R$.*

Proof. (a) Let B be a right coideal subalgebra. First, we prove the right B-freeness. Notice that $R \in ({}^H_H\mathcal{YD})^R_B$. The right B-freeness in (a) follows from Lemma 3.5. By Lemma 3.4 (i), the just proved result applied to the P of the lemma implies the left B-freeness[1].

(c) Let T be as in (b), and set $B = {}^{\operatorname{co} T}R$. We see that the natural left T-comodule structure $R \to T \otimes R$ on R is right B-linear. The base extension along $B \to R$ induces a right B-linear and left T-colinear map $g : R \otimes_B R \to T \otimes R$. This is induced from the canonical isomorphism $R \otimes R \xrightarrow{\simeq} R \otimes R$, and hence is a surjection. Note that T is also connected.

Since R is left B-free as was shown in (a), the left T-comodule socle of $R \otimes_B R$ equals $B \otimes_B R$. It follows that g is injective, and hence bijective, since it restricts to id_R on the left T-comodule socles.

The bijection g together with the left B-freeness of R shows that R is an injective cogenerator (or equivalently, faithfully coflat) as a left T-comodule; see also [Ma3, Proposition 1.4(1)]. The desired one-to-one correspondence follows just as in the ordinary situation; see [Ma3, Proposition 1.4(2)].

(d) Let B and T correspond to each other. The left T-injectivity of R allows the inclusion $k \to R$ of left T-comodules to extend to a unit-preserving left T-colinear map $h : T \to R$. The right B-linearization of h

$$h_B : T \otimes B \to R, \ h_B(t \otimes b) = h(t)b$$

is right B-linear and left T-colinear and injective, since it restricts to id_B on the T-comodule socles. It is an isomorphism, since $T \otimes B$ is T-injective.

(b) Let T be as in (b). We see from Part (d) that R is left T-cofree. Lemma 3.4 (ii) shows that R is also right T-cofree. □

Corollary 3.7. *Let H be a cosemisimple Hopf algebra, let R, T be connected braided Hopf algebras in ${}^H_H\mathcal{YD}$, such that T is a quotient of R. Then $R\#H$ is left and right cofree over $T\#H$. In particular, it is left and right coflat.*

Proof. As H is cosemisimple, the coalgebra surjection $\mathrm{id} \otimes \varepsilon : T\#H \to T$ is a cosemisimple coextension, that is a left or right $T\#H$-comodule is injective

[1]One can define the analogous category ${}_B({}^H_H\mathcal{YD})^R$. But, it is impossible to discuss as above, because for $M \in {}_B({}^H_H\mathcal{YD})^R$, the right R-comodule $B \otimes M^{\operatorname{co} R}$ is not isomorphic to a direct sum of copies of B, and so $\operatorname{soc}(B \otimes M^{\operatorname{co} R}) = M^{\operatorname{co} R}$ may not be true.

if it is injective as a T-comodule. Since R is left T-cofree as a T-comodule and so left T-injective, it follows that $R\#H$, being left T-injective, is left $T\#H$ injective. Note that the coradical of $T\#H$ is isomorphically liftable to the coradical of $R\#H$, since both of them coincide with H. It follows by the left version of [Sc, Theorem 4.2] that there is a left $T\#H$-colinear and right ${}^{\operatorname{co} T\#H}(R\#H)$-linear isomorphism

$$R\#H \simeq T\#H \otimes {}^{\operatorname{co} T\#H}(R\#H).$$

By switching the sides one can present $R\#H$, $T\#H$ as smash products $H\#R'$, $H\#T'$ of braided Hopf algebras R', T' in $\mathcal{YD}_H^H$, such that T' is a quotient of R', and prove that R' is right (and left) T'-injective, which shows as above that there is a right $T\#H$-colinear and left $(R\#H)^{\operatorname{co} T\#H}$-linear isomorphism $R\#H \simeq (R\#H)^{\operatorname{co} T\#H} \otimes (T\#H)$, by [Sc, Theorem 4.2]. $\square$

Corollary 3.8. *Let R be a connected braided Hopf algebra in ${}^H_H\mathcal{YD}$ and let $B \subseteq R$ be a left coideal subalgebra. Let $L = R\#H$, then $B\#1$ is a left coideal subalgebra of L and L is a left $B\#1$-free module.*

If $Y = \mathcal{S}(B\#1)$, then L is a right Y-free module. In particular, it is right Y-faithfully flat.

Proof. The formula for $\Delta_{R\#H}$ shows that $B\#1$ is a left coideal subalgebra. Now, the first statement follows since L is left R-free and R is left B-free by Proposition 3.6 (a). The second is straightforward. $\square$

4. The shape of all possible deformations

Let H be as in (1.1) and A be a Hopf algebra whose coradical is isomorphic to H. In the first part of this section, we assume that H is also semisimple (*e. g.* when the characteristic of $\Bbbk$ is 0) and analyze the structure of A.

A fundamental information is that there exists a coalgebra H-bimodule projection $\Pi : A \to H$ such that $\Pi_{|H} = \operatorname{id}_H$ [AMŞ, Theorem 5.9.c)]. Hence A is a Hopf bimodule coalgebra over H via the left and right multiplication and the coactions $\rho_L = (\Pi \otimes \operatorname{id})\Delta$ and $\rho_R = (\operatorname{id} \otimes \Pi)\Delta$. Let $P_0 = 0$, $P_1 = \{x \in A : \Delta(x) = \rho_L(x) + \rho_R(x)\}$ and

$$P_n = \{x \in A : \Delta(x) - \rho_L(x) - \rho_R(x) \in \sum_{i=1}^{n-1} P_i \otimes P_{n-i}\}.$$

Then $P_n = A_{[n]} \cap \ker \Pi$ [AN, Lemma 1.1]. Clearly, P_n is a Hopf sub-bimodule of $A_{[n]}$ and $A_{[n]}/A_{[n-1]} = P_n/P_{n-1}$.

The canonical projection $\pi_n : A_{[n]} \to A_{[n]}/A_{[n-1]}$ is a Hopf bimodule map and it has a section ι_n since H is semisimple and cosemisimple. Therefore

$$A \simeq H \oplus \bigoplus_{n\geq 1} \iota_n(P_n/P_{n-1})$$

as Hopf bimodule. We extend π_n to be 0 in $\bigoplus_{m>n} \iota_m(P_m/P_{m-1})$, $n > 0$, and set $\pi_0 = \Pi$. We shall generally omit ι_m.

We recall the structure of $\operatorname{gr} A$. As vector spaces, $\operatorname{gr} A(n) = A_{[n]}/A_{[n-1]} = P_n/P_{n-1}$. The multiplication and comultiplication of $\operatorname{gr} A$ are

$$\pi_n(x)\pi_m(y) = \pi_{n+m}(xy) \quad \text{and}$$

$$\Delta_{\operatorname{gr} A}(\pi_n(x)) = \sum_{i=0}^{n} \pi_i(x_{(1)}) \otimes \pi_{n-i}(x_{(2)}), \quad x \in A_{[n]}, y \in A_{[m]}.$$

By abuse of notation, π_0 denotes the projection $\operatorname{gr} A \twoheadrightarrow H$ with kernel $\oplus_{n>0} \operatorname{gr} A(n)$ and $\pi_{0|H} = \operatorname{id}$. Then $\operatorname{gr} A$ is a Hopf bimodule over H via the left and right multiplication and the coactions $(\pi_0 \otimes \operatorname{id})\Delta_{\operatorname{gr} A}$, $(\operatorname{id} \otimes \pi_0)\Delta_{\operatorname{gr} A}$.

It is well-known that $\operatorname{gr} A \simeq (\operatorname{gr} A)^{\operatorname{co} H} \# H$ as Hopf algebras. In [AMŞ, Theorem 5.23], it is shown that $A^{\operatorname{co} H}$ is a coalgebra in ${}^H_H\mathcal{YD}$ such that $A \simeq A^{\operatorname{co} H} \# H$ as coalgebras and the multiplication in A is recovered with an extra structure on $A^{\operatorname{co} H}$, see also [S2].

Lemma 4.1. $\pi_n : \iota_n(P_n/P_{n-1}) \to \operatorname{gr} A(n)$ *is an isomorphism of Hopf bimodules over H for all n. Therefore* $A^{\operatorname{co} H} \simeq (\operatorname{gr} A)^{\operatorname{co} H}$ *in* ${}^H_H\mathcal{YD}$.

Proof. If $x \in \iota_n(P_n/P_{n-1})$ and $h \in H$, then $\pi_n(hx) = \pi_0(h)\pi_n(x)$ and $\pi_n(xh) = \pi_n(x)\pi_0(h)$. Also,

$$\begin{aligned}(\operatorname{id} \otimes \pi_0)\Delta^{\operatorname{gr}}(\pi_n(x)) &= \sum_{i=0}^{n} \pi_i(x_{(1)}) \otimes \pi_0 \circ \pi_{n-i}(x_{(2)}) = \pi_n(x_{(1)}) \otimes \pi_0(x_{(2)}) \\ &= (\pi_n \otimes \operatorname{id})(\operatorname{id} \otimes \Pi)\Delta(x) = (\pi_n \otimes \operatorname{id})\rho(x).\end{aligned}$$

Analogously, π_n is a left comodule map. The last assertion is easy. □

Remark 4.2. Assume that the dimension of A is finite. If the isomorphism $A^{\operatorname{co} H} \simeq (\operatorname{gr} A)^{\operatorname{co} H}$ in Lemma 4.1 is also of coalgebras, then $A \simeq \operatorname{gr} A$ as coalgebras; thus A is a cocycle deformation of $\operatorname{gr} A$ by [S1, Corollary 5.9].

We fix $V \in {}^H_H\mathcal{YD}$ with $\dim V < \infty$. Let $\mathcal{B}(V) = T(V)/\mathcal{J}(V)$ be the Nichols algebra of V see [AS2] and set $\mathcal{T}(V) = T(V)\#H$.

Definition 4.3. A *lifting map* is an epimorphism $\phi : \mathcal{T}(V) \to A$ of Hopf algebras such that $\phi_{|H} = \operatorname{id}_H$ and $\phi_{|V\#H} : V\#H \to P_1$ is an isomorphism of Hopf bimodules over H.

Proposition 4.4. [AV1, Proposition 2.4] *Let A be a Hopf algebra whose coradical is a Hopf subalgebra isomorphic to H. Then A is a lifting of* $\mathcal{B}(V)$ *over H if and only if there exists a lifting map* $\phi : \mathcal{T}(V) \to A$. □

The case $H = \Bbbk\Gamma$, Γ an abelian group, in the above proposition has been previously considered, see for instance [Kh, He].

From now on, *we assume that A is a lifting of* $\mathcal{B}(V)$ *over H with lifting map* $\phi : \mathcal{T}(V) \to A$. In particular, V is a submodule of A in ${}^H_H\mathcal{YD}$.

Let $\mathbb{B}^n_{\mathcal{J}}$ be a basis of $\mathcal{J}^n(V)$ and extend it to a basis $\mathbb{B}^n \cup \mathbb{B}^n_{\mathcal{J}}$ of $V^{\otimes n}$. We still denote by $\mathbb{B}^n$ the basis of the quotient $\mathcal{B}^n(V)$. Then $\mathbb{B} = \bigcup_n \mathbb{B}^n$ is a basis of $\mathcal{B}(V)$, $\mathbb{B}_{\mathcal{J}} = \bigcup_n \mathbb{B}^n_{\mathcal{J}}$ is a basis of $\mathcal{J}(V)$. Let $\mathbb{B}_H$ be a basis of H.

Remarks 4.5. By Lemma 4.1 we have that:

(a) $\{\phi(x) - \Pi(\phi(x)) : x \in \mathbb{B}^n_{\mathcal{J}}\} \subset P_{n-1}$.

(b) $\{\phi(x)h - \Pi(\phi(x))h : x \in \mathbb{B}^i, h \in \mathbb{B}_H, 0 < i \leq n\}$ is a basis of P_n.

(c) $\phi(\mathbb{B}^2)H = \iota_2(P_2/P_1)$ and $A_{[2]} \simeq (\mathcal{B}(V)\#H)_{[2]}$ as coalgebras.

(d) $\{\phi(x)h : x \in \mathbb{B}, h \in \mathbb{B}_H\}$ is a basis of A. Let $\iota : A \to \mathcal{T}(V)$ be the linear map identifying this basis of A with $\mathbb{B}\#H$.

The shape of the liftings is given by the following proposition. If $M \subset T(V)$ is a Yetter-Drinfeld submodule, we define the ideal

$$\mathcal{I}_M = \langle m - \iota\phi(m) \,|\, m \in M\rangle.$$

Proposition 4.6. *Let $M \subset T(V)$ be a Yetter-Drinfeld submodule which generates $\mathcal{J}(V)$. If $\mathcal{I}_M$ is a Hopf ideal, then $A = \mathcal{T}(V)/\mathcal{I}_M$.*

Proof. Let $A' := \mathcal{T}(V)/\mathcal{I}_M$; since $\mathcal{I}_M$ is contained in the kernel of the lifting map ϕ, we have an epimorphism $A' \twoheadrightarrow A$ and $\mathcal{I}_M \cap (\Bbbk \oplus V)\#H = 0$. Hence the coradical of A' is H by [Mo, Corollary 5.3.5]. Then $\operatorname{gr}(A') \simeq R\#H$ where $R \simeq T(V)/J$ for a braided Hopf ideal $J \subseteq \mathcal{J}(V)$. Clearly $M \subset J$, cf. Remark 4.5 (a), then $J = \mathcal{J}(V)$ and $\dim(A'_{[n]}/A'_{[n-1]}) = \dim(A_{[n]}/A_{[n-1]})$ for all $n \in \mathbb{N}$, hence the proposition follows. □

If there are no ambiguities, we identify $(\Bbbk \oplus V)\#H$ with its image by ϕ omitting the map ι. We explore a case where the hypothesis of Proposition 4.6 is satisfied.

Definition 4.7. A submodule M of $T(V)$ in ${}^H_H\mathcal{YD}$ is *compatible with* ϕ when

$$\Delta(\phi(m)) = \phi(m) \otimes 1 + m_{(-1)} \otimes \phi(m_{(0)}) \text{ for all } m \in M.$$

Assume $M \subset T(V)$ is compatible with ϕ. For $m \in M$, we may see $\phi(m)$ as an element of $(\Bbbk \oplus V)\#H$. Fix a basis $\{m_i\}_{1 \leq i \leq r}$ of M and let $\{c_{ij}\}_{i,j} \subset H$ be the set of *comatrix elements* associated to M and $\{m_i\}_{1 \leq i \leq r}$, that is

$$(m_i)_{(-1)} \otimes (m_i)_{(0)} = \sum_j c_{ij} \otimes m_j, \quad 1 \leq i \leq r. \tag{4.1}$$

If M is simple, then the set $\{c_{ij}\}_{i,j}$ is linearly independent and spans a simple coalgebra. Next lemma helps us to describe the image $\phi(M)$.

Lemma 4.8. *Let $M \subset T(V)$ be compatible with ϕ. Then*

(a) *$\pi_1 \circ \phi_{|M} : M \to V$ is a morphism in ${}^H_H\mathcal{YD}$.*

(b) *Assume that M is simple and $V \simeq M^m \oplus P$ with m maximum. Then there exist $\lambda_1, \dots, \lambda_m \in \Bbbk$ such that*

$$\pi_1 \circ \phi_{|M} \simeq \lambda_1 \operatorname{id}_M \oplus \cdots \oplus \lambda_m \operatorname{id}_M \oplus 0.$$

(c) *For $\{m_i\}_{1 \leq i \leq r}$, $\{c_{ij}\}_{i,j}$ as in (4.1) there exist $a_1, \dots, a_r \in \Bbbk$ such that*

$$(\pi_0 \circ \phi)(m_i) = a_i - \sum_{j=1}^{r} a_j c_{ij} \qquad i = 1, \dots, r.$$

(d) *Let* $\Theta : A \to A'$ *be an isomorphism of Hopf algebras and let* $\phi' : \mathcal{T}(V) \to A'$ *be a lifting map. If there is no* $v \in V$ *such that* $h \cdot v = \varepsilon(h)v$ *for all* $h \in H$*, then* $\Theta\phi(V) = \phi'(V)$.

Proof. (a) Clearly $\phi(M) \subset A_{[1]}$. Since $\pi_1 \circ \phi_{|M}$ is a morphism of bicomodules over H by Lemma 4.1, $(\pi_1 \circ \phi)(M) \subset V$. (b) is a particular case of (a). We prove (c). Recall that ${}^H_H\mathcal{YD}$ is a semisimple category, so $M = \bigoplus_{l=1}^n M_l$ where each M_l is a simple H-comodule. If M is a simple H-comodule, then (c) follows from [AV1, Lemma 2.1] since

$$\Delta((\pi_0 \circ \phi)(m_i)) = (\pi_0 \circ \phi)(m_i) \otimes 1 + \sum_j c_{ij} \otimes (\pi_0 \circ \phi)(m_j).$$

Otherwise the same argument can be applied on each simple summand M_l.

(d) If we consider $A_{[1]}$ as a right H-comodule via the projection $(\varepsilon\#1)\circ\Theta$, then $\Theta\phi(V) \subset (\Bbbk \oplus \phi'(V))\#1$. Now if we consider $A'_{[1]}$ as a left H-module via $\operatorname{ad}\circ\Theta$, then (d) follows by hypothesis. □

Lemma 4.8 has been refined in the copointed case, *i.e.* when H is the function algebra on a finite group, in [GIV, Lemma 3.1].

Lemma 4.9. *Let* $M, N \subset T(V)$ *be Yetter-Drinfeld submodules.*

(a) *If* M *is included in the homogeneous component of minimum degree of* $\mathcal{J}(V)$*, then* M *is compatible with* ϕ.

(b) *Assume that* M *is compatible with* ϕ*,* $\mathcal{I}_M$ *is a Hopf ideal and*

$$\Delta(n) - n \otimes 1 - n_{(-1)} \otimes n_{(0)} \in \mathcal{I}_M \otimes \mathcal{T}(V) + \mathcal{T}(V) \otimes \mathcal{I}_M \tag{4.2}$$

holds for every $n \in N$*. Then* N *is compatible with* ϕ *and* $\mathcal{I}_{M\oplus N}$ *is a Hopf ideal.*

Proof. (a) By hypothesis, $M \subset \mathcal{P}(T(V))$ and then M is compatible with ϕ.

(b) Since $\mathcal{I}_M \subset \ker\phi$, N is compatible with ϕ by (4.2). Moreover, applying Lemma 4.8 (a) and (c), we see that $\langle m - \phi(m)\rangle_{m\in N}$ is a Hopf ideal in $\mathcal{T}(V)/\mathcal{I}_M$ by (4.2). Hence $\mathcal{I}_{M\oplus N}$ is a Hopf ideal of $\mathcal{T}(V)$. □

Definition 4.10. A *good module of relations* is a graded Yetter-Drinfeld submodule $M = \bigoplus_{i=1}^t M^{n_i} \subset T(V)$ where $M^{n_i} \subseteq T^{n_i}(V)$, with $M^{n_i} \neq 0$ and $n_i < n_{i+1}$ for all i, which generates $\mathcal{J}(V)$ such that the Yetter-Drinfeld submodules $\bigoplus_{i=1}^s M^{n_i}, M^{n_{s+1}} \subset T(V)$ satisfy (4.2) for all $s = 1, \dots, t-1$.

Now we describe the liftings of $\mathcal{B}(V)$ over H when $\mathcal{J}(V)$ is generated by a good module of relations.

Theorem 4.11. *Let* A *be a lifting of* $\mathcal{B}(V)$ *over* H*, with lifting map* ϕ*. Let* M *be a good module of relations for* $\mathcal{B}(V)$*. Then* $A \simeq \mathcal{T}(V)/\mathcal{I}_M$.

Proof. Follows from Proposition 4.6 and Lemma 4.9. □

Theorem 4.11 characterizes the liftings in the case in which the relations are deformed by elements in the first term of the coradical filtration. This is the case in [AS3, AG2, AV1, GGI, FG] (actually in those papers, the relations are deformed by elements in the zeroth term of the coradical filtration). However, there exist examples in which this does not hold, see Example 4.12 below, also [He, GIV].

Example 4.12. [GIV, Theorem 5.4] Set $\Bbbk = \mathbb{C}$ and let $\mathbb{F}_5$ denote the finite field of 5 elements. Consider the *affine rack* $X = (\mathbb{F}_5, 2)$ and the constant cocycle $q \equiv -1$. The Nichols algebra $\mathcal{B}(X, q)$, computed in [AG1], has dimension 1280 and can be presented by generators $x_0, \dots, x_4$ and relations

$$\begin{aligned} &x_i^2, \qquad x_i x_j + x_{2j-i} x_i + x_{3i-2j} x_{2j-i} + x_j x_{3i-2j} \qquad 0 \le i, j \le 4, \\ &x_1 x_0 x_1 x_0 + x_0 x_1 x_0 x_1. \end{aligned} \tag{4.3}$$

Let C_8 be the cyclic group of order 8 and let t denote a generator. Consider C_8 acting on $\mathbb{Z}_5$ by $t \cdot i = 2i$, $i \in \mathbb{Z}_5$, and set $\Gamma = \mathbb{Z}_5 \rtimes_2 C_8$. Then $\mathcal{B}(X, q)$ can be realized in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$. Let $\mathcal{H} = \mathcal{B}(X, q)\#\Bbbk\Gamma$. Set $g_i = i \times t \in \Gamma$, $i \in \mathbb{Z}_5$. Let V be the linear span of $\{x_0, \dots, x_4\}$. If L is a deformation of $\mathcal{B}(X, q)$, then there exist scalars $\lambda_1, \lambda_2, \lambda_3 \in \Bbbk$ such that L is the quotient of $T(V)\#\Bbbk\Gamma$ by the ideal generated by

$$\begin{aligned} &x_0^2 - \lambda_1(1 - g_0^2), \qquad x_0 x_1 + x_2 x_0 + x_3 x_2 + x_1 x_3 - \lambda_2(1 - g_0 g_1), \\ &x_1 x_0 x_1 x_0 + x_0 x_1 x_0 x_1 - s_X - \lambda_3(1 - g_0^2 g_1 g_2), \end{aligned}$$

for $s_X = \lambda_2 (x_1 x_0 + x_0 x_1) + \lambda_1 g_1^2 (x_3 x_0 + x_2 x_3) - \lambda_1 g_0^2 (x_2 x_4 + x_1 x_2) + \lambda_2 \lambda_1 g_0^2 (1 - g_1 g_2) \in L_{[2]}$. Hence, the relation $x_1 x_0 x_1 x_0 + x_0 x_1 x_0 x_1$ of $\mathcal{B}(X, q)$ is not deformed by elements in the first term of the coradical filtration.

4.1. **The shape of the Hopf algebra $L(A, \mathcal{H})$.** Let H, V be as in (1.1), (1.2) and $\mathcal{B}$ be a pre-Nichols algebra over V. We set $\mathcal{H} = \mathcal{B}\#H$ and let $\pi : \mathcal{T}(V) \to \mathcal{H}$ be the canonical projection. Consider $\mathcal{T}(V)$ as an $\mathcal{H}$-comodule algebra via $(\mathrm{id} \otimes \pi)\Delta_{\mathcal{T}(V)}$. Let $A \in \mathrm{Gal}(\mathcal{H})$ be provided with a projection $\tau : \mathcal{T}(V) \to A$ which is a morphism of comodule algebras.

We shall need explicit presentations of the Hopf algebra $L(A, \mathcal{H})$.

Proposition 4.13. *Let $\wp = (\tau \otimes \tau)(\mathrm{id} \otimes \mathcal{S})\Delta_{\mathcal{T}(V)} \in \mathrm{Alg}(\mathcal{T}(V), A \otimes A^{\mathrm{op}})$. Then $L(A, \mathcal{H}) = \wp(\mathcal{T}(V))$.*

Proof. On one hand, observe that $(\mathrm{id} \otimes \mathcal{S})\Delta_{\mathcal{T}(V)} : \mathcal{T}(V) \to L(\mathcal{T}(V), \mathcal{T}(V))$ is a Hopf algebra isomorphism. On the other hand, there is a Hopf algebra map $L(\mathcal{T}(V), \mathcal{T}(V)) \to L(A, \mathcal{H})$ making the following diagram commutative:

$$\begin{array}{ccc} L(\mathcal{T}(V), \mathcal{T}(V)) & \hookrightarrow & \mathcal{T}(V) \otimes \mathcal{T}(V) \\ \downarrow & & \downarrow {\scriptstyle \tau \otimes \tau} \\ L(A, \mathcal{H}) & \hookrightarrow & A \otimes A, \end{array}$$

which gives rise to

$$\begin{array}{ccc} L(\mathcal{T}(V),\mathcal{T}(V))\otimes\mathcal{T}(V) & \xrightarrow{\operatorname{can}^{-1}} & \mathcal{T}(V)\otimes\mathcal{T}(V) \\ \downarrow & & \downarrow{\scriptstyle \tau\otimes\tau} \\ L(A,\mathcal{H})\otimes A & \xrightarrow{\operatorname{can}^{-1}} & A\otimes A. \end{array}$$

Hence $\tau\otimes\tau$ restricts to a surjection $L(\mathcal{T}(V),\mathcal{T}(V)) \twoheadrightarrow L(A,\mathcal{H})$ and the proposition follows. □

4.2. **The graded Hopf algebra associated to a cocycle deformation.** Let H, V be as in (1.1), (1.2) and let $\mathcal{B}$ be a pre-Nichols algebra over V. We set $\mathcal{H} = \mathcal{B}\# H$ and let $\sigma : \mathcal{H}\otimes\mathcal{H} \to \Bbbk$ be a 2-cocycle. Let $\mathfrak{F} = (F_n)_{n\geq 0}$ be the filtration of $\mathcal{H}_\sigma$ induced by the graduation of $\mathcal{H}$. Then $\operatorname{gr}_{\mathfrak{F}} \mathcal{H}_\sigma = \mathcal{H}_\sigma = \mathcal{H}$ as coalgebras. Notice that, if $\mathcal{B}$ is a Nichols algebra, then $\mathfrak{F}$ coincides with the coradical filtration.

The items (a) and (b) of the next proposition are [MO, Theorem 2.7, Corollary 3.4], see also [AFGV, Theorem 3.8].

Proposition 4.14. (a) *There is an isomorphism of graded Hopf algebras* $\operatorname{gr}_{\mathfrak{F}} \mathcal{H}_\sigma \simeq \mathcal{B}'\# H_\sigma$, *for* $\mathcal{B}'$ *a pre-Nichols algebra over* $V' \in {}^{H_\sigma}_{H_\sigma}\mathcal{YD}$. *Here* V' *is the* H*-comodule* V *with action*

$$x \rightharpoonup_\sigma v = \sigma(x_{(1)}, v_{(-1)})\,(x_{(2)} \rightharpoonup v_{(0)})_{(0)}\,\sigma^{-1}((x_{(2)} \rightharpoonup v_{(0)})_{(-1)}, x_{(3)}) \tag{4.4}$$

for $x \in H_\sigma$, $v \in V$; *here* V *is identified with a subspace of* $T(V)\# H$, *with* H*-action given by the adjoint. Furthermore, the product in* $\mathcal{B}'$ *is given by* $x\cdot y = \sigma(x_{(-1)}, y_{(-1)})x_{(0)}y_{(0)}$, *for* $x, y \in \mathcal{B}'$ *homogeneous.*

(b) *With the notation in* (a), *if* $\mathcal{B} = \mathcal{B}(V)$ *is the Nichols algebra of* V, *then* $\mathcal{B}' = \mathcal{B}(V')$.

(c) *Let* $\mathcal{A} \in \operatorname{Cleft}(\mathcal{H})$ *with section* $\gamma : \mathcal{H} \to \mathcal{A}$ *and consider the induced cocycle* $\sigma(x\otimes y) = \gamma(x_{(1)})\gamma(y_{(1)})\gamma^{-1}(x_{(2)}y_{(2)})$, $x, y \in \mathcal{H}$, *see* (2.7). *Assume* $\gamma_{|H} \in \operatorname{Alg}(H, \mathcal{A})$. *Then* $\operatorname{gr}_{\mathfrak{F}} \mathcal{H}_\sigma \simeq \mathcal{B}\# H$.

(d) *Suppose that* $H = \Bbbk G$, G *a finite group. In particular,* $H_\sigma = H$. *Let* $\{x_1, \dots, x_\theta\}$ *be a basis of* V *with* $x_i \in V^{g_i}$, $g_i \in G$, $1 \leq i \leq \theta$. *If*

$$\sigma(g, g_i) = \sigma(g g_i g^{-1}, g), \quad g \in G,\ 1 \leq i \leq \theta, \tag{4.5}$$

then $V' = V \in {}^H_H\mathcal{YD}$.

Proof. (a) By Remarks 2.1 (d), $\operatorname{gr}_{\mathfrak{F}} \mathcal{H}$ is generated by $H_\sigma \oplus (F_1/H_\sigma)$. Then $\operatorname{gr}_{\mathfrak{F}} \mathcal{H}_\sigma \simeq \mathcal{B}'\# H_\sigma$, where $\mathcal{B}'$ is a pre-Nichols algebra over $V' := \mathcal{B}'^1$. Since the comultiplication is unchanged, $V' = V$ as H_σ-comodules and in $\operatorname{gr}_{\mathfrak{F}} \mathcal{H}_\sigma$

$$x \rightharpoonup_\sigma v = x_{(1)} \cdot_\sigma v \cdot_\sigma \mathcal{S}_\sigma(x_{(2)})$$

for all $x \in H_\sigma$, $v \in V'$. Using that $\Delta(v) = v\otimes 1 + v_{(-1)}\otimes v_{(0)}$, (4.4) follows as in the proof of [MO, Theorem 2.7]. Finally, if $x \in \mathcal{B}^n$ and $y \in \mathcal{B}^m$, then $x\cdot_\sigma y = \sigma(x_{(-1)}, y_{(-1)})x_{(0)}y_{(0)}$ plus terms of degree lesser than $m+n$. (b) follows since the coalgebra structure is unchanged. (c) follows since

$\sigma_{|H\otimes H} = \varepsilon \otimes \varepsilon$. For (d), the cocommutativity implies $H_\sigma = H$ and plugging (4.5) into (4.4), we have $V' = V$. □

5. The strategy for computing cocycle deformations

Let H, V be as in (1.1), (1.2). We explain how to compute cocycle deformations of $\mathcal{B}(V)\#H$ that are liftings of $\mathcal{B}(V)$ over H; depending on the context, this may eventually lead to all liftings. We fix a minimal set $\mathcal{G}$ of homogeneous generators of $\mathcal{J}(V)$; $\mathcal{G}$ is finite by assumption.

Assume $\mathcal{L}$ is a cocycle deformation, say $\mathcal{L} = (\mathcal{B}(V)\#H)_\sigma$. We seek for conditions for $\mathcal{L}$ to be a lifting of $\mathcal{B}(V)$ over H. Let $\mathcal{A}$ be a $\mathcal{B}(V)\#H$-Galois object such that $\mathcal{L} \simeq L(\mathcal{A}, \mathcal{B}(V)\#H)$. By Proposition 4.14 (b), one has $\operatorname{gr}\mathcal{L} \simeq \mathcal{B}(V')\#H_\sigma$, $V' \in {}^{H_\sigma}_{H_\sigma}\mathcal{YD}$. If $\sigma_{|H\otimes H} \overset{*}{=} \varepsilon \otimes \varepsilon$, then $\operatorname{gr}\mathcal{L} \simeq \mathcal{B}(V)\#H$ by (2.5) and (4.4). Now the equality $*$ is achieved when the object $\mathcal{A}$ is cleft with a section $\gamma : \mathcal{B}(V)\#H \to \mathcal{A}$ that satisfies $\gamma_{|H} \in \operatorname{Alg}(H, \mathcal{A})$, by Proposition 4.14 (c). In conclusion, we look for cleft objects $\mathcal{A}$ with a section $\gamma : \mathcal{B}(V)\#H \to \mathcal{A}$ satisfying this property.

5.1. Adapted stratifications.

5.1. Adapted stratifications. A stratification of $\mathcal{G}$ is a decomposition as a disjoint union $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \dots \cup \mathcal{G}_N$. For $0 \le k \le N$, we set

$$\mathcal{B}_0 := T(V), \qquad \mathcal{H}_0 = T(V)\#H = \mathcal{T}(V),$$
$$\mathcal{B}_k := T(V)/\langle \mathcal{G}_0 \cup \mathcal{G}_1 \cup \dots \cup \mathcal{G}_{k-1}\rangle, \qquad \mathcal{H}_k = \mathcal{B}_k\#H.$$

Clearly $\mathcal{H}_{N+1} = \mathcal{B}(V)\#H$. Let $\pi_k : \mathcal{T}(V) \to \mathcal{H}_k$ be the canonical projection. If $k < N$, then $\mathcal{G}_k$ identifies with its image in $\mathcal{B}_k$, by minimality of $\mathcal{G}$.

We say that the stratification $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \dots \cup \mathcal{G}_N$ is *adapted* when it satisfies the following properties:

(1) *$\mathcal{G}_k$ is a basis of a Yetter-Drinfeld submodule of $\mathcal{P}(\mathcal{B}_k)$*; then $\langle \mathcal{G}_k\rangle$ is a Hopf ideal of $\mathcal{B}_k$, $\langle \mathcal{G}_k\rangle\#H$ is a Hopf ideal of $\mathcal{H}_k$ and therefore $\mathcal{H}_{k+1} \simeq \mathcal{H}_k/\langle \mathcal{G}_k\rangle\#H$ is a Hopf algebra.
(2) *$\mathcal{G}_N$ is a basis of a Yetter-Drinfeld submodule of $\mathcal{B}_N$ and $\Bbbk\langle \mathcal{G}_N\rangle$ is a left coideal subalgebra of $\mathcal{B}_N$, but not necessarily $\mathcal{G}_N \subset \mathcal{P}(\mathcal{B}_N)$.*

Examples 5.1. (a) A standard choice is to take $\mathcal{G}_k$ such that (the image of) $\mathcal{G}_k$ is a basis of the subspace of $\mathcal{P}(\mathcal{B}_k)$ generated by all its homogeneous elements of degree ≥ 2, for all k.
(b) Assume that $H = \Bbbk\Gamma$, Γ a finite abelian group, or more generally that V is a direct sum of one-dimensional Yetter-Drinfeld modules. We may choose an adapted stratification with $\operatorname{card}\mathcal{G}_j = 1$ for all j.

It is not always possible to choose a stratification $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \dots \cup \mathcal{G}_N$ in which $\operatorname{card}\mathcal{G}_j = 1$ for each j, see the next example.

Example 5.2. Keep the notation in Example 4.12. The adapted stratification of $\mathcal{B}(X,q)$ considered in [GIV, Theorem 5.4] is:

$$\mathcal{G}_0 = \{x_i^2 : i \in \mathbb{F}_5\},$$
$$\mathcal{G}_1 = \{x_ix_j + x_{2j-i}x_i + x_{3i-2j}x_{2j-i} + x_jx_{3i-2j} : i,j \in \mathbb{F}_5\},$$
$$\mathcal{G}_2 = \{x_1x_0x_1x_0 + x_0x_1x_0x_1\}.$$

The Yetter-Drinfeld submodule of $\mathcal{B}_k$ generated by $\mathcal{G}_k$ is simple, $k = 0,1,2$.

5.2. **The strategy.** Fix $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \dots \cup \mathcal{G}_N$ an adapted stratification. The strategy consists of the following steps.

(1) *Cleft extensions of bosonizations of pre-Nichols algebras.* We shall construct recursively a set $\Lambda_k \subset \operatorname{Cleft}(\mathcal{H}_k)$, for all $k = 0, \dots, N+1$. We start with $\mathcal{H}_0 = \mathcal{T}(V) = T(V)\#H$ and the cleft object $\mathcal{A}_0 = \mathcal{T}(V)$ with section $\gamma_0 = \mathrm{id}$. Set $\Lambda_0 = \big\{\mathcal{T}(V)\big\} \subset \operatorname{Cleft}(\mathcal{H}_0)$.

The recursive step is done in one of the following ways, for $k = 0, \dots, N$:

(1a) We compute the algebra of left coinvariants $X_k := {}^{\operatorname{co}\mathcal{H}_{k+1}}\mathcal{H}_k$. Then we compute $\operatorname{Alg}^{\mathcal{H}_k}_{\mathcal{H}_k}(X_k, \mathcal{A}_k)$, for each $\mathcal{A}_k \in \Lambda_k$. *For each* $\psi \in \operatorname{Alg}^{\mathcal{H}_k}_{\mathcal{H}_k}(X_k, \mathcal{A}_k)$, *we collect* $\mathcal{A}_k/\mathcal{A}_k\psi(X_k^+)$ *in* Λ_{k+1}.

Theorem 3.2 may be useful to deal with the computation of right coideal subalgebras in this step. However, it usually happens that the computation of X_k, and *a fortiori* that of $\operatorname{Alg}^{\mathcal{H}_k}_{\mathcal{H}_k}(X_k, \mathcal{A}_k)$, is too hard. In such case, we take an alternative route.

(1b) We consider the subalgebra $Y_k := \Bbbk\langle \mathcal{S}(\mathcal{G}_k)\rangle = \mathcal{S}(\Bbbk\langle \mathcal{G}_k\rangle)$ of $\mathcal{H}_k$. Since $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \dots \cup \mathcal{G}_N$ is an adapted stratification, $\Bbbk\langle \mathcal{G}_k\rangle$ is a left coideal subalgebra of $\mathcal{H}_k$; hence Y_k is a right coideal subalgebra of $\mathcal{H}_k$. Also,

$$\mathcal{H}_{k+1} = \mathcal{H}_k/\langle Y_k^+\rangle.$$

We then compute $\operatorname{Alg}^{\mathcal{H}_k}(Y_k, \mathcal{A}_k)$, for each $\mathcal{A}_k \in \Lambda_k$. *We collect* $\mathcal{A}_k/\langle \varphi(Y_k^+)\rangle$ *in* Λ_{k+1}, *for each* $\varphi \in \operatorname{Alg}^{\mathcal{H}_k}(Y_k, \mathcal{A}_k)$ *with*

$$\langle \varphi(Y_k^+)\rangle \neq \mathcal{A}_k. \tag{5.1}$$

The alternative (1a) has the advantage to the alternative (1b) to avoid the checking of (5.1). Note that

- $\Lambda_k \subset \operatorname{Cleft}(\mathcal{H}_k)$ by Theorem 3.1 in (1a) or Theorem 3.3 in (1b).

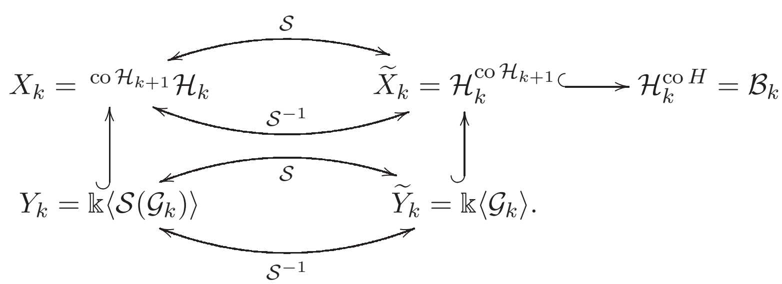
Indeed, this holds for $k = 0$ and a recursive argument applies since the coradical of the successive quotients remains unchanged. We can apply Theorem 3.1 because $\mathcal{H}_k$ is $\mathcal{H}_{k+1}$-coflat, by Corollary 3.7. This also implies that $\mathcal{H}_k$ is faithfully flat over X_k, by Theorem 3.2. As $X_k = N(Y_k)$, see Remark 5.4, we can also apply Theorem 3.3.

(2) *Deformations of pre-Nichols algebras.* We next compute the Hopf algebras $L(\mathcal{A}_k, \mathcal{H}_k)$, for $\mathcal{A}_k \in \Lambda_k$, $0 < k \leq N+1$. These are new examples of Hopf algebras; they are quotients of $\mathcal{T}(V)$ by Propositions 4.13 and 5.8, which can be computed using Proposition 5.10 and Corollary 5.12.

(3) *Exhaustion.* The Hopf algebras $L(\mathcal{A}_{N+1}, \mathcal{H}_{N+1})$ for $\mathcal{A}_{N+1} \in \Lambda_{N+1}$ are liftings of $\mathcal{B}(V)$ over H, by Proposition 4.14 (b) and (c). We need to check whether this family of Hopf algebras is an exhaustive list of liftings of $\mathcal{B}(V)$ over H; for this Theorem 4.11 might apply under suitable conditions.

Remark 5.3. A similar strategy is already proposed by Günther in [Gu, page 4399] to compute the cleft objects of a pointed Hopf algebra $\overline{H}$ which is a quotient of a pointed Hopf algebra H for which $\operatorname{Cleft}(H)$ is known. He suggests to *choose an easy decomposition* $H = H_1 \twoheadrightarrow H_2 \twoheadrightarrow \cdots \twoheadrightarrow H_n = \overline{H}$ in such a way that $\operatorname{Cleft}(H_{i+1})$ is *easily computable* from $\operatorname{Cleft}(H_i)$ using [Gu, Theorems 4 & 8]. He does not, however, investigates how to find that decomposition or when the method applies, nor relates this process with the lifting procedure or the classification problem.

5.3. **Comments on X_k and Y_k.** Let $\widetilde{X}_k = \mathcal{H}_k^{\operatorname{co}\mathcal{H}_{k+1}}$, $\widetilde{Y}_k = \Bbbk\langle\mathcal{G}_k\rangle$. The next picture describes the relation between these subalgebras and the subalgebras of $\mathcal{H}_k$ which are involved in the steps (1a) and (1b).



Indeed, $\mathcal{S}^2(\mathcal{G}_k) = \mathcal{G}_k$, for $k < N$. First, if $x \in \mathcal{P}(\mathcal{B}_k)$, then $\Delta_{\mathcal{H}_k}(x) = x \otimes 1 + x_{(-1)} \otimes x_{(0)}$, hence $\mathcal{S}(x) = -\mathcal{S}(x_{(-1)})x_{(0)}$ and $\mathcal{S}^2(x) = \operatorname{ad}\mathcal{S}(x_{(-1)})(x_{(0)})$. So, $\mathcal{G}_k$, being a Yetter-Drinfeld submodule of $\mathcal{P}(\mathcal{B}_k)$, is stable under $\mathcal{S}^2$.

Remark 5.4. $X_k = N(Y_k)$, cf. page 7.

Indeed, let B be the subalgebra generated by $h_{(1)}y\mathcal{S}(h_{(2)})$, $h \in \mathcal{H}_k$, $y \in \widetilde{Y}_k$. By Corollary 3.8, $\mathcal{H}_k$ is right $N(Y_k)$-faithfully flat. Now we invoke Theorem 3.2: $\mathcal{I}(N(Y_k)) = \langle\mathcal{G}_k\rangle$; but $\mathcal{X}(\langle\mathcal{G}_k\rangle) = X_k$, as $\mathcal{H}_k$ is $\mathcal{H}_{k+1} \simeq \mathcal{H}_k/\langle\mathcal{G}_k\rangle\#H$-coflat by Corollary 3.7.

As said, the computation of the algebra of coinvariants X_k might be hard; a potentially easier instance is when $X_k = Y_k$. We analyze when this could happen in the following Remark.

Remark 5.5. The following are equivalent:

(1) $X_k = Y_k$;
(2) Y_k is normal;
(3) For all $y \in \mathcal{G}_k$, $x \in V$,

$$\operatorname{ad}_r(x)(\mathcal{S}(y)) = \mathcal{S}(x_{(-1)})\mathcal{S}(y)x_{(0)} - \mathcal{S}(x_{(-1)})x_{(0)}\mathcal{S}(y) \in Y_k; \tag{5.2}$$

(4) $x\mathcal{S}(y) - \mathcal{S}(y)x \in Y_k$, for all $y \in \mathcal{G}_k$, $x \in V$.

Proof. (1) $\Rightarrow$ (2): X_k is normal. (2) $\Rightarrow$ (1): $Y_k = N(Y_k) = X_k$ by Remark 5.4. Clearly, (2) $\Rightarrow$ (3). (3) $\Rightarrow$ (2): We have to prove that $\mathrm{ad}_r(x)(z) \in Y_k$ for all $x \in \mathcal{H}_k$, $z \in Y_k$. Since $\mathrm{ad}_r(x)(zz') = \mathrm{ad}_r(x_{(1)})(z)\,\mathrm{ad}_r(x_{(2)})(z')$, it is enough to consider $z \in \mathcal{S}(\mathcal{G}_k)$; since $\mathrm{ad}_r(xx')(z) = \mathrm{ad}_r(x')\,\mathrm{ad}_r(x)(z)$, it is enough to consider $x \in H$ or $x \in V$. If $x \in H$, $u = \mathcal{S}^{-1}(x)$ and $y \in \mathcal{G}_k$, then $\mathrm{ad}_r(x)(\mathcal{S}(y)) = \mathcal{S}(\mathrm{ad}_\ell(u)(y)) = \mathcal{S}(u \cdot y) \in \mathcal{S}(\mathcal{G}_k)$. It only remains the case $z \in \mathcal{S}(\mathcal{G}_k)$ and $x \in V$, which is (5.2).

(3) $\Rightarrow$ (4): $x\mathcal{S}(y) = x_{(-2)}\mathcal{S}(x_{(-1)})x_{(0)}\mathcal{S}(y) \overset{(5.2)}{=} x_{(-2)}\mathcal{S}(x_{(-1)})\mathcal{S}(y)x_{(0)} + Y_k = \mathcal{S}(y)x + Y_k$. The converse implication is similar. □

5.4. **Properties of $\mathcal{A}_{k+1}$.** Fix $k \geq 0$ and $\mathcal{A}_{k+1} \in \Lambda_{k+1}$ which is a quotient of $\mathcal{A}_k \in \Lambda_k$. We collect some information about the algebra $\mathcal{A}_{k+1}$. We start with some general considerations.

Remarks 5.6. Let H be a Hopf algebra and $\mathcal{B}$ be a Hopf algebra in ${}^H_H\mathcal{YD}$. Set $\mathcal{H} = \mathcal{B}\#H$ with projection and inclusion maps $\mathcal{H} \underset{\iota}{\overset{\pi}{\rightleftarrows}} H$. Let $\mathcal{A} \in \mathrm{Cleft}(\mathcal{H})$ with section $\gamma : \mathcal{H} \to \mathcal{A}$; assume that $\gamma_{|H} \in \mathrm{Alg}(H, \mathcal{A})$. Both $\mathcal{H}$ and $\mathcal{A}$ are H-comodules via π. Then

(a) $\mathcal{A}$ is a cleft extension of H. Moreover $\mathcal{A} \simeq \mathcal{E}\#H$, where $\mathcal{E} = \mathcal{A}^{\mathrm{co}\,H}$, and $p : \mathcal{A} \to \mathcal{E}$, $p(x) = x_{(0)}\gamma^{-1}\iota\pi(x_{(-1)})$ is an H-module projection.
(b) Let $S \subseteq \mathcal{E}$ be an H-submodule. Then $\langle S\rangle_{\mathcal{A}} = \langle S\rangle_{\mathcal{E}}\#H$ and consequently $\mathcal{A}/\langle S\rangle_{\mathcal{A}} \simeq (\mathcal{E}/\langle S\rangle_{\mathcal{E}})\#H$.
(c) Let $I \subset \mathcal{A}$ be an ideal and H-subcomodule. Then $I = I^{\mathrm{co}\,H}\#H$.
(d) Let $S \subset \mathcal{A}$ be an H-submodule and H-subcomodule. Then $\langle S\rangle_{\mathcal{A}} = \langle p(S)\rangle_{\mathcal{E}}\#H$.

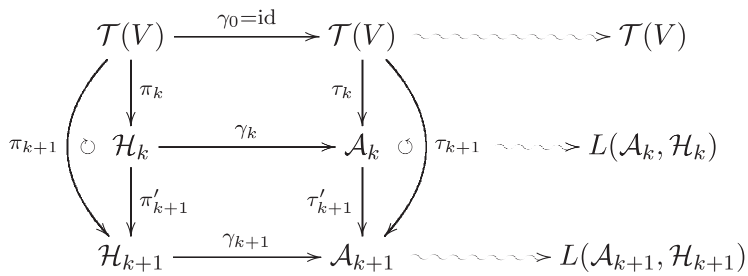
Proof. (a) Clearly, γ is an H-colinear map and thus $\gamma\iota : H \to \mathcal{A}$ is a section. Hence $\mathcal{A} \simeq \mathcal{E}\#_\sigma H$ and $\sigma = \varepsilon$ since $\gamma_{|H} \in \mathrm{Alg}(H, \mathcal{A})$, cf. (2.7). Last sentence is [Mo, Lemma 7.2.6]. (b) is easy. (c) $\langle I^{\mathrm{co}\,H}\rangle_{\mathcal{A}} = \langle I^{\mathrm{co}\,H}\rangle_{\mathcal{E}}\#H \subset I$ by (b). If $x \in I$, then $p(x) \in I^{\mathrm{co}\,H}$ and thus $x = p(x_{(0)})\gamma\iota(\pi(x_{(1)})) \in \langle I^{\mathrm{co}\,H}\rangle_{\mathcal{A}}$. Finally $\langle I^{\mathrm{co}\,H}\rangle_{\mathcal{E}} = I^{\mathrm{co}\,H}$. (d) By hypothesis, $p(S)$ is an H-submodule. Then $S \subseteq \langle p(S)\rangle_{\mathcal{A}} = \langle p(S)\rangle_{\mathcal{E}}\#H \subseteq \langle S\rangle_{\mathcal{A}}$, by (b), and the equality holds. □

Lemma 5.7. *Let $H \subset \mathcal{H}$ be Hopf algebras where H is finite-dimensional and semisimple. If $\mathcal{A} \in \mathrm{Cleft}(\mathcal{H})$, then $\mathcal{A}$ is an injective object in $\mathcal{YD}^{\mathcal{H}}_H$.*

Proof. Recall that $\mathcal{YD}^{\mathcal{H}}_H \simeq \mathcal{M}^L$, with $L = \mathcal{H} \bowtie_\tau H^{*\,\mathrm{cop}}$, see [M, Exercise 7.2.16]. Here $\tau = \sum_i \mathcal{S}(e^i) \otimes e_i \in H^{*\,\mathrm{cop}} \otimes \mathcal{H}$, for dual bases $\{e_i\}$, $\{e^i\}$ of H and H^*. In particular, $L \simeq \mathcal{H} \otimes H^{*\,\mathrm{cop}}$ as algebras and there is a Hopf algebra projection $L \twoheadrightarrow \mathcal{H}$, see *loc. cit.* for details.

As $\mathcal{A}$ is cleft, it is $\mathcal{H}$-coflat. As $L \twoheadrightarrow \mathcal{H}$ is a cosemisimple coextension, then $\mathcal{A}$ is also L-coflat and the lemma follows. □

The following is a snapshot of the Strategy:



Here π_k, π_{k+1} and π'_k are the natural projections of Hopf algebras and τ'_k is the natural projection of algebras which is also $\mathcal{H}_{k+1}$-colinear via π'_{k+1}. The epimorphisms $\tau_k : \mathcal{T}(V) \to \mathcal{A}_k$ are defined recursively as follows: If $k = 1$, we take $\tau_1 = \tau'_1$. Given τ_k, we set $\tau_{k+1} = \tau'_{k+1}\tau_k$. Notice that each τ_k is a morphism of right $\mathcal{H}_k$-comodule algebras.

The sections γ_k, γ_{k+1} are introduced in the next proposition.

Proposition 5.8. (a) *The Miyashita-Ulbrich action* (3.1) *on* $\mathcal{A}_k$ *is:*

$$a \leftharpoonup \pi_k(x) = \tau_k(\mathcal{S}(x_{(1)}))a\tau_k(x_{(2)}), \qquad a \in \mathcal{A}_k,\ x \in \mathcal{T}(V). \tag{5.3}$$

(b) *There is a section* $\gamma_k : \mathcal{H}_k \to \mathcal{A}_k$ *such that* $\gamma_{k|H} \in \mathrm{Alg}(H, \mathcal{A}_k)$, $\gamma_{k|H} = \tau_{k|H}$ *and* $\gamma_k(xh) = \gamma_k(x)\gamma_k(h)$ *for all* $x \in \mathcal{B}_k$, $h \in H$.

(c) *If* H *is semisimple, then the above* $\gamma_k : \mathcal{H}_k \to \mathcal{A}_k$ *is a morphism of right* H*-modules and* $\gamma_k(hx) = \gamma_k(h)\gamma_k(x)$ *for all* $x \in \mathcal{B}_k$, $h \in H$.

(d) $\mathcal{A}_k$ *is a cleft extension of* H, $\mathcal{A}_k \simeq \mathcal{E}_k \# H$, *where* $\mathcal{E}_k = \mathcal{A}_k^{\mathrm{co}\,H}$ *and*

$$\tau_k(T(V)\#1) = \gamma_k(\mathcal{B}_k\#1) = \tau'_k(\mathcal{E}_{k-1}\#1) = \mathcal{E}_k\#1.$$

In particular, $\mathcal{E}_k \simeq \mathcal{E}_{k-1}/(\ker \tau'_k)^{\mathrm{co}\,H}$.

Proof. (a) Let $x \in \mathcal{T}(V)$. We compute

$$\begin{aligned}\mathrm{can}(\tau_k(\mathcal{S}(x_{(1)})) \otimes \tau_k(x_{(2)})) &= \tau_k(\mathcal{S}(x_{(1)}))\tau_k(x_{(2)})_{(0)} \otimes \tau_k(x_{(2)})_{(1)} \\ &= \tau_k(\mathcal{S}(x_{(1)}))\tau_k(x_{(2)}) \otimes \pi_k(x_{(3)}) = 1 \otimes \pi_k(x).\end{aligned}$$

Then $\mathrm{can}^{-1}(1 \otimes \pi_k(x)) = \tau_k(\mathcal{S}(x_{(1)})) \otimes \tau_k(x_{(2)})$ and (5.3) follows by (3.1).

(b) First, we may choose $\gamma_0 = \mathrm{id}$, then the statement holds trivially for $k = 0$. We now proceed by induction, assume it holds for k. The inductive step follows as [Sc, Theorem 4.2] in this setting: Notice that $\mathcal{A}_k$ is an injective $\mathcal{H}_{k+1}$-comodule, since $\mathcal{H}_k$ is $\mathcal{H}_{k+1}$-coflat and $\mathcal{A}_k$ is $\mathcal{H}_k$-coflat. Thus, as $\gamma_{k|H} : H \to \mathcal{A}_k$ is $\mathcal{H}_{k+1}$-colinear, there exists a $\mathcal{H}_{k+1}$-colinear map $\omega : \mathcal{H}_{k+1} \to \mathcal{A}_k$ such that $\omega_{|H} = \gamma_{k|H}$. By [T1, Lemma 14] ω is convolution-invertible, since its restriction to H is. Also $\tau'_{k+1}\,\omega_{|H} \in \mathrm{Alg}(H, \mathcal{A}_{k+1})$. Then the section $\gamma_{k+1} : \mathcal{H}_{k+1} \to \mathcal{A}_{k+1}$ is defined by

$$xh \longmapsto \tau'_{k+1}\,\omega(x)\,\tau'_{k+1}\,\omega(h), \qquad x \in \mathcal{B}_{k+1},\ h \in H.$$

Note that $\gamma_{k+1}(h) = \tau'_{k+1}\,\omega(h) = \tau'_{k+1}\gamma_k(h) = \tau'_{k+1}\tau_k(h) = \tau_{k+1}(h)$, $h \in H$.

(c) If H is semisimple, then $\mathcal{A}_k$ is injective in ${}\mathcal{YD}_H^{\mathcal{H}_k}$ by Lemma 5.7. The proof of (b) *mutatis mutandis* shows the first claim. If $x \in \mathcal{B}_{k+1}$, $h \in H$, then $\gamma_{k+1}(hx) = \gamma_{k+1}(\mathrm{ad}_r(\mathcal{S}^{-1}(h_{(2)}))(v)h_{(1)}) = \gamma_{k+1}(h)\gamma_{k+1}(x)$.

(d) By Remark 5.6 (a), $\mathcal{A}_k$ is H-cleft and $\mathcal{A}_k \simeq \mathcal{E}_k \# H$. As $\mathcal{B}_k\#1 = \mathcal{H}_k^{\mathrm{co}\,H}$, $T(V)\#1 = \mathcal{T}(V)^{\mathrm{co}\,H}$, $\mathcal{E}_{k-1}\#1 = \mathcal{A}_{k-1}^{\mathrm{co}\,H}$, we have $\tau_k(T(V)\#1)$, $\gamma_k(\mathcal{B}_k\#1)$, $\tau'_k(\mathcal{E}_{k-1}\#1) \subseteq \mathcal{E}_k\#1$ since all γ_k, τ_k and τ'_k are H-colinear. The equality and the last assertion of (d) follow from Remark 5.6 (c). $\square$

We fix the following setting:

- We denote by $u_i \in \mathcal{P}(\mathcal{B}_k)$, $1 \le i \le n$, the elements of $\mathcal{G}_k$. We set $v_i = \mathcal{S}(u_i) \in Y_k$, $1 \le i \le n$. Let U, resp. W, be the linear span of $\{u_i\}_{1\le i\le n}$, resp. $\{v_i\}_{1\le i\le n}$. Notice that $U \in {}^H_H\mathcal{YD}$, $W \in \mathcal{YD}^H_H$.
- Let $\{e_{ij}\}_{1\le i,j\le n} \subset H$ be the set of comatrix elements associated to U and $\{u_i\}_{1\le i\le n}$, see (4.1). Then
$$\Delta(v_i) = \sum_{j=1}^{n} v_j \otimes \mathcal{S}(e_{ij}) + 1 \otimes v_i, \quad 1 \le i \le n.$$
- We set C the subcoalgebra of H generated by $\{\mathcal{S}(e_{ij})\}_{1\le i,j\le n}$.
- We fix a section $\gamma_k : \mathcal{H}_k \to \mathcal{A}_k$ such that $\gamma_k(xh) = \gamma_k(x)h$, $x \in \mathcal{B}_k$, $h \in H$, see Proposition 5.8. If H is semisimple we assume moreover that γ_k is H-linear.

By Proposition 5.8 (b) we can identify H with $\tau_k(H) = \gamma_k(H) \subset \mathcal{A}_k$.

Lemma 5.9. (a) *Let $\varphi \in \mathrm{Alg}^{\mathcal{H}_k}(Y_k, \mathcal{A}_k)$. There are $\{c_i\}_{1\le i\le n} \subset \Bbbk$ with*

$$\varphi(v_i) = \gamma_k(v_i) + \sum_{j=1}^{n} c_j\, \mathcal{S}(e_{ij}). \tag{5.4}$$

(b) *If H is semisimple and φ is H-linear, then $(\varphi - \gamma_k)_{|W} : W \to \mathcal{A}_k$ is a morphism in $\mathcal{YD}^H_H$ whose image is contained in C.*

Proof. (a) As H is cosemisimple, $U = \bigoplus_l M_l$ where each M_l is a simple H-comodule. We can assume that each u_i belongs to some M_l, up to changing the basis of U. For each i, we restrict to the subcomodule M_l with $u_i \in M_l$ and consider the corresponding comatrix elements $\{e_{ij}\}_{i,j}$; they are linearly independent. To simplify the notation, assume $U = M_l$ is simple.

Set $b_i = \varphi(v_i) - \gamma_k(v_i)$ for $1 \le i \le n$. Since φ and γ_k are $\mathcal{H}_k$-colinear, $\rho(b_i) = \sum_{j=1}^n b_j \otimes \mathcal{S}(e_{ij})$. Then $\{b_1, \dots, b_n\} \subset H$ since H is the socle of $\mathcal{A}_k$. Moreover, $\{b_1, \dots, b_n\} \subset C$. Now, we write $b_\ell = \sum_{i,j=1}^n c_{ij}^\ell \mathcal{S}(e_{ij})$ for all $1 \le \ell \le n$, where $c_{ij}^\ell \in \Bbbk$. We have

$$\Delta(b_\ell) = \sum_{i,j} c_{ij}^\ell \Delta(\mathcal{S}(e_{ij})) = \sum_{i,j,s} c_{ij}^\ell\, \mathcal{S}(e_{sj}) \otimes \mathcal{S}(e_{is}),$$
$$\rho(b_\ell) = \sum_{s} b_s \otimes \mathcal{S}(e_{\ell s}) = \sum_{i,j,s} c_{ij}^s\, \mathcal{S}(e_{ij}) \otimes \mathcal{S}(e_{\ell s}).$$

Recall that $\Delta(b_\ell) = \rho(b_\ell)$ since $b_\ell \in H$, then $c^\ell_{ij} = 0$, if $\ell \neq i$, and hence $c^\ell_{\ell j} = c^s_{sj}$ for all $1 \leq \ell, s \leq n$. Therefore we set $c_j = c^\ell_{\ell j}$ for each $1 \leq j \leq n$. (b) follows from Proposition 5.8 (c). □

5.5. **The shape of $L(\mathcal{A}_{k+1}, \mathcal{H}_{k+1})$.** We keep the setting above and also:

- We fix $\varphi \in \mathrm{Alg}^{\mathcal{H}_k}(Y_k, \mathcal{A}_k)$ with $c_j \in \Bbbk$, $1 \leq j \leq n$, as in (5.4).
- We assume that $\mathcal{A}_{k+1} = \mathcal{A}_k / \langle \varphi(v_i) \rangle_{1 \leq i \leq n} \neq 0$, then $\mathcal{A}_{k+1} \in \Lambda_{k+1}$.
- We fix a Hopf algebra $\mathcal{L}_k$ such that $\mathcal{A}_k$ is a $(\mathcal{L}_k, \mathcal{H}_k)$-biGalois object. Let $\vartheta : L(\mathcal{A}_k, \mathcal{H}_k) \to \mathcal{L}_k$ be the isomorphism in (2.11).
- We identify $H \hookrightarrow \mathcal{L}_k$ as a Hopf subalgebra via $\vartheta(\mathrm{id} \otimes \mathcal{S})\Delta$ since $\tau_{k|H} = \gamma_{k|H} = \mathrm{id}_H$.

We now describe $L(\mathcal{A}_{k+1}, \mathcal{H}_{k+1})$ as a quotient of $\mathcal{L}_k$.

Proposition 5.10. $L(\mathcal{A}_{k+1}, \mathcal{H}_{k+1}) \simeq \mathcal{L}_k / \langle \widetilde{v}_i - c_i + \sum_{j=1}^n c_j \mathcal{S}(e_{ij}) \rangle_{1 \leq i \leq n}$ *where* $\widetilde{v}_i \in \mathcal{L}_k$, $1 \leq i \leq n$, *is such that*

$$\widetilde{v}_i \otimes 1_{\mathcal{A}_k} = \sum_t \gamma_k(v_t)_{(-1)} \otimes \gamma_k(v_t)_{(0)} \gamma_k^{-1} \mathcal{S}(e_{it}) + 1 \otimes \gamma_k^{-1}(v_i). \tag{5.5}$$

Proof. We may assume $\mathcal{L}_k = L(\mathcal{A}_k, \mathcal{H}_k) \subset \mathcal{A}_k \otimes \mathcal{A}_k$. The general case follows by applying ϑ. Set

$$\overline{v}_i = (\gamma_k \otimes \gamma_k^{-1})\Delta(v_i), \qquad E_{ij} = (\gamma_k \otimes \gamma_k^{-1})\Delta(\mathcal{S}(e_{ji}))$$

for all $1 \leq i, j \leq n$. Let $J = \left\langle \overline{v}_i - c_i + \sum_{j=1}^n c_j E_{ji} \right\rangle_{1 \leq i \leq n}$, notice that this is a Hopf ideal since $(\gamma_k \otimes \gamma_k^{-1})\Delta$ is an (injective) coalgebra map by (2.8) and (2.9). Set $\mathcal{L}_{k+1} = L(\mathcal{A}_k, \mathcal{H}_k)/J$. We have to show that $L(\mathcal{A}_{k+1}, \mathcal{H}_{k+1}) \simeq \mathcal{L}_{k+1}$. By [S1, Theorem 3.5], it suffices to prove the following statement.

Claim. $\mathcal{A}_{k+1}$ is a $(\mathcal{L}_{k+1}, \mathcal{H}_{k+1})$-biGalois object.

Set $I = \langle \varphi(v_i) \rangle_{1 \leq i \leq n} \subset \mathcal{A}_k$. Let $\lambda : \mathcal{A}_k \to L(\mathcal{A}_k, \mathcal{H}_k) \otimes \mathcal{A}_k$ be the coaction as in (2.9). Then $\lambda(I) \subset L(\mathcal{A}_k, \mathcal{H}_k) \otimes I + J \otimes \mathcal{A}_k$ and thus λ induces a coaction $\lambda' : \mathcal{A}_{k+1} \to \mathcal{L}_{k+1} \otimes \mathcal{A}_{k+1}$ such that $\mathcal{A}_{k+1}$ is a left $\mathcal{L}_{k+1}$-comodule algebra. Indeed, it is straightforward to see that

$$
\begin{aligned}
\lambda(\varphi(v_i)) &= \lambda_k\Big(\gamma_k(v_i) + \sum_{j=1}^{n} c_j\, \mathcal{S}(e_{ij})\Big)\\
&= \sum_t \Big(\gamma_k(v_t) + \sum_{j=1}^{n} c_j\, \mathcal{S}(e_{tj})\Big) \otimes \operatorname{can}^{-1}(1\otimes \mathcal{S}(e_{it})) + 1 \otimes \operatorname{can}^{-1}(1 \otimes v_i)\\
&= \sum_{t,s} \gamma_k(v_t) \otimes \gamma_k^{-1}\mathcal{S}(e_{st}) \otimes \gamma_k\mathcal{S}(e_{is}) + \sum_s 1 \otimes \gamma_k^{-1}(v_s) \otimes \gamma_k\mathcal{S}(e_{is})\\
&\quad + \sum_{s,t,j} c_j\, \mathcal{S}(e_{tj}) \otimes \gamma_k^{-1}\mathcal{S}(e_{st}) \otimes \gamma_k\mathcal{S}(e_{is}) + 1 \otimes 1 \otimes \gamma_k(v_i)\\
&= \sum_s \Big(\overline{v}_s + \sum_j c_j E_{js} - c_s\Big) \otimes \mathcal{S}(e_{is}) + 1 \otimes 1 \otimes \Big(\gamma_k(v_i) + \sum_s c_s \mathcal{S}(e_{is})\Big)
\end{aligned}
$$

This also shows that $\mathcal{A}_{k+1}$ is a $(\mathcal{L}_{k+1}, \mathcal{H}_{k+1})$-bicomodule algebra. Let $\operatorname{can} : \mathcal{A}_k \otimes \mathcal{A}_k \to L(\mathcal{A}_k, \mathcal{H}_k) \otimes \mathcal{A}_k$ be the Galois map. To conclude, we need to show that

$$
\operatorname{can}(I \otimes \mathcal{A}_k + \mathcal{A}_k \otimes I) = J \otimes \mathcal{A}_k + L(\mathcal{A}_k, \mathcal{H}_k) \otimes I.
$$

Indeed, by the above computation, $\operatorname{can}(I \otimes \mathcal{A}_k + \mathcal{A}_k \otimes I) \subseteq J \otimes \mathcal{A}_k + L(\mathcal{A}_k, \mathcal{H}_k) \otimes I$. Now, we show that $\operatorname{can}^{-1}(J \otimes \mathcal{A}_k + L(\mathcal{A}_k, \mathcal{H}_k) \otimes I) \subseteq I \otimes \mathcal{A}_k + \mathcal{A}_k \otimes I$, using (2.10). To this end, it is enough to check that $\operatorname{can}^{-1}((\overline{v}_i - c_i + \sum_{j=1}^n c_j E_{ji}) \otimes \mathcal{A}_k) \subseteq I \otimes \mathcal{A}_k + \mathcal{A}_k \otimes I$ since the other inclusion is straightforward. This is a consequence of the following computation:

$$
\begin{aligned}
\operatorname{can}^{-1}\Big((\overline{v}_i - c_i + \sum_{j=1}^{n} c_j E_{ji}) \otimes a\Big) &= \sum_t \gamma_k(v_t) \otimes \gamma_k^{-1}\mathcal{S}(e_{it})a + 1 \otimes \gamma_k^{-1}(v_i)a\\
&\quad - 1 \otimes c_i a + \sum_{j,t} c_j\, \gamma_k\mathcal{S}(e_{tj}) \otimes \gamma_k^{-1}\mathcal{S}(e_{it})a\\
&\overset{(\star)}{=} \sum_t \Big(\gamma_k(v_t) + \sum_j c_j\, \mathcal{S}(e_{tj})\Big) \otimes \gamma_k^{-1}\mathcal{S}(e_{it})a\\
&\quad - 1 \otimes \Big(\sum_t \gamma_k(v_t)\gamma_k^{-1}\mathcal{S}(e_{it}) + c_i\Big)a\\
&\overset{(\star\star)}{=} \sum_t \Big(\gamma_k(v_t) + \sum_j c_j\, \mathcal{S}(e_{tj})\Big) \otimes \gamma_k^{-1}\mathcal{S}(e_{it})a\\
&\quad - 1 \otimes \sum_t \Big(\gamma_k(v_t) \sum_j c_j\, \mathcal{S}(e_{tj})\Big)\gamma_k^{-1}\mathcal{S}(e_{it})a
\end{aligned}
$$

for each $a \in \mathcal{A}_k$. In the computation above, $(\star)$, resp. $(\star\star)$, follows since

$$\sum_t \gamma_k(v_t)\gamma_k^{-1}\mathcal{S}(e_{it}) + \gamma_k^{-1}(v_i) = 0, \quad \text{resp.}$$

$$\sum_t \Big(\gamma_k(v_t) + \sum_j c_j\, \mathcal{S}(e_{tj})\Big)\gamma_k^{-1}\mathcal{S}(e_{it}) = \sum_t \gamma_k(v_t)\gamma_k^{-1}\mathcal{S}(e_{it}) + \sum_j c_j\, \varepsilon(e_{ij}).$$

This ends the proof of the claim. □

5.6. **Adapted stratifications with a stratum of skew-primitive elements.** We add the following assumption to the setting of page 22:

- We assume that $\mathcal{G}_k$ is composed of skew-primitive elements in $\mathcal{H}_k$. Explicitly, $u_i \in \mathcal{P}_{g_i,1}(\mathcal{H}_k)$ for some $g_i \in G(\mathcal{H}_k) = G(H)$. In particular, $v_i = u_i g_i^{-1} \in \mathcal{P}_{1,g_i^{-1}}(\mathcal{H}_k)$, so $Y_k = \Bbbk\langle v_i\rangle_{1\le i\le n}$.

Remark 5.11. Let $\varphi \in \operatorname{Alg}^{\mathcal{H}_k}(Y_k, \mathcal{A}_k)$. Lemma 5.9 (a) in this context says that there exist $c_i \in \Bbbk$, $1 \le i \le n$, such that

$$\varphi(v_i) = \gamma(v_i) - c_i\, g_i^{-1}.$$

Let $\varphi \in \operatorname{Alg}^{\mathcal{H}_k}(Y_k, \mathcal{A}_k)$, $c_i \in \Bbbk$ be as in Remark 5.11. Assume that $\mathcal{A}_{k+1} = \mathcal{A}_k/\langle\varphi(v_i)\rangle_{1\le i\le n} = \mathcal{A}_k/\langle\gamma(u_i) - c_i\rangle_{1\le i\le n} \neq 0$. Let $\mathcal{L}_k$ be a Hopf algebra such that $\mathcal{A}_k$ is a $(\mathcal{L}_k, \mathcal{H}_k)$-biGalois object.

Proposition 5.10 is formulated in this context as follows, compare with [Gu, Lemma 11].

Corollary 5.12. *$L(\mathcal{A}_{k+1}, \mathcal{H}_{k+1}) \simeq \mathcal{L}_k/\langle\widetilde{u}_i - c_i(1 - g_i)\rangle_{1\le i\le n}$, where $\widetilde{u}_i \in \mathcal{P}_{g_i,1}(\mathcal{L}_k)$ is such that*

$$\widetilde{u}_i \otimes 1_{\mathcal{A}_k} = \gamma_k(u_i)_{(-1)} \otimes \gamma_k(u_i)_{(0)} - g_i \otimes \gamma_k(u_i). \tag{5.6}$$

Proof. Follows by Proposition 5.10. The fact that $\widetilde{u}_i \in \mathcal{P}_{g_i,1}(\mathcal{L}_k)$ follows since $\widetilde{u}_i = \vartheta\big((\gamma_k \otimes \gamma_k^{-1})\Delta(u_i)\big)$ and $(\gamma_k \otimes \gamma_k^{-1})\Delta$ is a coalgebra map. □

5.6.1. *A stratum generated by one-dimensional submodules.* In this part we refine the previous setting as follows:

- We assume there is a family of YD-pairs $(g_i, \chi_i) \in G(H) \times \operatorname{Alg}(H, \Bbbk)$ cf. (2.2), $i = 1, \dots, n$, such that $u_i \in \mathcal{P}(\mathcal{B}_k)_{g_i}^{\chi_i} - 0$. We also assume that u_i is homogeneous of degree $d_i \ge 2$. We identify u_i with $u_i\#1 \in \mathcal{P}_{g_i,1}(\mathcal{H}_k)$. Recall that $v_i = u_i g_i^{-1}$.

For completeness, we include the proof of the following well-known result.

Lemma 5.13. *Assume $\operatorname{char}\Bbbk = 0$. Let $q_i = \chi_i(g_i)$, $N_i = \operatorname{ord} q_i$. Then $\Bbbk\langle v_i\rangle$ is either a polynomial algebra or a polynomial algebra truncated at N_i (in case $N_i \ge 2$).*

Clearly, q_i is a root of 1. If $q_i = 1$, then $\Bbbk\langle v_i\rangle$ is always a polynomial algebra. For $q_i \neq 1$, it is possible to check whether $\Bbbk\langle v_i\rangle$ is truncated in specific examples.

Proof. Notice that $\Bbbk\langle v_i\rangle$ is a Hopf algebra in ${}^H_H\mathcal{YD}$, with braiding determined by $c(v_i\otimes v_i)=q_i^{-1}(v_i\otimes v_i)$. Consider the polynomial algebra $\Bbbk[T]$ as a braided Hopf algebra with the analogous braiding. Then the kernel of the epimorphism $\varsigma:\Bbbk[T]\to\Bbbk\langle v_i\rangle$, given by $T\mapsto v_i$, is an homogeneous Hopf ideal of the braided Hopf algebra $\Bbbk[T]$ spanned by a primitive element. Thus $\ker\varsigma=0$ or $\langle T^M\rangle$ for some M, but T^M is primitive only when $M=1$ or $M=\operatorname{ord}q_i^{-1}=N_i$. □

In the following lemma we study the set $\mathrm{Alg}^{\mathcal{H}_k}_{\mathcal{H}_k}(X_k,\mathcal{A}_k)$, which is required for Step (1a) of the Strategy.

Lemma 5.14. *Let $\psi_1,\psi_2\in\mathrm{Alg}^{\mathcal{H}_k}_{\mathcal{H}_k}(X_k,\mathcal{A}_k)$. Fix j, $1\le j\le n$, with $\chi_{j|G(H)}\neq\varepsilon$.*

(a) *$\psi_1(v_j)=\psi_2(v_j)$.*
(b) *If H is semisimple, then $\psi_1(v_j)=\psi_2(v_j)=\gamma(v_j)$.*

Proof. (a) Since $(\gamma(v_j)\leftharpoonup t^{-1}-\chi_j(t)\gamma(v_j))g_j\in\mathcal{A}_k^{\operatorname{co}\mathcal{H}_k}=\Bbbk$, we see that there exists a map $a_j:G(H)\to\Bbbk$ such that

$$\gamma(v_j)\leftharpoonup t^{-1}=\chi_j(t)\gamma(v_j)+a_j(t)g_j^{-1},\qquad t\in G(H). \tag{5.7}$$

Note that $a_j(1)=0$ and $a_j(ts)=a_j(s)+\chi_j(s)a_j(t)$. Let $\varphi_i=\psi_{i|Y_k}\in\mathrm{Alg}^{\mathcal{H}_k}(Y_k,\mathcal{A}_k)$. Then $\psi_i(v_j)=\gamma(v_j)-c_j^{(i)}g_j^{-1}$ by Remark 5.11, for some $c_j^{(i)}\in\Bbbk$, $i=1,2$. As ψ_i is $\mathcal{H}_k$-linear, we have

$$0=\psi_i(v_j)\leftharpoonup t^{-1}-\chi_j(t)\psi_i(v_j)\overset{(5.7)}{=}\left(a_j(t)+\chi_j(t)c_j^{(i)}-c_j^{(i)}\right)g_j^{-1}.$$

Thus $a_j(t)=(1-\chi_j(t))c_j^{(i)}$. If $\chi_{j|G(H)}\neq\varepsilon$, then $c_j^{(1)}=c_j^{(2)}=\dfrac{a_j(t)}{1-\chi_j(t)}$ for $t\in G(H)$ with $\chi_j(t)\neq 1$. (b) $a_j=0$ by Proposition 5.8 (c). □

Now, we study the set $\mathrm{Alg}^{\mathcal{H}_k}(Y_k,\mathcal{A}_k)$ for Step (1b) in the Strategy.

Lemma 5.15. *Let $\varphi_i\in\mathrm{Alg}^{\mathcal{H}_k}(Y_k,\mathcal{A}_k)$ and such that $\langle\varphi_i(Y_k^+)\rangle\neq\mathcal{A}_k$, $i=1,2$. Fix j, $1\le j\le n$, with $\chi_{j|G(H)}\neq\varepsilon$.*

(a) *$\varphi_1(v_j)=\varphi_2(v_j)$.*
(b) *If H is semisimple, then $\varphi_1(v_j)=\varphi_2(v_j)=\gamma(v_j)$.*

Proof. Notice that $t\varphi_i(v_j)t^{-1}-\chi_j(t)\varphi_i(v_j)\in\langle\varphi_i(v_j)\rangle$. The computation in Lemma 5.14 shows that $a_j(t)=(1-\chi_j(t))c_j^{(i)}$ since $\langle\varphi_i(v_j)\rangle\subseteq\langle\varphi_i(Y_k)^+\rangle$. □

5.7. **Some tools for diagonal braidings.** Let V be a vector space with a basis $\xi_1,\dots,\xi_\theta$. Let $N_i\in\mathbb{N}$, $1\le i\le\theta$. Let $\Omega=(\omega_{ij})_{1\le i,j\le\theta}\in\Bbbk^{\theta\times\theta}$ such that $\omega_{ii}=1$ and $\omega_{ij}\omega_{ji}=1$ for every i,j. Consider the *quantum linear space* associated to Ω, that is

$$\Bbbk_\Omega[\xi_1,\dots,\xi_\theta]=T(V)/I_\Omega,\qquad\text{for}\qquad I_\Omega=\langle\xi_i\xi_j-\omega_{ij}\xi_j\xi_i\rangle_{1\le i,j\le\theta}.$$

The following well-known lemma is useful to deal with the Strategy for braidings of diagonal type.

Lemma 5.16. *Fix* $(\lambda_i)_{1\le i\le\theta} \in \Bbbk^\theta$ *such that*

$$\lambda_i = 0, \qquad \text{when } \omega_{ij}^{N_i} \neq 1 \text{ for some } j. \tag{5.8}$$

Let $S \subseteq I_\Omega$ *and set* $S' = S \cup \{\xi_i^{N_i} - \lambda_i\}_{1\le i\le\theta}$. *Then* $T(V)/\langle S'\rangle \neq 0$.

Proof. It suffices to consider $S = I_\Omega$, as $T(V)/\langle S'\rangle \twoheadrightarrow T(V)/\langle I'_\Omega\rangle$. We can assume that $\lambda_i = 0$, $i = 1,\dots,k$ and $\lambda_i \neq 0$, if $i > k$, for some $0 \le k \le \theta$. Set $I = \langle I_\Omega \cup \{\xi_i^{N_i}\}_{1\le i\le k}\rangle$; this is a proper $\mathbb{N}_0^\theta$-graded ideal and therefore the quotient $\Bbbk_\Omega[\xi_1,\dots,\xi_\theta]/\langle\xi_i^{N_i}\rangle_{1\le i\le k} = T(V)/I \neq 0$.

By (5.8) the elements $\xi_i^{N_i}$, $k+1 \le i \le \theta$ are central in $\Bbbk_\Omega[\xi_1,\dots,\xi_\theta]$. Moreover, the subalgebra $P \subset T(V)/I$ generated by their images is a polynomial algebra. We consider the 1-dimensional representation $M = \Bbbk$ of P given by $\xi_i^{N_i}\cdot 1 = \lambda_i$ and $M' = T(V)/I \otimes_P M$ the induced representation of $T(V)/I$. Notice that the algebra map $T(V)/I \to \operatorname{End} M'$ factors through $T(V)/\langle I'_\Omega\rangle$ and hence this algebra is nonzero. □

We will also make use of the following remark.

Remark 5.17. Let $\{r_i\}_{i\in I}$ be a family of monomials in $T(V)$ and set $J = \langle r_i\rangle_{i\in I}$. Then (the image of) the set of monomials in $T(V)$ that do not contain an r_i as a subword is a linear basis of $T(V)/J$. Indeed, a basis of $T(V)$ is given by the collection of all monomials. This set can be splitted in two subsets: the monomials containing an r_i as a subword and those that do not. The first subset is a linear basis of J.

5.8. **An example of diagonal type.** Assume $\Bbbk = \mathbb{C}$. Let $\zeta \in \Bbbk$ be a primitive 9^{th}-root of unity. We apply our strategy to classify the liftings of the Nichols algebra associated to the diagram

$$\overset{-\zeta}{\circ} \overset{\zeta^7}{\text{———}} \overset{\zeta^3}{\circ} \tag{5.9}$$

of [H, Table 1, row 9]. Consider a matrix $(q_{ij})_{1\le i,j\le 2}$ corresponding to (5.9), that is $q_{11} = -\zeta$, $q_{22} = \zeta^3$ and $q_{12}q_{21} = \zeta^7$. Let Γ be a finite group such that there is a realization of this braiding, *i.e.* there are $g_1, g_2 \in \Gamma$, $\chi_1,\chi_2 \in \widehat{\Gamma}$ with $\chi_j(g_i) = q_{ij}$, $1 \le i,j \le 2$. Set $H = \Bbbk\Gamma$ and $V \in {}^H_H\mathcal{YD}$ the associated Yetter-Drinfeld module: V has a basis $\{x_1,x_2\}$ with $x_i \in V_{g_i}^{\chi_i}$, $i = 1,2$. Let

$$\begin{aligned} &x_{12} = x_1x_2 - q_{12}x_2x_1, && x_{112} = x_1x_{12} - q_{11}q_{12}x_{12}x_1,\\ &x_{1112} = x_1x_{112} - q_{11}^2q_{12}x_{112}x_1, && x_{122} = x_{12}x_2 - q_{12}q_{22}x_2x_{12},\\ &x_{1,122} = x_1x_{122} - q_{11}q_{12}^2x_{122}x_1. \end{aligned} \tag{5.10}$$

By [An, Example 2.5], $\mathcal{B}(V)$ is presented by generators x_1, x_2 and relations

$$x_1^{18} = x_2^3 = x_{12}^{18} = x_{1112} = x_{1,122} - a\,x_{12}^2 = 0,$$

for $a = \zeta^7 q_{12}(1+\zeta)^{-1}$. We fix the following stratification:

(5.11)
$$\mathcal{G}_0 = \{x_1^{18}, x_2^3\}, \quad \mathcal{G}_1 = \{x_{1,122} - a\, x_{12}^2\}, \quad \mathcal{G}_2 = \{x_{1112}\}, \quad \mathcal{G}_3 = \{x_{12}^{18}\}.$$

Set $\mathcal{H} = \mathcal{H}_4 = \mathcal{B}(V)\#H$. Let $\lambda_1, \lambda_2 \in \Bbbk$ be subject to:

(5.12) $$\lambda_1 = 0 \quad \text{if } \chi_1^{18} \neq \varepsilon, \qquad \lambda_2 = 0 \quad \text{if } \chi_2^3 \neq \varepsilon.$$

Let $\mathcal{A}(\lambda_1, \lambda_2)$ be the quotient of $\mathcal{T}(V)$ by the ideal generated by

(5.13) $$x_1^{18} - \lambda_1, \qquad x_2^3 - \lambda_2, \qquad x_{1,122} - a\, x_{12}^2, \qquad x_{1112}.$$

Remark 5.18. $\mathcal{A}(\lambda_1, \lambda_2) \neq 0$.

Proof. By Remark 5.6 (b), $\mathcal{A}(\lambda_1, \lambda_2) \simeq T(V)/J\#H$ where J is the ideal generated by the relations (5.13). Set $\xi_i = x_i$, $i = 1, 2$, $\omega_{12} = q_{12} = \omega_{21}^{-1}$. Then, in the notation of Lemma 5.16, $S := \{x_{1,122} - a\, x_{12}^2, x_{1112}\} \subset I_\Omega$ by (5.10). Condition (5.8) is tantamount to (5.12), and $J = \langle S' \rangle$. Thus Lemma 5.16 applies. □

Let $\mathcal{L}(\lambda_1, \lambda_2)$ be the quotient of $\mathcal{T}(V)$ by the ideal generated by

$$x_1^{18} - \lambda_1(1 - g_1^{18}), \qquad x_2^3 - \lambda_2(1 - g_2^3), \qquad x_{1,122} - a\, x_{12}^2, \qquad x_{1112}.$$

Notice that this is a Hopf ideal, as $\mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2 \subseteq \mathcal{P}(T(V))$.

We will now follow the strategy in Subsection 5.2 in order to find all the liftings of $\mathcal{B}(V)$ over Γ. We stick to the notation therein.

Proposition 5.19. (a) *$\mathcal{A}(\lambda_1, \lambda_2)$ is a right $\mathcal{H}_3$-Galois object with coaction induced by the comultiplication in $\mathcal{T}(V)$. Moreover,*

$$\Lambda_3 = \big\{\mathcal{A}(\lambda_1, \lambda_2) : \lambda_1, \lambda_2 \textit{ as in } (5.12)\big\}.$$

(b) $L(\mathcal{A}(\lambda_1, \lambda_2), \mathcal{H}_3) \simeq \mathcal{L}(\lambda_1, \lambda_2)$.

(c) *Let $\lambda_3 \in \Bbbk$ be subject to:*

(5.14) $$\lambda_3 = 0 \quad \textit{if } \chi_1^{18}\chi_2^{18} \neq \varepsilon.$$

Then the algebra $\mathcal{A}(\lambda_1, \lambda_2, \lambda_3) := \mathcal{A}(\lambda_1, \lambda_2)/\langle x_{12}^{18} - \lambda_3\rangle$ is a right $\mathcal{H}$-Galois object. Moreover,

$$\Lambda_4 = \big\{\mathcal{A}(\lambda_1, \lambda_2, \lambda_3) : \lambda_1, \lambda_2 \textit{ as in } (5.12) \textit{ and } \lambda_3 \textit{ as in } (5.14)\big\}.$$

Proof. Following the Strategy in 5.2, we start by constructing the set Λ_1. For this we use (1b). Consider the subalgebra Y_0 of $\mathcal{H}_0 = \mathcal{T}(V)$ generated by $x_1^{18} g_1^{-18}$ and $x_2^3 g_2^{-3}$. This is a free associative algebra in two generators. Then we have $\operatorname{Alg}^{\mathcal{H}_0}(Y_0, \mathcal{A}_0) \cong \Bbbk^2$, since every map is determined by its value on $x_1^{18} g_1^{-18}$ and $x_2^3 g_2^{-3}$ and these values must be $x_1^{18} g_1^{-18} - \lambda_1\, g_1^{-18}$ and $x_2^3 g_2^{-3} - \lambda_2\, g_2^{-3}$, for some $\lambda_1, \lambda_2 \in \Bbbk$, by Remark 5.11. Then Λ_1 is the set of all algebras $\mathcal{A}_1(\lambda_1, \lambda_2)$ obtained as $\mathcal{T}(V)/\langle x_1^{18} - \lambda_1, x_2^3 - \lambda_2\rangle$, for λ_1, λ_2 subject to (5.12) by Lemma 5.15. Indeed, these algebras are nonzero since they project over $\mathcal{A}(\lambda_1, \lambda_2)$ which is nonzero by Remark 5.18. We denote by y_1, y_2 the images of the generators x_1, x_2 in each one of these quotients.

For Λ_2 we use again (1b). Set $Y_1 = \Bbbk\langle (x_{1,122} - a\,x_{12}^2)g_1^{-2}g_2^{-2}\rangle \subset \mathcal{H}_1$. Notice that $\chi_1^2\chi_2^2(g_1^2g_2^2) = \zeta^8$ so $\chi_1^2\chi_2^2 \neq \varepsilon$. It follows that Y_1 is a polynomial algebra. Indeed, by Lemma 5.13 we need to check that $z = \left(x_{1,122} - a\,x_{12}^2\right)^9 \neq 0$. Now, z is a linear combination of monomials containing $(x_1^2x_2^2)^9$ with coefficient 1. So it is nonzero by Remark 5.17. Lemma 5.15 implies that

$$\left(x_{1,122} - a\,x_{12}^2\right) g_1^{-2}g_2^{-2} \mapsto \gamma_1\left(\left(x_{1,122} - a\,x_{12}^2\right) g_1^{-2}g_2^{-2}\right),$$

is the unique possible map in $\mathrm{Alg}^{\mathcal{H}_1}(Y_1, \mathcal{A}_1)$. Let $y_{1,122}$, $y_{12} \in \mathcal{A}_1$ be defined as in (5.10). It is easy to see that

$$\gamma_1\left(\left(x_{1,122} - a\,x_{12}^2\right) g_1^{-2}g_2^{-2}\right) = \left(y_{1,122} - a\,y_{12}^2\right) g_1^{-2}g_2^{-2}.$$

Indeed, $\gamma_1\left(\left(x_{1,122} - a\,x_{12}^2\right) g_1^{-2}g_2^{-2}\right) = \left(y_{1,122} - a\,y_{12}^2\right) g_1^{-2}g_2^{-2} - c\,g_1^{-2}g_2^{-2}$, for some $c \in \Bbbk$ by $\mathcal{H}_1$-colinearity but $c = 0$ because γ_1 is H-linear. Then Λ_2 is composed of the algebras $\mathcal{A}_2(\lambda_1, \lambda_2) = \mathcal{A}_1(\lambda_1, \lambda_2)/\langle y_{1,122} - a\,y_{12}^2\rangle$ with λ_1, λ_2 subject to (5.12). These are nonzero as they project over $\mathcal{A}(\lambda_1, \lambda_2)$.

For Λ_3 we also use (1b). Set $Y_2 = \Bbbk\langle x_{1112}g_1^{-3}g_2^{-1}\rangle \subset \mathcal{H}_2$. We have that $\chi_1^3\chi_2(g_1^3g_2) = -\zeta^6$, so $\chi_1^3\chi_2 \neq \varepsilon$. As above, Y_2 is a polynomial algebra. Again, by Lemma 5.13 it is enough to check that $x_{1112}^6 \neq 0$. For this, let F be the set composed of the monomials x_1^{18}, x_2^3 and those appearing in the expression of $x_{1,122} - a\,x_{12}^2$. Set $J = \langle F\rangle \subset T(V)$, then there exists a projection $\mathcal{H}_2 \twoheadrightarrow T(V)/J$. As x_{1112}^6 is a linear combination of monomials containing $(x_1^3x_2)^6$ with coefficient 1, Remark 5.17 implies $x_{1112}^6 \neq 0$ in $T(V)/J$ and hence it is nonzero in $\mathcal{H}_2$. Thus Lemma 5.15 implies that $x_{1112}g_1^{-3}g_2^{-1} \mapsto \gamma_2(x_{1112}g_1^{-3}g_2^{-1})$ is the unique possible map in $\mathrm{Alg}^{\mathcal{H}_2}(Y_2, \mathcal{A}_2)$. Also, it is easy to see that $\gamma_2(x_{1112}g_1^{-3}g_2^{-1}) = y_{1112}g_1^{-3}g_2^{-1}$, for y_{1112} defined as in (5.10). We have already seen that the quotients $\mathcal{A}_2/\langle y_{1112}\rangle = \mathcal{A}(\lambda_1, \lambda_2)$ are nonzero. We obtain that Λ_3 is the set of all the algebras $\mathcal{A}(\lambda_1, \lambda_2)$ and (a) follows. Now (b) holds by Corollary 5.12.

For Λ_4, we use (1a), since $\Bbbk\langle x_{12}^{18}\rangle$ is a normal subalgebra of $\mathcal{H}_2$. This follows because x_{12}^{18} is in the center of $\mathcal{H}_3$, which can be proved using [GAP], see also [AAGI]. By Lemma 5.13, X_3 is a polynomial algebra. Using [GAP] again[2], we see that

$$\rho_3(y_{12}^{18}) = y_{12}^{18} \otimes 1 + g_1^{18}g_2^{18} \otimes x_{12}^{18}. \tag{5.15}$$

Hence, $\gamma_3(x_{12}^{18}) = y_{12}^{18} + c$, for some $c \in \Bbbk$. Now, if $\chi_1^{18}\chi_2^{18} \neq \varepsilon$, then $c = 0$ and there is a unique map in $\mathrm{Alg}_{\mathcal{H}_3}^{\mathcal{H}_3}(X_3, \mathcal{A}_3)$, determined by $x_{12}^{18}g_1^{-18}g_2^{-18} \mapsto y_{12}^{18}g_1^{-18}g_2^{-18}$ by Lemma 5.14. On the other hand, if $\chi_1^{18}\chi_2^{18} = \varepsilon$, then it follows by Remark 5.11 that $\mathrm{Alg}_{\mathcal{H}_3}^{\mathcal{H}_3}(X_3, \mathcal{A}_3) \cong \Bbbk$, since for each $\lambda_3 \in \Bbbk$, $x_{12}^{18}g_1^{-18}g_2^{-18} \mapsto y_{12}^{18}g_1^{-18}g_2^{-18} - \lambda_3\,g_1^{-18}g_2^{-18}$ induces an algebra morphism $X_3 \to \mathcal{A}_3$ in $\mathcal{YD}_{\mathcal{H}_3}^{\mathcal{H}_3}$. Hence (c) follows. $\square$

[2] This coaction is computed with [GAP] using the method described in [GIV, Appendix].

Let $\mathcal{L}(\lambda_1,\lambda_2,\lambda_3)$ be the quotient of $\mathcal{L}(\lambda_1,\lambda_2)$ by the ideal generated by

$$x_{12}^{18}-\lambda_3(1-g_1^{18}g_2^{18}).$$

In the next theorem we show that this is a Hopf ideal and that the family of Hopf algebras $\mathcal{L}(\lambda_1,\lambda_2,\lambda_3)$ exhausts the list of liftings of $\mathcal{B}(V)$ over Γ. In particular, every lifting is a cocycle deformation of $\mathcal{B}(V)\#\Bbbk\Gamma$.

Theorem 5.20. (a) $\mathcal{L}(\lambda_1,\lambda_2,\lambda_3)$ *is a cocycle deformation of* $\mathcal{H}$.
(b) $\mathcal{L}(\lambda_1,\lambda_2,\lambda_3)$ *is a lifting of* $\mathcal{B}(V)$ *over* Γ.
(c) *Reciprocally, if* L *is a lifting of* $\mathcal{B}(V)$ *over* Γ*, then there are* $\lambda_1,\lambda_2,\lambda_3$ *such that* $L\simeq\mathcal{L}(\lambda_1,\lambda_2,\lambda_3)$.

Proof. (a) We use [GAP] as in (5.15) to see that

$$\gamma_3(x_{12}^{18})_{(-1)}\otimes\gamma_3(x_{12}^{18})_{(0)}-g_1^{18}g_2^{18}\otimes\gamma_3(x_{12}^{18})=x_{12}^{18}\otimes 1. \tag{5.16}$$

Then x_{12}^{18} satisfies (5.6). Hence, $L(\mathcal{A}(\lambda_1,\lambda_2,\lambda_3),\mathcal{H})\simeq\mathcal{L}(\lambda_1,\lambda_2,\lambda_3)$, by Corollary 5.12.

(b) follows by Proposition 4.14 (b) and (d).

(c) Let $\phi:\mathcal{T}(V)\to L$ be a lifting map. If $r\in\mathcal{G}_0\cup\mathcal{G}_1\cup\mathcal{G}_2$, then r is $(g(r),1)$-primitive for $g(r)\in\Gamma$, hence $\phi(r)\in L_1$. Let $\chi_r\in\widehat{\Gamma}$ be the character from the Γ-action on r. Now, the pair $(\chi_r,g(r))$ is different from (χ_i,g_i), $i=1,2$ and thus $\phi(r)\in\Bbbk\Gamma$ by Lemma 4.8 (b), see also [AS3, Lemma 6.1]. Indeed, $\chi_1^{18}(g_1^{18})=\chi_2^3(g_2^3)=1$ and we have already seen that $\chi_1^2\chi_2^2(g_1^2g_2^2)=\zeta^8$, $\chi_1^3\chi_2(g_1^3g_2)=-\zeta^6$. Then there exist $\lambda_1,\lambda_2\in\Bbbk$ such that ϕ factorizes through $\mathcal{L}(\lambda_1,\lambda_2)$. By equation (5.16) and Corollary 5.12, x_{12}^{18} is $(g_1^{18}g_2^{18},1)$-primitive in $\mathcal{L}(\lambda_1,\lambda_2)$. Also, $\phi(x_{12}^{18})\in\Bbbk\Gamma$ again by Lemma 4.8 (b). Hence, there exists $\lambda_3\in\Bbbk$ such that ϕ factorizes through $\mathcal{L}(\lambda_1,\lambda_2,\lambda_3)$ and induces an isomorphism since both algebras have dimension $\dim\mathcal{B}(V)|\Gamma|$. □

5.9. **A question.** Set $\mathcal{A}_k\in\operatorname{Cleft}(\mathcal{H}_k)$. To find $\mathcal{A}_{k+1}\in\operatorname{Cleft}(\mathcal{H}_{k+1})$ we can either apply Theorem 3.1 or Theorem 3.3. As said in Subsection 5.2 both alternatives present a hard computational obstacle, namely the computation of X_k or the checking of $\langle\varphi(Y_k^+)\rangle\neq\mathcal{A}_k$. Hence we need an *intermediate Günther's Theorem* exploiting the benefits of both alternatives. That said, we collect from the examples enough evidence to change alternative (1b) by

(1c) Compute Y_k and then $\operatorname{Alg}_H^{\mathcal{H}_k}(Y_k,\mathcal{A}_k)$.

Actually, in many examples we see that not only $\varphi\in\operatorname{Alg}_H^{\mathcal{H}_k}(Y_k,\mathcal{A}_k)$ induces a nonzero algebra $\mathcal{A}_{k+1}$ but also that any $\mathcal{A}_{k+1}\in\operatorname{Cleft}(\mathcal{H}_{k+1})$, and hence any $\psi\in\operatorname{Alg}_{\mathcal{H}_k}^{\mathcal{H}_k}(X_k,\mathcal{A}_k)$ is determined by $\varphi=\psi_{|Y_k}\in\operatorname{Alg}^{\mathcal{H}_k}(Y_k,\mathcal{A}_k)$. Furthermore, as $\mathcal{H}_k$ is Y_k-faithfully flat, see Corollary 3.8,

(1) X_k is the subalgebra generated by $Y_k\cdot\mathcal{H}_k$, see Remark 5.4.

Question. Is there a general setting in which any $\varphi\in\operatorname{Alg}_H^{\mathcal{H}_k}(Y_k,\mathcal{A}_k)$ extends to $\psi\in\operatorname{Alg}_{\mathcal{H}_k}^{\mathcal{H}_k}(X_k,\mathcal{A}_k)$ with $\psi_{|Y_k}=\varphi$?

Assume that H is finite-dimensional and semisimple. Then evidence of a positive answer is given by (1) above and the fact that

(2) $\mathcal{A}_k$ is an injective object in $\mathcal{YD}_H^{\mathcal{H}_k}$, see Lemma 5.7.

So, any $\varphi \in \mathrm{Hom}_H^{\mathcal{H}_k}(Y_k, \mathcal{A}_k)$ extends to $\psi \in \mathrm{Hom}_{\mathcal{H}_k}^{\mathcal{H}_k}(X_k, \mathcal{A}_k)$ with $\psi_{|Y_k} = \varphi$.

As a last word, we recall that:

(3) There is an H-linear section $\gamma_k : \mathcal{H}_k \to \mathcal{A}_k$ with $\gamma_{k|H} = \mathrm{id}_H$.

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