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# Uniqueness condition for an auto-logistic model ☆

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ABSTRACT

Auto-logistic models are widely used to describe binary images texture and spatial presence–absence data. There exist some techniques, like Gibbs sampler algorithm among others, that allow simulating the process, but its performance depends on the model global properties at $\mathbb{Z}^2$. Under general conditions there will be at least a global distribution which conditionals are Gibbs specifications. The present work establishes sufficient conditions on the parameters of an auto-logistic model, in order to ensure the distribution's uniqueness.
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## 1. Introduction

In his doctoral thesis, Ising (in 1925) laid the foundation of the Random fields in Statistical Physics (Ising, 1924, 1925). He presented a ferromagnetic model where particles fixed in a lattice, each one associated with a spin value $+1$ or $-1$, interact with their nearest neighbors. Besag applied this idea for the first time to image processing (Besag, 1974). Statistical modeling of images is a valuable tool and there is a wide range of interests in image processing in several areas of knowledge. Ising and Besag models take into account the dependence between nearest pixels. This kind of models is called Markov random fields. Moreover, by means of Hammersley–Clifford Theorem (Winkler, 1995), it is shown that Markov random fields presents a Gibbs distribution. There are algorithms that simulate Gibbs distribution being Gibbs sampler is the most popular one (see Bustos and Ojeda (1994), Winkler (1995)). Gibbs sampler generates a Markov chain of images converging to a realization of the Gibbs model using its local dependence. The convergence holds if there is only one global distribution. At finite lattice there is only one global distribution, it can be obtained through the Brook expansion (see Besag (1974), Bartolucci and Besag (2002), At $\mathbb{Z}^2$.) under general condition existence is demonstrated, but uniqueness is not trivial (see Georgii (1988)). There are a lot of papers on this topic (see, for instance, Albeverioa et al. (1997), Betz and Lorinczi (2003), Kepa and Kozitsky (2007), Weitz (2005)). Dobrushin's condition theorem provides sufficient conditions for uniqueness. However, the verification of this condition is not straightforwardly attained and furthermore it is actually model dependent. Auto-logistic models handle the dependence of spatial binary data like binary images indicating presence–absence of something. These kinds of data appear in several areas like biology and geoscience. The present work presents a theorem giving a sufficient (but not necessary) condition that ensures uniqueness in the auto-logistic model for 4 and 8 neighbors. This condition does not involve any external field and constrains interaction parameters to a bounded region (in $\mathbb{R}^2$ or $\mathbb{R}^4$). Preliminary and in progress works suggest that this theorem could be extended to larger neighborhoods as well as to 3D lattices. This goal might be attained by increasing the dimension of the vector parameters in order to include new neighbors. This article is organized in different parts, Section 2 is dedicated to main required definitions, Section 3 is the description of the proposed

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model and condition bases; whereas Section 4 presents and demonstrates the theorem. Finally, Section 5 is devoted to a discussion about theorem condition.

## 2. Theoretical framework

Consider the lattice $S \subseteq \mathbb{Z}^2$, not necessarily finite. Let us introduce the following definitions as assumptions

$E = \{0, 1\}$; its $\sigma$-algebra $\mathscr{E} = \mathscr{P}(E)$, the set of all subset of $E$;
$\lambda : \mathscr{E} \to \mathbb{N} \cup \{0\}$, the counting measure ($\lambda(A) \doteq \# A$);
$x = \{x_s\}_{s\in S} \in E^S$ is a binary image;
$X = \{X_s\}_{s\in S}$ is the underlying stochastic process;
$V \subseteq S$; $x_V \doteq \{x_s\}_{s\in V} \in E^V$; $\mathscr{E}^V$ is the product $\sigma$-algebra;
$A \subseteq S$; $\xi = \{\xi_s\}_{s\in A} \in E^A$; if $A \subseteq S \setminus V$, then $\xi x_V \doteq \{(\xi x_V)_s\}_{s\in A\cup V} \in E^{A\cup V}$, where $(\xi x_V)_s = \xi_s$ for $s \in A$, and $(\xi x_V)_s = x_s$ for $s \in V$;
$\mathscr{F}_V \doteq \left\{B \times E^{S\setminus V} : B \in \mathscr{E}^V\right\} \subseteq \mathscr{F} \doteq \mathscr{E}^S$;
$\mathscr{S} \doteq \left\{\Lambda \subset \mathbb{Z}^2 : 1 \le \#(\Lambda) < \infty\right\}$;
$\Phi \doteq (\Phi_\Lambda)_{\Lambda\in\mathscr{S}}$ is a potential. That is $\Phi_\Lambda(x) = \phi_\Lambda(x_\Lambda)$ where $\phi_\Lambda$ is a real function $\mathscr{E}^\Lambda$-measurable and there exists a energy function $H_\Lambda \doteq \sum_{\Delta\in\mathscr{S}\cap\Lambda} \Phi_\Delta$;
$\gamma = (\gamma_\Lambda)_{\Lambda\in\mathscr{S}}$ is the Gibbs specification (for $\Phi$) that is
$\gamma_\Lambda(A|x) \doteq \frac{\sum_{\xi\in E^\Lambda} 1_A(\xi x_{S\setminus\Lambda})\exp(-H_\Lambda(\xi x_{S\setminus\Lambda}))}{\sum_{\xi\in E^\Lambda}\exp(-H_\Lambda(\xi x_{S\setminus\Lambda}))}$, $1_A$ indicator function, $A \in \mathscr{F}$;
$\mathscr{G}(\gamma) \doteq \{\mu : \mu(A \cap B) = \int_B \gamma_\Lambda(A|\,)d\mu,\ \forall B \in \mathscr{F}_{S\setminus\Lambda}\}$ is the set of global Gibbs measures $\mu$ in $E^S$ such that $\gamma_\Lambda(A|x)$ is the probability of $A$ with respect to $\mu$ conditional to $x_{S\setminus\Lambda}$.

Since $E$ is finite, $(E, \mathscr{E})$ is a standard Borel space, then $\#\mathscr{G}(\gamma) \ge 1$ holds (see Theorem 4.23 in Georgii (1988)).

## 3. Auto-logistic model and Dobrushin's condition

Consider the potential

$$\Phi_\Lambda(x) = \begin{cases} \beta_i x_t x_{t+v_i} & \Lambda = \{t, t+v_i\} \\ \beta_0 x_t & \Lambda = \{t\} \\ 0 & \text{otherwise,} \end{cases}$$

where $t \in S$, $i = 1, \ldots, g$ ($g = 2$, first order and $g = 4$, second order), $v_1 = (0, 1)$, $v_2 = (1, 0)$, $v_3 = (1, 1)$, and $v_4 = (-1, 1)$.

For $s \in S$ the following definitions are introduced

$$\begin{aligned}\gamma_s^0(B|x) &\doteq \gamma_{\{s\}}(B \times E^{S\setminus\{s\}}|x) \\ &= \frac{\sum_{\xi\in B}\exp\left(-H_{\{s\}}\left(\xi x_{S\setminus\{s\}}\right)\right)}{\sum_{\xi\in E}\exp\left(-H_{\{s\}}\left(\xi x_{S\setminus\{s\}}\right)\right)} \\ &= \frac{\sum_{\xi\in B}\exp\left(-\sum_{\Lambda\in\{s\}\cap\mathscr{S}}\Phi_\Lambda\left(\xi x_{S\setminus\{s\}}\right)\right)}{\sum_{\xi\in E}\exp\left(-\sum_{\Lambda\in\{s\}\cap\mathscr{S}}\Phi_\Lambda\left(\xi x_{S\setminus\{s\}}\right)\right)}, \quad B \in \mathscr{E}.\end{aligned}$$

Then, the local characteristic is[1]

$$\pi(x_s|x_{-s}) = \gamma_s^0(\{x_s\}|x) = \frac{e^{-x_s(\beta_0+\sum_{i=1}^g \beta_i(x_{s+v_i}+x_{s-v_i}))}}{e^{-(\beta_0+\sum_{i=1}^g \beta_i(x_{s+v_i}+x_{s-v_i}))}+1}. \tag{3.1}$$

We note that $\gamma_s^0(.|x)$ depends on $x_{\partial s}$, with $\partial s = \{s \pm v_i\}_{i=1}^g$ neighborhood of $s$. For $g = 4$, $\partial s =$

| | | |
|---|---|---|
| $s - v_3$ | $s - v_1$ | $s + v_4$ |
| $s - v_2$ | | $s + v_2$ |
| $s - v_4$ | $s + v_1$ | $s + v_3$ |

.

$\beta = (\beta_0, \beta_1, \beta_2, \beta_3, \beta_4)$ is the parameter vector for the second order model (8 neighbors) and $\beta = (\beta_0, \beta_1, \beta_2)$ is the parameter vector for the first order model (4 neighbors). Now, $\beta_0$ is the external field parameter. Then, if $\beta_0 = 0$ we say the

[1] Following some usual notation, where $x_{-s} \doteq x_{S\setminus\{s\}}$.

model does not have an external field. Therefore, in absence of external field, we have $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$ and $\beta = (\beta_1, \beta_2)$ in the 8 and 4 neighbors cases, respectively. It is then clear that $g = 2$ is the particular case of $g = 4$ when $\beta_3 = \beta_4 = 0$, and $\Phi$ is translation-invariant (i.e. $\beta$ does not depend on $t \in S$). It is remarkable that we cannot consider the auto-logistic regression model, since the external field depends on $t$ (Hughes et al., 2011; Wang and Zheng, 2013).

We introduced now the Uniform Distance between the probabilities $\mu$ and $\tilde{\mu}$ by

$$d(\mu, \tilde{\mu}) \doteq \sup\{|\mu(B) - \tilde{\mu}(B)| : B \in \mathscr{E}\}.$$

For $s$ and $t$ in $S$ we define the Dobrushin interdependence between $s$ and $t$

$$C_{s,t} \doteq \sup\left\{d(\gamma_s^0(\cdot|x), \gamma_s^0(\cdot|w)) : x_{S\setminus t} = w_{S\setminus t}\right\}.$$

We note that $C_{s,t} = 0$ if $t \notin \partial s$.

Finally,

$$\alpha(\gamma) \doteq \sup_{s\in S}\left\{\sum_{t\in S} C_{s,t}\right\}.$$

**Definition 3.1.** Let $\widetilde{\gamma}$ be an specification. $\widetilde{\gamma}$ will be said to satisfy Dobrushin's condition if $\widetilde{\gamma}$ is quasilocal[2] and $\alpha(\widetilde{\gamma}) < 1$.

Since $E$ is finite then $\phi$ is $\lambda$-admissible and then $\gamma$ is quasilocal (see Eq. (2.7) and Proposition 2.24 in Georgii (1988)). If $\alpha(\gamma) < 1$ then $\gamma$ meets Dobrushin's condition and $\#\mathscr{G}(\gamma) = 1$ (see Theorem 8.7 in Georgii (1988)).

## 4. Uniqueness theorem

**Theorem 4.1.** *Let $\gamma$ be the Gibbs specification for $\Phi$. Then*

$$2\sum_{l=1}^{g} \tanh(|\beta_l|/4) < 1 \Rightarrow \#\mathscr{G}(\gamma) = 1. \tag{4.1}$$

**Proof.** To prove theorem it is enough to check that $\alpha(\gamma) \le 2\sum_{l=1}^{g} \tanh(|\beta_l|/4)$.

Let $s \in S$, $t \in \partial s$, $x$ and $w$ in $E^S$ such that $x_{S\setminus t} = w_{S\setminus t}$.

If $x_t = w_t$, then $x = w$ and $d(\gamma_s^0(\cdot|x), \gamma_s^0(\cdot|w)) = 0$.

If $x_t = 1 - w_t$, where $t = s + v_l$ or $t = s - v_l$, for $1 \le l \le g$. Without loss of generality, it may be assumed that $t = s - v_l$. Then $x_r = w_r$, $r \ne s - v_l$ and $x_{s-v_l} = 1 - w_{s-v_l}$.

Hence,

$$\begin{aligned} d(\gamma_s^0(\cdot|x), \gamma_s^0(\cdot|w)) &= \sup\left\{|\gamma_s^0(A|x) - \gamma_s^0(A|w)| / A \in \mathscr{E}\right\}, \\ &= \max\left\{|\gamma_s^0(A|x) - \gamma_s^0(A|w)| / A = \emptyset, E, \{0\}, \{1\}\right\}, \end{aligned}$$

and

$$|\gamma_s^0(\emptyset|x) - \gamma_s^0(\emptyset|w)| = |0 - 0| = 0,$$

$$|\gamma_s^0(E|x) - \gamma_s^0(E|w)| = |1 - 1| = 0,$$

$$|\gamma_s^0(\{1\}|x) - \gamma_s^0(\{1\}|w)| = |\gamma_s^0(\{0\}|x) - \gamma_s^0(\{0\}|w)|,$$

(because $\gamma_s^0(\{1\}|x) = 1 - \gamma_s^0(\{0\}|x)$), therefore

$$\begin{aligned} d(\gamma_s^0(\cdot|x), \gamma_s^0(\cdot|w)) &= \left|\gamma_s^0(\{1\}|x) - \gamma_s^0(\{1\}|w)\right|, \\ &= \left|\frac{e^{-(\beta_0 + \sum_{i=1}^{g} \beta_i(x_{s+v_i} + x_{s-v_i}))}}{e^{-(\beta_0 + \sum_{i=1}^{g} \beta_i(x_{s+v_i} + x_{s-v_i}))} + 1} - \frac{e^{-(\beta_0 + \sum_{i=1}^{g} \beta_i(w_{s+v_i} + w_{s-v_i}))}}{e^{-(\beta_0 + \sum_{i=1}^{g} \beta_i(w_{s+v_i} + w_{s-v_i}))} + 1}\right|, \\ &= \left|\frac{1}{1 + e^{\beta_0 + \beta_l(x_{s+v_l} + x_{s-v_l}) + \sum_{i\ne l} \beta_i(x_{s+v_i} + x_{s-v_i})}} - \frac{1}{1 + e^{\beta_0 + \beta_l(w_{s+v_l} + w_{s-v_l}) + \sum_{i\ne l} \beta_i(w_{s+v_i} + w_{s-v_i})}}\right|, \\ &= \left|\frac{1}{1 + e^{\beta_0 + \beta_l(x_{s+v_l} + x_{s-v_l}) + \sum_{i\ne l} \beta_i(x_{s+v_i} + x_{s-v_i})}} - \frac{1}{1 + e^{\beta_0 + \beta_l(x_{s+v_l} + (1 - x_{s-v_l})) + \sum_{i\ne l} \beta_i(x_{s+v_i} + x_{s-v_i})}}\right|, \end{aligned}$$

[2] See Georgii (1988).

$$
\begin{aligned}
&= \left| \frac{1}{1+e^{\beta_l x_{s-v_l}} e^{\theta}} - \frac{1}{1+e^{\beta_l (1-x_{s-v_l})} e^{\theta}} \right|^{3} \\
&= \left| \frac{e^{\beta_l (1-x_{s-v_l})} - e^{\beta_l x_{s-v_l}}}{e^{-\theta} + e^{\beta_l (x_{s-v_l})} + e^{\beta_l (1-x_{s-v_l})} + e^{\beta_l} e^{\theta}} \right|, \\
&= \frac{|1-e^{\beta_l}|}{e^{\beta_l} e^{\theta} + e^{-\theta} + e^{\beta_l} e^{0} + e^{-0}}, \\
&\leq \frac{|1-e^{\beta_l}|}{e^{\beta_l} e^{-\beta_l/2} + e^{-(-\beta_l/2)} + e^{\beta_l} + 1}, \\
&= \frac{|1-e^{\beta_l}|}{(1+e^{\beta_l/2})^2} = \frac{(1+e^{\beta_l/2})|1-e^{\beta_l/2}|}{(1+e^{\beta_l/2})^2} \\
&= \frac{|1-e^{\beta_l/2}|}{1+e^{\beta_l/2}} = \frac{e^{|\beta_l|/2}-1}{e^{|\beta_l|/2}+1} = \tanh(|\beta_l|/4)
\end{aligned}
$$

(since $e^{\beta_l} e^{-\beta_l/2} + e^{\beta_l/2} \leq e^{\beta_l} e^{z} + e^{-z}, z \in \mathbb{R}$).

Summarizing, $C_{s,s-v_l} \leq \tanh(|\beta_l|/4)$ and $\sum_{t\in\partial s} C_{s,t} \leq \sum_{l=1}^{g} 2\tanh(|\beta_l|/4),\ \forall s \in S$, then

$$\alpha(\gamma) = \sup_{s\in S} \left\{ \sum_{t\in\partial s} C_{s,t} \right\} \leq 2 \sum_{l=1}^{g} \tanh(|\beta_l|/4). \quad \blacksquare$$

**Remark 4.1.** The counterpart is false ($\not\Leftarrow$).

**Proof.** Identifying 0 with $-1$, first order Auto-logistic model with $\beta_0/2 = \beta_1 = \beta_2$ is the $(-1)$-normalized Ising model for $\beta_1/4$ and without an external field (see Example 3.3.33 in Bustos and Guerrero (2011)). The $\beta_1 = \beta_2 = 1.6$ case does not match theorem conditions. But there is uniqueness because $\beta_1/4 < \beta_c = (\log(1+\sqrt{2}))/2 = 0.4402$ (Ising critical parameter, see page 100 of Georgii (1988) and example in Comets (1997)).

## 5. Discussion

Theorem 4.1 provides a region for interaction parameters. It was called uniqueness region, which graphical schema is outlined in Fig. 1 for the first order model. A lot of binary textures could characterized by auto-logistic models through his

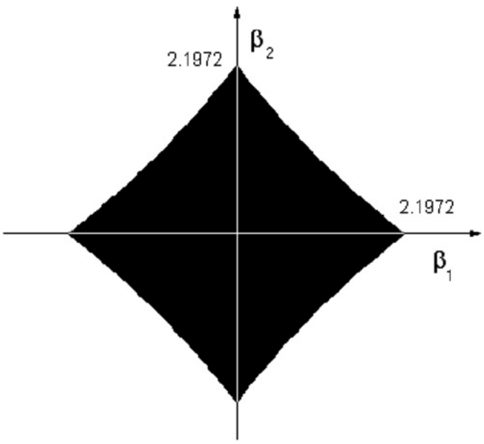


**Fig. 1.** Uniqueness region.

parameters (see Cross and Jain (1983), Derin and Elliott (1987), Schröder et al. (1997)). The uniqueness region constrains parameters and ensure uniqueness but bound models diversity. There are a lot of textures, like the one in Fig. 2, that might be not characterized by an auto-logistic model if parameter values has to lie in the uniqueness region. Images in Fig. 2

(a) 500 iterations of Gibbs sampler. (b) 8000 iterations of Gibbs sampler.

**Fig. 2.** Images of size 64 × 64, from an auto-logistic model with $\beta = (20, -20, -20, 10, 10)$.

(a) 500 iterations of Gibbs sampler. (b) 8000 iterations of Gibbs sampler.

**Fig. 3.** Images of size 64 × 64, from an auto-logistic model with $\hat{\beta}_{NR} = (0.66, -0.66, -0.66, 0.33, 0.32)$.

(a) 500 iterations of Gibbs sampler. (b) 8000 Gibbs iterations.

**Fig. 4.** Images of size 64 × 64, from an auto-logistic model with $\hat{\beta}_{SA} = (0.82, -0.95, -0.69, -0.07, 0.23)$.

correspond to an auto-logistic model with $\beta = (20, -20, -20, 10, 10)$. Image in Fig. 2(a) was generated with 500 iterations of Gibbs sampler and image in Fig. 2(b) was generated with 8000 iterations of the same algorithm.

The $\beta$ vector of image in Fig. 2(b) was estimated by maximizing pseudo-likelihood function within Uniqueness region. As assessment it results $\hat{\beta}_{NR} = (0.66, -0.66, -0.66, 0.33, 0.32)$ and $\hat{\beta}_{SA} = (0.82, -0.95, -0.69, -0.07, 0.23)$. $\hat{\beta}_{SA}$ was obtained using the Newton–Raphson method (see Johansson (2001)) and $\hat{\beta}_{NR}$ using Simulated annealing (see Bustos and Ojeda (1994), Winkler (1995)). Images in Fig. 3 were generated with a Gibbs sampler and $\hat{\beta}_{NR}$. Images in Fig. 4 were generated with Gibbs sampler and $\hat{\beta}_{SA}$. Differences between images in Fig. 2 and images in Figs. 3 and 4 are noticeable. Thus, pointing

[3] $\theta = \beta_0 + \beta_l(x_{s+v_l}) + \sum_{i \neq l} \beta_i(x_{s+v_i} + x_{s-v_i})$.

out limitations for the constrained model. However uniqueness region avoids the phenomenon known in Statistical Physics as Phase transition (i.e. $\#\mathscr{G}(\gamma) > 1$). Eq. (4.1) suggests that the theorem could be extended to larger neighborhoods and to 3D lattices increasing the dimension of $\beta$. Let us introduce the following examples, in order to include new neighbors

- at $\mathbb{Z}^2$, $g = 6$, $\partial s = \begin{array}{|c|c|c|c|c|}\hline & & s-v_5 & & \\ \hline & s-v_3 & s-v_1 & s+v_4 & \\ \hline s-v_6 & s-v_2 & & s+v_2 & s+v_6 \\ \hline & s-v_4 & s+v_1 & s+v_3 & \\ \hline & & s+v_5 & & \\ \hline \end{array}$; and $\beta = (\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)$.
- at $\mathbb{Z}^3$ (3D), $g_{3D} = 5$, $\partial s = \begin{array}{|c|c|c|}\hline & & \\ \hline & s-v_5 & \\ \hline & & \\ \hline \end{array} \times \begin{array}{|c|c|c|}\hline s-v_3 & s-v_1 & s+v_4 \\ \hline s-v_2 & & s+v_2 \\ \hline s-v_4 & s+v_1 & s+v_3 \\ \hline \end{array} \times \begin{array}{|c|c|c|}\hline & & \\ \hline & s+v_5 & \\ \hline & & \\ \hline \end{array}$; and $\beta = (\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_{3D,5})$.

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