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SURVEY PAPER

ICME international survey on teachers working and learning through collaboration: June 2016

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Abstract This article presents preliminary results from a survey commissioned for ICME 13 (2016) focusing on "Teachers Working and Learning Through Collaboration". It takes as a starting point a previous survey, commissioned for ICME 10 in 2004 that focused on Mathematics Teacher Education. The current survey focuses centrally on teachers involved in collaborations, sometimes in formal settings of professional development, but also in a more diverse range of collaborative settings including research initiatives. The roles of teachers involved in the collaboration, survey methods, decisions and limitations are described. While some of the findings to date resonate with those of the earlier survey, other findings highlight characteristics and issues relating to the differing ways in which teachers collaborate, either with other teachers or the various 'others', most notably mathematics teacher educator researchers. The roles and relationships that contribute to learning in such collaborations, as well as theories and methodologies found in survey sources, are a focus of the findings presented here. Studies rarely theorised collaboration, and few of those that did so reported explicitly on how their theoretical frame shaped the design of research methodologies/approaches guiding activities with teachers. One significant outcome has been the difficulty of relating teachers' learning to collaboration within a project, although many initiatives report developments in teaching, teacher learning and students' learning.

Keywords Mathematics teaching · Teacher collaboration · Teacher professional development · Teacher learning · Mathematics teaching development · Lesson study · Community of practice · Community of inquiry

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1 Introduction

1.1 The survey team and its task

The authors of this paper represent a Survey Team charged with conducting a survey on the topic *teachers working and learning through collaboration.* This particular emphasis zooms into the wider professional development scene to focus on the learning that occurs when pre-service and in-service teachers of mathematics work together *collaboratively*, and moreover on its implications for the mathematics learning of students which motivates their teaching. It expands on the previous survey developed for and presented at ICME 2004 (Adler, Ball, Krainer, Lin, and Novotna 2005), of which we say more below.

The notion of *mathematics teachers' working and learning through collaboration* is not new, but gains increasingly more attention in educational research and practice, particularly after the report about Lesson Study in Japan from the TIMSS classroom video study (Stigler, Gonzales, Kawanaka, Knoll, and Serrano 1999). Across education

systems, mathematics teachers work and learn through various forms of collaboration, which contribute to learning and development in differing ways. Efforts to understand what teachers do in and for improving their teaching and expertise have led to ever-increased interest in exploring and examining different activities, processes, and the nature of differing collaborations through which mathematics teachers work and learn.

At the same time, new theoretical perspectives have also been developed and proposed about mathematics teachers working and learning (e.g., practice-based professional education of teachers, communities of teachers working with communities of researchers and evolving in their professional practices, working and learning in collaborative groups, spontaneous or institutionally based). In the light of these developments, it becomes important for the international mathematics education community to survey and synthesise current research and development on the working and learning of mathematics teachers through collaboration. Understanding the current landscape will be an important step to pointing out future directions.

The team consists of researchers from different education systems around the world whose work is related to the topic of the survey. Team members are, or have been, teachers themselves, have worked extensively with other teachers in professional development settings and have themselves experienced and valued collaborative practice and gained professionally from working collaboratively with others in the field. In addition, they have synthesised, theorised and published from their experiences and research findings. Their task in this survey has been to reveal and explore the nature, extent, purpose and outcomes of collaborative activity in mathematics teaching around the world. To this task they bring a range of languages—English, French, Italian, Japanese, Portugese and Spanish—these have helped with the range of papers accessed, but also limited this range.

The team is expected to report results of the survey at ICME 13 in 2016 in Hamburg. This paper presents a preliminary insight into the findings of the survey.[1] It describes the methodology undertaken in a systematic search of associated literature, which has involved searching for articles published in research and professional journals and conference proceedings. This search has proved to be a major task, revealing a huge literature base and challenging the team to be clear about what to include and what to leave out, as well as how to organise and synthesise what is included. However, we are aware that this data is still not representative of the complete landscape for our focus: there is very much collaborative work between teachers of mathematics that is not published in these sources. We are addressing this wider scene in other ways, which will be communicated at ICME and through further writing. Meanwhile, this preliminary insight provides our overview of the data emerging from our systematic searches and adumbrates more detailed interrogations of what we see to be key parts of this data.

1.2 The meaning of collaboration

Before we report on our findings to date, we consider the meaning of "collaboration". The word collaboration comes from the Latin word "collaborāre" and means "to work jointly with others". However, the meaning "to work jointly with others" could also be related to the word "cooperation". Indeed, the Oxford English Dictionary gives the principal sense of collaborate as: "To work in conjunction with another or others, to co-operate". Peter-Koop, Santos-Wagner, Breen, and Begg (2003) suggest that both terms are related to the idea of "working together" for describing joint activities of individuals and/or institutions. However, Morris and Miller-Stevens (2016) point out that the term cooperation (deriving from *Latin* word cooperari) usually indicates that individuals contribute to various aspects of a particular task or a set of tasks that are accompanied by a well defined and concrete plan. In contrast, the term collaboration could be understood as a relational system of people or as an emergent "process characterised by unpredictability that implies negotiations and decisions" (Gray, 1989 cited by Morris and Miller-Stevens 2016).

1.3 Teachers collaborating

Thus, collaboration implies co-working (working together) and can also imply co-learning (learning together). It involves teachers in joint activity, common purpose, critical dialogue and inquiry, and mutual support in addressing issues that challenge them professionally. It helps them in reflecting on their role in school and in society.

By using the term "working", we include all the dimensions of teaching that include and go beyond face to face activity with students in the classroom: we include the didactics and pedagogy of creating the classroom environment for students to learn mathematics; the evaluation of students' mathematical learning through classroom activity and

[1] The paper is preliminary in two main respects. The first is a practical issue: despite extending the size of our original team, with colleagues who aided us in extremely valuable ways, the weight of papers and impossibility of reading them all as thoroughly as they deserved means that there may be important aspects and issues that are not as well represented at we might wish. The second is the matter of teachers' voice and teachers' associated learning related to collaboration. We shall be looking further into this as we analyse the narratives that we have collected informally. Finally, we shall be presenting our work at ICME 13 and will invite further discussion with interested colleagues at ICME 13 and beyond.

summative assessment; the professional development activity through which teachers become more knowledgeable about teaching; the institutional demands of working in a school and with national curricula and assessment; the societal demands of parents, employers and politicians; and so on.

Ponte, Segurado, and Oliveira (2003), in the context of a collaborative research project, involving two teacher educators and a teacher, point out that "the study of questions about classroom dynamics and teachers' professional knowledge requires the active involvement of teachers committed to a deep analysis of their own practices as well as those of researchers interested in teaching" (Ponte et al. 2003, p. 88). Thus, the collaboration may develop between peers, for example, between teachers working on the same project. It can also take place between people with different roles and status, for example, between teachers and researchers, between teachers and students, between teachers and parents, or even within teams that integrate teachers who teach different subjects. The collaboration occurs in such a way that the joint work involves careful negotiation, joint decision-making, effective communication and mutual learning in an enterprise that focuses on promoting professional dialogue. For Ponte et al. the success of collaborative work depends very much on the setting up of common goals and on responding to the different needs of all participants.

For the purpose of this survey, we consider teachers who are working together as collaborating for some specific aims, which could be directed towards: improving students' learning; improving their professional role in the school; learning to use new resources (e.g., technological tools); creating a professional network within the school or region; and discussing institutional reforms and demands around the curriculum, the national evaluations system, etc.

As we have said, the collaboration could extend beyond groups of teachers to include: teacher educators; researchers; parents; policy makers; heads of department or senior school leaders; and regional/provincial/municipal advisors. There is evidence of the growing institutionalisation of collaboration as some countries develop professional centres to promote particular approaches. For example, in Japan, each prefecture has a Teacher Education Centre to promote Lesson Study; in Thailand a government-funded strategy is being developed for STEM teachers (Isoda, Stephens, Ohara, and Miyakawa 2007; Inprasitha, Isoda, Wang-Iverson, and Yeap 2015). For the purpose of this paper, we refer to the participants who work with teachers in these ways as 'others'.

Although the scope of the survey is restricted to *mathematics* teachers working and learning through collaboration, here we acknowledge briefly that much has been written about teachers' (in general) collaborations. For example, Cordingley and colleagues conducted three systematic reviews of international studies between 1988 and 2004 that had been reported in English and that related to the impact of collaborative CPD (Continuing Professional Development) on classroom teaching and learning (Cordingly, Bell, Rundell, Evans, and Curtis 2003; Cordingly, Bell, Evans, and Firth 2005a; Cordingly, Bell, Thomason, and Firth 2005b), finding, for example, that "In all but one of the studies the teachers involved in the CPD interventions changed or substantially developed aspects of their teaching following the CPD intervention" (Cordingly et al. 2005a). A second body of research to acknowledge is that of computer supported collaborative learning (CSCL) and computer supported collaborative working (CSCW). This research domain "is an emerging branch of the learning sciences concerned with studying how people can learn together with the help of computers." (Stahl, Koschmann, and Suthers 2006) and, as teachers of mathematics increasingly use computer technologies for their collaboration, this literature may become increasingly important.

In education systems across the world it is not difficult to notice contrasting practices in which mathematics teachers work and learn in diverse ways through various forms of collaborations. Efforts to understand what teachers do in and for improving their teaching and expertise have led to an increased interest in exploring and examining different activities, processes, and the nature of various collaborations through which mathematics teachers are engaged to work and learn.

1.4 The meaning of community

Many of the papers that address collaborations involving teachers also speak about *communities* of teachers who are collaborating. This is often loosely defined and not aligned with particular theoretical ideas. *Community* is used colloquially to mean groups of people who engage together socially, professionally, corporately, or officially. Communities can be formal or informal. Phrases are used such as "the local community", "the Community of Actuaries", and the "Oil and Gas Community of Best Practice". When used in these ways the community membership is usually understood, although the boundaries of membership are sometimes not well defined. However, the community is usually seen to have some joint purpose—living in the same region, belonging to the same society—and some stability over time. We would usually not talk about a community "standing at the bus stop, waiting for the bus", as this is temporary and changing.

The new Springer Encyclopedia of Mathematics Education speaks of communities of teachers working together:

> [T]eachers can become members of communities of colleagues in the same school, in a network of

schools or in a teacher education program (as community of practice, in the sense of Wenger 1998) or in a research program (as community of inquiry, Jaworski 2006). They can participate in these communities in synchronous and asynchronous activities aimed at sharing materials, designing curricular plans, doing teaching experiments, collecting data for assessment and discussing results. In the second case, they can organise their classroom activities in ways that combine face-to-face interactions with distance ones, mediated by these infrastructures (Sinclair and Robutti 2014, p. 598).

1.5 Focuses and research questions

It is timely for the international mathematics education community to survey, synthesise, and propose new directions for research that is focused on mathematics teachers working and learning through collaboration. The survey team will address the specific scope and foci of relevant work that has been developing over the last decade in different education systems around the world.

Broad research questions, guiding the analysis are:

- What is the *nature* of collaborative working (to include the different roles that teachers can play) and how does this relate to situation, culture and context?
- Who are the *people* who engage collaboratively to promote the effective learning and teaching of mathematics, what are their roles, and how do they relate to each other within the different communities?
- What *methodological* and *theoretical perspectives* are used to guide and inform collaborative working and learning?
- What *learning* can be observed and how does it relate to collaboration?

2 Preliminary examples

In accord with the preliminary nature of this paper (as mentioned above) many of the references below are brief and compact. The reader is likely to want more. Unfortunately space precludes this being possible in all the cases we mention. So, in order to help the reader create images on which to base many of these examples, in this section we offer three examples of collaborative activity between teachers and others. These are chosen to be significant in their own right as examples of collaborative activity and learning; they include themes that recur in the later sections of the paper and are indicative in some way of the many examples contained within. They illustrate very different forms of collaboration. The first is established nationally; it has developed over more than 100 years in Japan where teachers work together as a normal part of their everyday activity. The second is sponsored nationally, but highly individual in its outcomes. Teachers volunteer to participate jointly, to work together in some form of innovative practice, sometimes with other researchers, and to report on their learning. The third is theoretically motivated and critically evaluated. It involves a partnership between teachers and didacticians and explores the ways such a partnership contributes to learning and development.

2.1 Example 1: lesson study in Japan

Japanese Lesson Study is the most widely known form of collaborative activity in which teachers work together and with others to improve their practice. However, there are few research-based publications in English on lesson study conducted in Japanese classrooms (Japan Society of Mathematics Education 2000). Lesson Study is a culturally embedded practice within Japanese education (Stephens and Isoda 2007). Thus we think it is important to give a brief account of the history of lesson study in Japan and of its characteristics in the original Japanese context.

The origins of Lesson Study go back to the rapid, top-down establishment of a modern education system in Japan. Starting in 1872, the Ministry of Education created a centralised school system from elementary to university level, organised around Normal Schools that prepared teachers and provided professional development based on observation of teaching methods in the classroom. The government Normal Schools developed resources, curriculum content, and textbooks for teachers. By the 1880s, the search for innovation of teaching methods led to publication of new teacher guidebooks that drew on the ideas of J. H. Pestalozzi, a Swiss educator, in order to foster a more dialogical and argumentational style of classroom communication. Professional development to help teachers master this approach was led by teachers from the Tokyo Normal School and the lesson structure created during this period differs little from today's classroom lessons in Japan (S. Shimizu and Chino 2015).

Between World Wars I and II there was a period of bottom-up Lesson Study in which teachers and mathematics educators at experimental schools worked together to develop and disseminate shared approaches and textbooks that become an origin of the theories for the mathematics curriculum (Japan Society of Mathematics Education 2010). Major themes of lesson study were related with the influence of Kline's movement for integration of subjects, and the major products were textbooks for mathematising. After World War II, within a more democratic structure, schools began to set the themes by themselves to lead

regular cycles of curriculum reform, and a known product is the Japanese Teaching Method "Open or Problem-Solving Approach."

A feature of school-based lesson study is the cycle of collaborative processes, Jugyou-kenkyu, involving teachers' planning, teaching and observation of the research lesson in an open class setting, and formal post-lesson discussion and reflection. The origins of Jugyou-kenkyu is much closer to national lesson study meetings and personal lesson studies which lead the school based one and do not always follow the same processes. Such studies adopt goals for themes that are distinct from daily teaching practice. The themes function to challenge innovation in teaching and to establish theories of good practice and curriculum for developing students and classrooms. The resulting theories have then been disseminated to support wide classroom enculturation through supportive resources such as teachers' professional journals and guidebooks. In Japan, academic research publications that draw on empirical studies are published by academic journals, which are not limited to reproducing good practice.

Lesson Study in Japan builds on a long history of teacher-led professional development and is firmly oriented towards improving teaching practice and students' learning. Isoda (2015) argues that the aim of Lesson Study is to "produce better practice for developing children" (p. 87). Thus teachers need to use their analysis of students' understanding to develop and engage in further teaching on a shared curriculum, whereas researchers who observe the lesson do not necessarily have this responsibility.

2.2 Example 2: collaborative teacher projects in England

The National Centre for Excellence in Teaching Mathematics (NCETM) in England was launched in 2006. Funded by the government, one strand of its work has been to promote teacher inquiry as a process through which teachers problematise aspects of their work as a means to improving learners' mathematical outcomes—putting a "toe in the water" of classroom-based investigations of teaching practices. In teachers' interpretation of inquiry, theory was mainly implicit and was unlikely to have been informed by more theoretical interpretations of inquiry, such as "critical alignment" (Jaworski 2006). Joubert and Sutherland (2010) reported on the findings from the first 96 inquiries, which were undertaken between 2006 and 2010. Of relevance to this survey is their finding that teachers reported changes in their practice in terms of working with other teachers:

> Many grant holders discussed changes in their practice in terms of changed ways of working with other teachers in their schools, with other teachers in their local authorities or with other colleagues. (2010, p. 9).

In January 2012, the NCETM initiated Collaborative Teacher Projects (CTPs) which included requirements to involve at least two schools and an "expert other". It was recommended that there should be a focus on mathematical proficiency.

By June 2015, forty-nine of these projects were complete and reports were made publicly available on the NCETM website. Twenty-three projects involved only primary schools and eight involved only secondary schools. The remaining eighteen projects involved both primary and secondary schools, typically a secondary school and a number of primary feeder schools.

The projects fell into three main types, those aiming to:

- produce an artefact such as a calculation policy or resources to support teaching (19);
- plan, implement and evaluate an intervention (a research study) usually in the classroom (12); and
- explicitly provide professional development for other teachers, such as by adopting a lesson study approach (18).

In terms of the mathematical focus of the collaboration, for many the mathematical content was *calculation* or *arithmetic* (16). A further nine focused on *fractions*, four on *algebra* and two on *division*. Also included were, for example, *number facts*, *number lines*, "*little big maths*", *investigations* and *subtraction*. Some did not state a mathematical focus but were concerned with improving the experience of a particular group of students, such as low achievers at the end of primary school.

For many, particularly in the first group listed above, the CTP was designed to address a particular issue. Two issues appeared multiple times: transition between different phases of schooling, particularly between primary and secondary and inconsistency in teaching approaches across different schools or even within a school. A third prominent issue was progression; teachers' lack of understanding of what has come before and what will come afterwards. Policy documents, progression charts and so on were produced to provide guidance for teachers and schools.

The grant holders were asked to state what had been learned, what impact there had been on teachers' practice, what impact there had been on colleagues and learners. They were also asked to provide advice to teachers who may want to try something similar. Generally, it seems that the inquiries yielded positive results (perhaps to be expected in self-reports prepared for a funder), with the most of the reports providing detailed accounts of learning and impact, as requested. We focus here on what was learned about teaching and learning mathematics, for

which the findings fall into three main categories, related to (a) ways of teaching, (b) mathematical content/pedagogic knowledge and (c) issues surrounding consistency of teaching and progression.

Learning related to ways of teaching included, for example, the use of rich tasks and manipulables. Crucial, however, was learning related to teaching for understanding, such as is captured in the following quotation:

> The project has helped me understand that children need a conceptual understanding of maths. It is not enough to set them a set of questions to solve. Children need to question and investigate concepts (Dowling 2013, p. 3).

For (b), teachers reported learning about specific mathematics (related to the topic of the inquiry) such as fractions, operations on number and place value. For example, one teacher reported that 'teachers have a clearer understanding of the different representations of fractions' (Ellis 2013, p. 6).

In projects related to issues of consistency and progression, it seems that the key learning was about the experience of students in both primary and secondary schools. As one teacher reported, for example:

> Primary school teachers have a much improved understanding of secondary school pedagogy and a better awareness of the next stages in the mathematical development of the children that they are teaching (Heffernan 2013, p. 4).

2.3 Example 3: learning communities in mathematics in Norway

Learning Communities in Mathematics (LCM) was a developmental research project in Norway, involving a team of didacticians—mathematics educators within a university department (n = 12)—in partnership with eight schools from lower primary to upper secondary and 25–30 teachers. The project spanned 4 years, with three of these involving fieldwork in schools. Didacticians designed the project, attracted funding from the Research Council of Norway and invited schools to join. LCM involved an inquiry-based approach to developing mathematics learning and teaching within the schools and furthering knowledge about the developmental process. Theory of inquiry and community of inquiry (Cochrane-Smith and Lytle 2009; Jaworski 2006; Wells 1999) underpinned the project. The project reimbursed schools for the costs of a replacement teacher to enable project teachers to attend workshops at the university; it was expected that a minimum of three teachers would attend from each school and that the school leaders would support the project.

LCM was based on three *layers* of inquiry: inquiry in mathematical activity with students in the classroom; inquiry in mathematics teaching development by teachers and didacticians; inquiry in the research process involving didacticians and teachers. In workshops at the university, teachers and didacticians worked together on mathematical tasks, exploring the nature of inquiry in doing mathematics, and relating experiences to possibilities for using inquiry with students in classrooms. Teachers designed inquiry-based tasks for their students, often adapting tasks they had experienced in the workshops. Teachers and students worked together with inquiry-based tasks related to mathematical topics in the national curriculum, and, when teachers requested it, didacticians videoed a selection of lessons. Videos were used for reflective activity and research analysis.

The project involved teacher collaboration in workshops with didacticians and with teachers from own and other schools; and in their own school with teacher colleagues. Didacticians also collaborated as a team, inquiring critically into their own roles in the project (Cestari, Daland, Eriksen, and Jaworski 2006). Both teachers and didacticians brought important knowledge to the project, which created tensions but led to a developing degree of partnership (Jaworski & Goodchild 2006). Early activity was led by didacticians. Interviews with teachers after 2 years of LCM showed that teachers had initially expected didacticians to tell them how to teach through inquiry. They gradually came to learn that development involved both teachers and didacticians engaging in inquiry activity, and learning from experience and reflection—which could be taken as indicative of the socialisation of teachers into the didacticians' perspective. However, over time, teachers developed their own critical "voice", communicating with each other and with didacticians what they found possible or impossible in school settings and how workshops could be organised more effectively to achieve inquiry-based goals. In workshops, teachers presented reports to the project community on activity in their classroom and school, and shared issues which had arisen (Jaworski, Goodchild, Eriksen, and Daland 2011). After each workshop, didacticians reflected as a group on workshop activity and the extent to which what had been planned achieved its goals.

Data were collected from all activity, largely through audio or video recordings, and were stored in a special database organised to allow access by all didacticians in order to facilitate reporting and paper writing. A range of research questions guided research activity, which was largely conducted by didacticians, sometimes with teacher collaboration. Over the three years of the field work, the nature of development and issues in partnership between didacticians and teachers were acknowledged and reported (e.g., Jaworski 2008; Goodchild, Fuglestad & Jaworski

2013). The Research Council of Norway funded a further project building on the work of LCM and led by didacticians, entitled TBM (Teaching Better Mathematics). This took place in collaboration with a third project LBM (Learning Better Mathematics) led by schools (See for example, Carlsen 2010). Teachers who had developed their practice through LCM became guides to new teachers participating in LBM and TBM.

2.4 The three examples

The three examples differ from each other in significant ways. Lesson study has a long history of development as a nationally enacted mode of professional development in mathematics teaching, emulated in differing forms throughout the world. Teachers are centrally involved in the development and critique of lessons along with colleagues from higher education. The NCETM Project was a large-scale professional development programme involving well over 100 projects over a period of about 10 years and involving schools and teachers in primary and secondary phases. The projects themselves involved differing focuses and methodologies. All projects were school based and resulted in teachers' learning related to their particular focus in the project. The LCM project was a 4-year project involving teachers and didacticians in partnership focusing on developing the teaching of mathematics in project schools based on inquiry processes. Methodology involved developmental research in which both teachers and didacticians engaged and learned through their inquiry. What the three examples have in common is their collaborative activity with the aim to develop learning and teaching in classrooms for the benefit of students learning mathematics. All demonstrate professional learning for teachers and others in the absence of any traditional Professional Development format (see 3.1 below, and Adler et al. 2005; Simon 2008).

3 Methodology adopted for this survey

In this section we describe:

1. How our study is located with respect to the previous survey on a similar theme that was developed for and presented at ICME 2004 (Adler et al. 2005).
2. The approaches we have adopted for the identification and selection of the data that we analysed: i.e. the criteria for the selection of sources—journals, conference proceedings and books—and the criteria for the identification of data within these sources (specific keywords, reviewing the book/journal index, scrutinising the abstracts, and so on).
3. The two methodological approaches for the analysis of this data.

3.1 Locating the study alongside the previous survey

In preparation for ICME 2004, Jill Adler, Deborah Ball, Konrad Krainer, Fou-Lai Lin and Jarmila Novotna were asked to prepare a survey that focused on research into mathematics teacher education. They considered the learning of teachers involved in programmes for the *education* of teachers, and the work of researchers studying those teachers' activity and development. Whereas this earlier survey focused on teacher education, our survey takes as its focus the work carried out by teachers when they collaborate with each other and with 'others' (researchers, teacher educators etc.). In our case, we have found extensively that teacher collaborations arise in or from professional development activity led by 'others', principally educators or didacticians. Thus, there is a fine distinction between teacher education by 'others', and teachers' development through their own collaborative activity, which we seek to elaborate. More particularly, Adler and colleagues' focus was on "what the teachers learn or do when they are involved in teacher education programmes", whereas our focus is on: how the teachers are involved in the collaboration under scrutiny; the sorts of activities that take place; how teachers interact with others; and the nature of the learning that results. With respect to the interactions, we are interested in teachers as members of communities of teachers and 'others' who collaborate towards some specific aims.

Adler's team made four main claims:

Claim 1: Small-scale qualitative research predominates;

Claim 2: Most teacher education research is conducted by teacher educators studying the teachers with whom they are working;

Claim 3: Research in countries where English is the national language dominates the literature;

Claim 4: Some questions have been studied, not exhaustively, but extensively, while other important questions remain unexamined.

Predictably our survey of the literature yielded the same sorts of findings since our broad area of interest overlaps with the area surveyed by Adler et al. However, we have approached our survey in both similar and different ways as we explain below, and this has led to other findings which we report in Sects. 4 and 5.

This paper reports on our survey *of the literature*. However, our work also aims to represent collaborations of teachers of mathematics that may never be reported widely. We draw on ad-hoc knowledge of local and regional projects and initiatives. This data has been in the form of narratives solicited from projects around the world, which

are not necessarily described in the survey literature. This aspect of our work is not included in this paper but will form part of our report at the ICME conference and will be published elsewhere.

In the methodological approach to our survey, we share some common approaches with Adler's team. For example, we start with the type of source (journal article, book etc.), the title, the authors' names and the country of the study. We have also noted the language, the geographic-cultural area and contextual factors such as the phase of education in which the teachers worked. We looked closely at the aims or focus for the work with teachers and noted whether mathematics was central to this or not. If so, which aspects of mathematics and its pedagogy were considered? We also noted who the participants were, how they had been recruited, how the participants' roles and relationships had been established and if (and how) they had evolved over time.

The research perspectives of the different sources examined, including key questions guiding the research, are of course central to the survey and both surveys specifically report on this aspect. It concerns both the collaborative work of teachers and, simultaneously, developmental themes related to teachers' professionalism when involved in educational programmes. Consequently, where studies and projects involved links between different institutions, we sought to identify where the locus of responsibility lay for the research (i.e. government ministry, university, political/economic institutions such as the European Union, professional associations, schools, the local municipality etc.). This aspect is important as it relates to the design of the research methodology or training programme alongside its constituent development activities, which have consequences for the roles and responsibilities of the participants. More explicitly, the design of research and of professional development in which teachers are involved is strictly connected to the aims of the participating institutions, the funding and the choice of participants. We were also prepared to find developmental projects led by teachers which had no specific research focus, albeit perhaps in some evaluation of the programme. Some of the NCETM teacher inquiries (Sect. 2.2 above) provide examples of this sort of project.

3.2 Identifying the data sources

As with the previous survey, this survey focuses on the work of prospective and practising teachers of mathematics at all levels of education, from pre-school to tertiary education, albeit here with the added focus on *collaboration*. As the previous survey was in 2005, we looked largely at papers published in the ten years from 2005 to 2015. In terms of the literature review, we looked at papers in mathematics education journals, conference proceedings, books, and handbooks published during this period.

From this list of sources (see Appendix 1), we engaged with a hand-search to identify research to include within our data set in which collaboration is an *explicit and deliberate* part of the research design and which explores, and explicitly reports on, the influence of the collaboration with respect to the teachers' learning and/or working practices.

Also, with the identified sources, we used manual and automated strategies to select the papers that appeared to be concerned with the topic of the survey. One strategy was the automatic search of keywords that were prominent within the literature in this field. For example, collaborate, collaboration, teachers' professional development, projects, etc. We applied this search successively to titles, indexes and abstracts and full-texts. Alongside this, we sought papers from those researchers or institutions particularly active in this area.

This resulted in an initial set of sources that were considered to be broadly in scope. Successive discussions and email exchanges enabled the team to share and hone their search strategies and, in doing so, we focused more closely on research that was not only reporting on collaborative work with teachers but also attempted to theorise about the processes and products of such collaborations as a means to advance knowledge in this domain.

These iterations led to a more selective set of 316 sources[2] that aligned closely to the survey's focus, which we then explored to enable us to comment upon emerging themes and dimensions.

3.3 Collecting, collating and analysing the sources

The analysis of our sources was carried out using a two-level approach. First of all, we used a systematic approach to specific information (e.g. country, theoretical framework, methodology,…). This information was entered into a spreadsheet under a set of headings or variables (database fields). We noted particularly the language in which a paper was written, so that we could get a sense of the domination of papers in English (expressed by Adler et al. 2005). However, it should be noted that the production of this set of database fields was a cyclical process that occurred over several months as the survey team began to identify and read relevant sources. For example, we expanded the detail that we extracted concerning the participants and their motivations and respective roles within the reported

[2] Identification of the 316 sources led to various levels of analysis in just these sources, reported in this paper. Thus, when, inevitably, further papers were drawn to our attention, it was not possible to include them in this analysis. They will be included for later discussion/publication.

collaborative work. Where possible, direct quotations were entered in the spreadsheet (with reference to the page numbers) in order to avoid imposing our interpretations of the content of the papers at the identification stage.

Our list of sources consulted can be found in Appendix 1; a table summarising the database fields and describing the aims corresponding to the choice of each field can be found in Appendix 2.

After having completed the spreadsheet, we analysed the information collected within it, adopting what we call a synthetic approach. This analysis was aimed at the identification of fundamental themes (or areas of interest) that could frame the topic of collaboration as evidenced by the research and projects described in the papers. We elaborate this in Sect. 4.

4 The framework: rationale and findings

The spreadsheet fields (described in Appendix 2) were grouped into the following three themes:

Theme 1—Different contexts and features of mathematics teachers working in collaboration.

This theme introduces the typical contexts within which mathematics teachers work and learn together in collaboration and highlights the specific characteristics of the collaboration.

Theme 2—Theories and methodologies framing the studies.

This theme focuses on the methodological and theoretical aspects connected to the ways in which mathematics teachers are involved in collaborative work. We detail the methodological approaches as far as possible and expand on what we see as the key methodologies permeating the studies. We report on the theories/perspectives that frame the research carried out and, particularly, those that influence the specific methodologies of the collaboration.

Theme 3—Outcomes of collaborations.

This theme relates to the results or outcomes reported in the papers we collected. These could relate, for example, to the learning of the teachers or researchers or lessons learned in terms of the ways in which the collaborations were organised.

We acknowledge that these themes are not mutually exclusive in that they influence and shape each other. However, they provide an analytical overview of the very large and complex data set and offer an initial frame to facilitate the objective analysis of the data by a large and geographically dispersed survey team.

The three themes relate to the research questions (RQ) in Sect. 1 as follows: Theme 1 addresses RQs 1, and 2. Theme 2 addresses RQ 3. Theme 3 responds to all the RQs. We repeat these questions here.

1. What is the *nature* of collaborative working (to include the different roles that teachers can play) and how does this relate to situation, culture and context?
2. Who are the *people* who engage collaboratively to promote the effective learning and teaching of mathematics, what are their roles, and how do they relate to each other within the different communities?
3. What *methodological* and *theoretical perspectives* are used to guide and inform collaborative working and learning?
4. What *learning* can be observed and how does it relate to collaboration?

Having identified these themes, the sources were compared and contrasted in order to highlight possible transversal dimensions (and related sub-dimensions) connected to the themes. In the following sections these dimensions are introduced and described; we expand on the rationale for each theme and present our findings alongside examples from the survey data that are indicative of the international landscape.

4.1 Theme 1: different contexts and features

This theme introduces the typical contexts within which mathematics teachers work and learn together in collaboration and highlights the specific characteristics of the collaboration.

4.1.1 Global features

The geographical spread by continent of the collaborations that were reported in the sources is shown in Fig. 1.

Of course our search methodology has greatly influenced these particular findings. Although we endeavoured to carry out a comprehensive international survey by a team with a broad language base, we were unable to review publications outside of the previously stated languages. However, the data does reveal the distribution of the sources that we found, with European, Australian and North American research dominating the picture. Of these, 21 of the 316 sources (7 %) reported collaborations that involved more than one country, and 12 out of 316 (4 %) of the studies were intercontinental.

We noted that there were few sources from Asia. We were particularly surprised that, given the impact of Japanese Lesson Study outside of Japan, there were no research papers from Japan that discussed teachers' collaborative work during Lesson Study from the Japanese perspective. However, further discussion with Japanese colleagues, revealed that Lesson Study (as detailed historically in Example 1 in Sect. 2.1 above) is an integral component of the professional work of Japanese teachers that is so

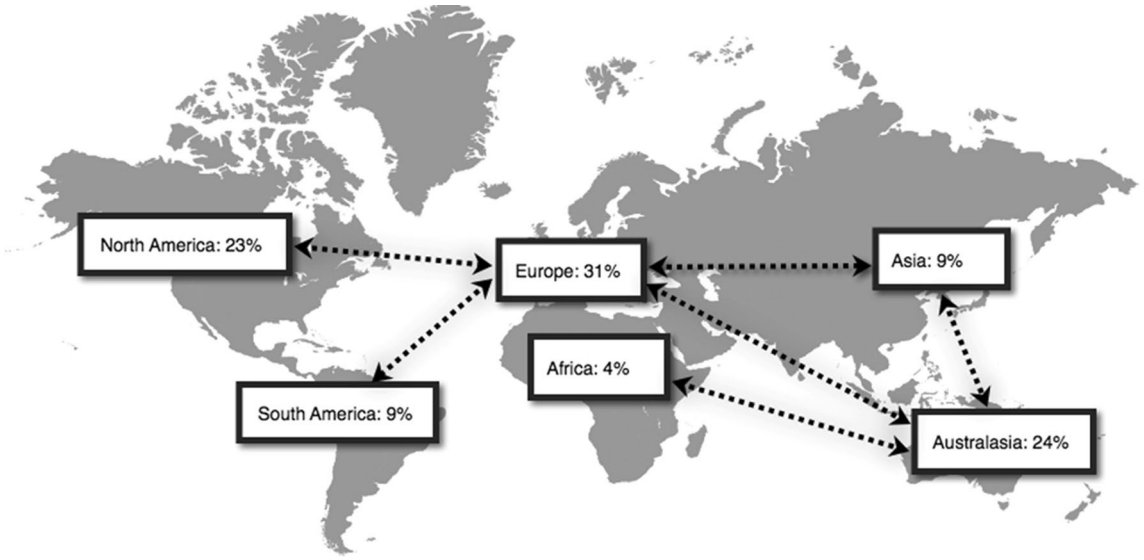

Fig. 1 The geographical spread of collaborations. The *arrows* indicate inter-continental collaborative projects

embedded in the institutional cultures of schools and universities for all subjects, it does not warrant a particular research focus.

There are also very few sources from Africa, and the majority of those are from South Africa. We suggest, from broader reading around the topic, and from knowledge of the topics that seem to be of interest in African research, three main reasons for this. The first is that, although more papers about collaborations involving teachers are published, almost all those we found concern teachers generally, not *mathematics* teachers specifically. The second is that it seems that collaborative ways of working (for teachers) are frequently so far removed from the cultural norms in many African countries that this does not take place. The third is that, to some extent at least, African researchers have other more immediate concerns and researching about collaboration may not have a high priority.

Across the world, it is also very apparent that teachers are working collaboratively on aspects of mathematics in all levels of schooling from "early years" through to tertiary and university level education. Although the global differences in school systems results in different terminology and timelines relating to levels of schooling, where this information was stated explicitly, we have classified the sources accordingly, leading to the graph in Fig. 2.

Across the world, younger children are mostly taught by non-specialist teachers who teach a wide range of subjects, whereas older children are more often taught by specialist mathematics teachers (who might also teach one or two other subjects). We have chosen to classify these two groups of teachers as primary and secondary, respectively. A small number of papers report on collaborations involving

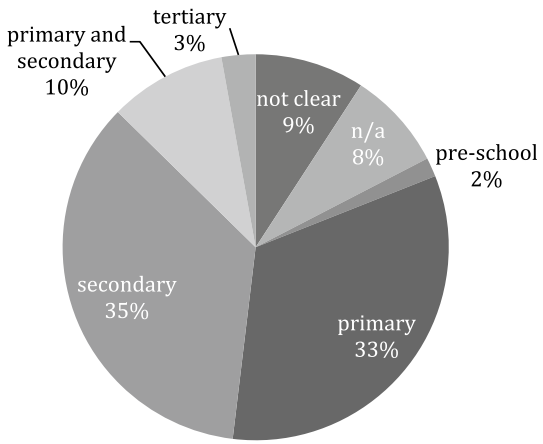

Fig. 2 Reported educational phase in which the teachers worked (n = 316)

teachers at university level (classified as "tertiary") or teachers of pre-school children (classified as "pre-school"). In some sources, the level of schooling in which the teachers work is not clear and we have classified them as "not clear". In other sources that focused more on theoretical notions related to collaboration and did not refer to specific collaborative work with teachers, the level of schooling is not relevant and we have classified these as "n/a".

Perhaps unsurprisingly, a large proportion of collaborations involve only primary teachers (33 %) or secondary teachers (35 %) with 10 % of papers reporting on collaborations involving both primary and secondary teachers. With regard to teachers at these two levels learning from each other one source makes a relevant point:

> This study suggests that professional development that supports growth for high school teachers may be different from elementary teachers. The interaction between high school teachers [secondary] and elementary teachers [primary] with their different expertise was critical to help a traditional high school teacher re-examine her own content knowledge and identity (Olsen and Kirtley 2005, p. 4).

The NCETM teacher enquiries described in Sect. 2.2, provide a number of examples of primary and secondary teachers working together to address the issue of progression or transition between these phases of schooling. For example, teachers reported that they had better understanding of curriculum expectations and school practices at different levels. Fourteen teachers stated that they had learned about what is taught, and how it is taught, at primary/secondary level, with some stating this had helped in terms of their ability to make the transition from primary to secondary level easier for children.

Far fewer sources report collaborations involving preschool and tertiary teachers (about 2 % each). In 8 % of the papers the schooling level of the teachers is not relevant and in quite a large number (9 %) it is not mentioned, nor could it be ascertained from the text or the survey team's knowledge of the author's (or authors') work.

4.1.2 The initiation, foci and aims of collaborations

4.1.2.1 Rationale A first interesting aspect to highlight is how the collaborations reported in the sources were initiated, the focus for the collaborative work and its respective aims. These were all notable variables that were relevant to analyse and, in particular, we were interested in how they might be inter-related.

4.1.2.2 Findings We report that the ways in which collaborations were *initiated* were wide and varied. In the following we provide examples of each form of collaboration, with brief details of one or more papers. Our purpose here is to signal the diversity of areas reported and provide very brief insights into research foci.

These included:

- Initiatives mandated by ministries and national/regional institutions (Nickerson and Moriarty 2005; Cooper, Baturo, and Grant 2006). For example, Nickerson and Moriarty report "The elementary mathematics teachers are part of a reform of mathematics instruction in the eight lowest-performing schools in a large urban school district in the western United States" (p. 113).
- Collaborations supported by ministries and national/regional institutions (Miyazaki 2015; White 2007). For example, White reports on a study contrasting teacher professional development in two related approaches to teacher professional learning in the Asia Pacific Region—one involving lesson study in Australia and the other involving Active Mathematics in Classrooms (AMIC) implemented in Brunei Darussalam.
- Research collaborations initiated by researchers (Carlsen 2010; Cavanagh 2012). For example, Carlsen reports on a research project in Norway that involved 10 schools from kindergarten to upper secondary and a team of didacticians over a four-year period (TBM/LBM noted Sect. 2.3 above). The paper addresses the role played by inquiry in orchestrating a mathematical activity in the kindergarten and shows how teachers and one didactician worked together to develop teaching practice.
- Professional development initiated by researchers/didacticians (Lin 2007; Goodchild et al. 2013; Gellert 2008; Ell and Meissel 2011). For example, Lin's paper focuses on a collaborative mentor study group, consisting of four mentors and the researcher, set up to enhance mentor development in Taiwan. By contrast, Ell and Meissel describe how teachers in seven rural primary schools in New Zealand decided to form their own teacher-led cluster to improve numeracy in the schools. They invited a local numeracy facilitator and a researcher to join the group to offer particular areas of expertise.
- Within-school collaborations that were both initiated and sustained by the teachers without the direct involvement of 'others'. For example in the research in three English secondary school mathematics departments as reported by Watson and De Geest (2014).

The different ways of initiating the projects often overlap. For example, a research project or a professional development approach introduced by a group of researchers could be the starting point of a wider national initiative, involving huge numbers of teachers. See, for instance, the introduction of Lesson Study as a top down initiative in Indonesia, raising tensions for teachers who were unprepared from a sociocultural perspective for the expected levels of collaboration (Kusanagi 2014).

It should be stressed that most of the sources do not declare how the collaboration between teachers and others was initiated. This could be due to the fact that, often, this kind of information is not considered part of the necessary aspects that should be discussed within the sources we scrutinised.

For the *foci* of the collaborations, we began by scanning the titles of the papers. A first impression of the landscape of teachers working and learning in collaboration can be gained from a simple "word cloud' of the titles of the 316 sources that comprised our data set (see Fig. 3). Following

Fig. 3 A frequency cloud of words that featured within the titles of sources (n = 316)

the methodology of Joubert (2013), we used the titles of the sources to begin to provide us with an understanding of the landscape of mathematics teachers' working and learning through collaboration. To do this, we took all the titles as data and "cleaned" this data by replacing similar words with the same word. For example, collaboration and collaborative were replaced with collaborate; teaching and teachers were replaced with teach and so on. The diagram below provides a visual representation of the main words used, with the size of the word proportional to the number of times it appeared.

This demonstrates that although mathematics, teach, develop, learn and professional all appear relatively frequently, which is unsurprising, the only other word/phrase that seems to be important in terms of frequency is lesson-study (which indicates the pervasive nature of LS). In part, this may be because many titles do not describe the key concepts in the papers and are rather designed to pique the interest of the reader. Two examples illustrate this: "The transition of a secondary mathematics teacher: from a reform listener to a believer" or "How pre-service teachers use experiments for understanding the circular billiard". The graphic does indicate that the words "community" and "collaborate" appear with a very similar frequency to "lesson study" and are also more frequent than many other words that were commonly found within the titles.

As we began to analyse the different foci for the collaborations in more depth, it became apparent that these fell into two broad categories. The first category concerned aspects of innovation about: *mathematical content* (algebraic thinking, teaching of fractions, geometry etc.), the development of *new curricula* (reform agenda, examination changes etc.), *different pedagogical approaches* (formative assessment, problem solving, discussion etc.), and the *integration of new tools and resources*, to include digital tools. The second category was more associated with the different practices designed to foster *teachers' professional learning*. This resonates with Simon's (2008) classification of teacher professional development efforts into two categories: those that focus on process and content goals whereby "teacher educators aim to promote particular mathematical and pedagogical concepts, skills and dispositions"; and those that focus on process goals, e.g. teacher inquiry and teacher research (Simon 2008, p. 17). Simon indicates specifically that "Lesson Study" fits into the second category. It has been interesting to address just how much of the reported learning through collaboration is instigated and led by CPD leaders. We return to this idea later.

When considering the aims for the collaborations, we were aware that there could be multiple and overlapping aims for the different participants. For example, teachers' aims might commonly be concerned with improving learners' mathematical outcomes, whereas researchers' aims might be more concerned with eliciting teachers' professional learning in the same context. However, the degree to which the aims of the different participants were articulated within the sources and, more importantly shared between participants, was only visible in the survey data if the authors had chosen to comment on these aspects. Muñoz-Catalán et al. (2010b), for example, when analysing the case study of a primary teacher participating in a collaborative project for professional development, chose to focus comments upon the teacher's aims for her involvement in the project.

Typical aims connected to the first category of foci (usually expressed by researchers or teacher educators) were to promote the development of teachers': (a) awareness of students' different learning trajectories; (b) necessary competencies to foster students' learning (in the phases of designing lessons, implementing in the classroom, the analysis of practice, reflecting on and redesigning); and (c) understanding of the ways in which the different teaching and learning resources support/inhibit learning.

As regards (a), Fried and Amit (2005), for example, report a project in Israel that involved 82 middle and secondary phase mathematics teachers from 31 schools who were "encouraged to discuss teaching approaches required by the students at each grade level and the relationships between the different stages of the development" (p. 419).

A collaboration that could be connected to (b) is the "Pendikan Matematika Realistik Indonesia" project (Brodie 2012), which aimed to implement a new way of teaching and learning mathematics through the development of exemplary curriculum materials. Another research project that focused on the teachers' development of important competencies to foster students' learning is described by Sembiring, Hadi, and Dolk (2008). The authors analysed the results of work carried out within a small professional learning community that specifically focused on improving teachers' discourse-based assessment practice from convergent formative assessment to more divergent formative assessment. We are aware of considerable research in the areas of the development of curriculum materials and of formative assessment. However, we address these only through our main focus of teacher collaborative working and learning. A possible way of supporting teachers to understand the ways in which different teaching and learning resources support/inhibit learning (c) might involve them in the design of activities, as active learners for their professional growth, as in the study reported by Chen and Chang (2012). Here, four teachers formed a professional learning community, led by a teacher educator researcher who focused on improving teachers' discourse-based assessment practice.

A typical aim connected to the second category of foci is to evaluate the implementation of specific processes and tools as professional development programs for mathematics teachers. Krammer, Ratzka, Klieme, Lipowsky, Pauli, and Reusser (2006) describe a project that aimed to examine the conditions and effectiveness of web-based professional development with classroom videos to support mutual exchange, shared reflection and reciprocal analyses of instruction. The use of video recordings to support collaborative work with teachers is discussed by Coles (2012), who focuses on the role of the discussion facilitator in such processes. Another interesting example is the initiative described in the paper by Fede, Civil, and Toscano (2014), who present the results of a study focused on the use of "Odyssey", a hybrid space intentionally constructed to engage prospective teachers and mentor teachers in joint explorations of mathematics.

If we consider sources where the aims are made explicit, only some of the collaborations state dual aims where, on one side, the researchers are researching the teachers' learning, and, on the other side, the teachers themselves are engaged in research. For example, Hunter and Back (2011), in the UK, highlight the participants' different objectives when involved in a Lesson Study process: on one side was the researchers' objective to explore Lesson Study as a means of sustainable PD and, on the other side, were the specific goals set by different groups of teachers for their classroom based research that were relevant to their school contexts. If we select sources where the teachers *are* involved in the research process, there are notable differences between the kinds of questions that are posed, by whom and the extent to which these are they negotiated with the teachers. In their collaboration between researchers, teachers and university students that took place over a 10 year period, Nacarato and Grando (2013), in Brazil, report how their research community was characterised by the identification of a common purpose and by a non-prescriptive and non-hierarchical relationship between participants (academics and teachers who were all counted as researchers). The authors emphasise the processes through which the members jointly developed materials, collected data, and analysed and systematised their results.

4.1.3 Collaborative ways of working and their conception

4.1.3.1 Rationale Whereas the previous dimension is mainly concerned with the focus and aims of the work with teachers, this second dimension relates to the collaborative approaches that are adopted within the communities. In particular, this dimension considers a further and important aspect to be scrutinised. How is collaborative work within the community both conceived and activated?

4.1.3.2 Findings A condition of the selection of sources was that they focus, in some way, on communities of teachers working with colleagues or 'others'. However, the reasons that underpin the conception and implementation of the collaborative work could be very different.

In some cases the development of these communities is a declared objective of the collaborations. Potari (2013), for example, in Greece, analyses the effects of a course specifically conceived to support teachers to construct bridges between teaching and research and develop an inquiry approach to mathematics teaching and learning. She stresses that the intentions from the side of the mathematics educator "are not to transmit knowledge from the research to the teachers but to form a community of inquiry where the teachers use research as a tool for their inquiry" (p. 509).

In other cases the development of such communities is a means to reach other objectives (that is, a methodological approach for teacher education). In the training program described by Martins and Santos (2012) in Portugal, for example, the collaborative work is aimed at developing teachers' abilities to reflect over time, through the stimuli from other people involved in these reflections (mentor, tutor, supervisor and critical friend).

Most of the methodological approaches to the creation of collaborative contexts for teachers presented within the papers we analysed have common characteristics. First of all, these approaches involve *cycles* of activities. Often these activities include: the study of specific materials (to include research literature), the design of classroom activities, the implementation of these activities, their analysis and a consequent re-design and re-implementation.

This focus on the analysis of teachers' practices reveals another fundamental common characteristic of the methodological approaches—the activation of processes of reflection. This is stressed, for example, by Martins and Santos (2012), who write "It is not only important to state that the teacher reflects, it matters that the teacher is aware that he is so doing, what must be considered within the process and of the intention underneath it. To that purpose several strategies may be adopted such as the intervention of a person stimulating it (mentor, tutor, supervisor, critical friend), reflections sharing within collaboration works or the use of written reflection" (p. 194).

Other important features are:

- the importance of fostering teachers' willingness to participate in teacher education projects and initiatives (e.g., Gestoso de Souza and Anunciato de Oliveira 2013).
- The fundamental role played by expert figures as reference models for teachers such as: other teachers; teacher educators; mathematicians; and researchers. In Hoek and Gravemeijer (2011), for example, this role is played by the researcher, who observes the teachers' practices and shares his interpretation of these practices with the teachers, who are involved in a subsequent re-planning of the lesson. In Nyaumwe (2009), the support in the reflection to be developed on the lessons taught by teachers is provided by peers.
- The fostering of a teacher's engagement, within the communities, characterised by challenge, solidarity and accountability (Brodie and Shalem 2011), but also trust and respect (Dawson 2008).
- The reference to theoretical lenses to introduce a specific topic or to support the teachers' analysis of their practices and the sharing of their reflections. In the study reported in Verhoef, Tall, Coenders, and van Smaalen (2014), for example, the teachers were asked to present and to discuss research papers in a Lesson Study team seminar as a starting point for the design of a research lesson.[3]

[3] In relation to our findings reported in 4.1.3 above and 4.1.5 below, we should have liked to probe more directly and deeply into the manner in which collaboration actually takes place, in order to strengthen the findings. Such consideration is highly relevant but demanding on space. Since we have already far exceeded our space limits on this paper, we reserve some issues and considerations for further publications.

The terminology used in the papers in relation to this dimension varies greatly from source to source: communities of practice, communities of inquiry, professional learning communities. Indeed, 91 out of 316 papers mention community or communities of some form in their abstract. Of these we explored whether they specifically mentioned the kind of community that was involved. A rough analysis showed 25 % each of the following categories in this data: Professional Learning Communities (sometimes labelled PLC; some including Lesson Study), Communities of Practice (often with reference to Wenger), Communities of Inquiry (often with reference to Wenger and/or Jaworski), and "Other communities (which included a diversity of characterisations including lesson study, video clubs, online communities, school communities). We expand on the nature of such communities later in Sect. 4.2.

The vast majority of the sources (about 80 %) explicitly present the research methodology for the work done. In some, but not all, cases, the descriptions are very clear. For example in their study, Marquesin and Nacaranto state their methodological choice as a qualitative paradigm, which focuses on the analysis of transcripts of initial interviews and reflective conversations; researcher-educator's field notes; transcription of audio recordings of meetings; and narratives produced by teachers (2011). We comment further on research methodology in Sect. 4.2 below.

4.1.4 The scale of collaborations (numbers of teachers and time-line)

4.1.4.1 Rationale We anticipated that the survey data would reveal differences in the scale of collaborations, as indicated by the numbers of teachers involved and the timeline for the collaborative work. We developed a graphic that indicates the reported numbers of participants and the time period for the collaborative work (See Fig. 4) that would facilitate a quantitative analysis and enable us to comment on the frequency of each type of collaboration.

The significance of the numbers involved relate to scale and influence of projects. Adler et al. reported that most papers they reviewed were about small-scale projects. Our review shows some movement to larger scale, especially in terms of more lengthy projects as we see below.

4.1.4.2 Findings Approximately three quarters of the 316 sources (238) provided explicit information about the numbers of participants *and/or* the duration of the collaboration. The breakdown of this information is shown in Fig. 5. Of course, some sources (25) were not about actual collaborations but were theoretical in nature, for example, the paper by de Carvalho Borba and Llinares (2012). There were 19 sources that did not provide this information.

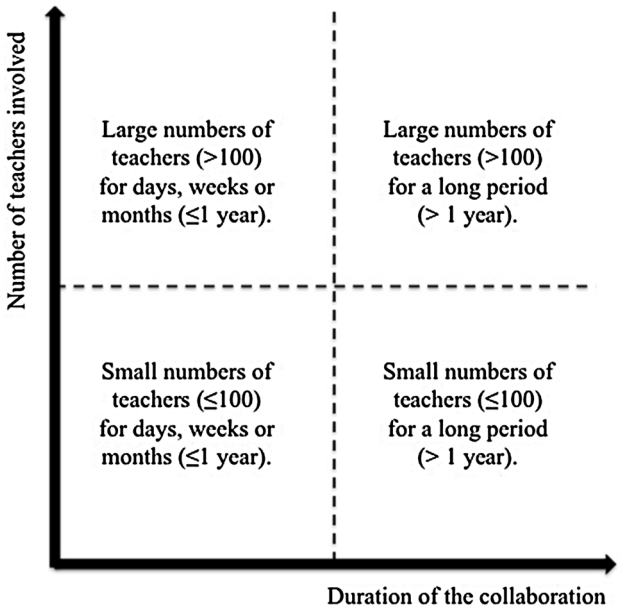

Fig. 4 A classification of the scale and duration of collaborations

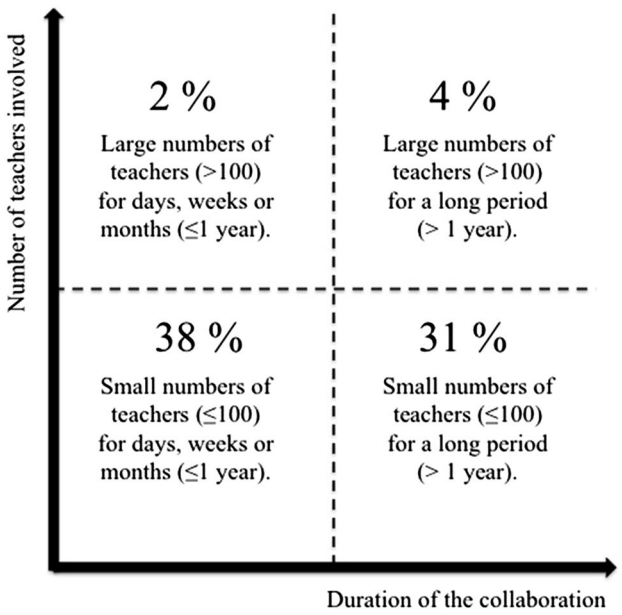

Fig. 5 The distribution of the scale *and* duration of collaborations in the sources, where explicitly reported (n = 238)

The remaining 34 of the 320 sources gave partial information—*either* the number of participants *or* the duration of the project. These were classified as shown in Table 1.

The categorisation of the different sources in this way indicates a slight increase in the proportion of larger scale collaborative projects (>100 participants) being reported (24 of 238 empirical studies, approx. 10 %) when compared to the 2004 ICME Survey findings (10 of 145 empirical studies, approx. 7 %). Furthermore, the longitudinal nature of approximately one-third of the reported collaborations suggests that sustained work with teachers is highly valued by the different communities.

The four examples that follow feature highly collaborative projects that were classified in each of the four quadrants of Fig. 3.

An example of collaborative project that could be situated in the bottom-left quadrant is that presented by Allmond and Huntly (2013), who reports of the experience of a Lesson Study group in Singapore, comprising seven mathematics teachers and a school leader that focused on the use of a model of productive mathematical noticing over a 6 week period.

Collaborative projects focused on Lesson Study could also be situated in the top-left quadrant. Lewis and Perry (2014), for example, present a study that involved thirty-nine teams of primarily elementary teachers in the USA. The PD programme was focused on the use the Lesson Study resource kit to improve facets of teachers' knowledge of fractions.

The collaborative work analysed by Da Ponte (2008) and carried out within the "Grupo de Trabalho de Investigação of the Associação de Professores de Matemática" with the aim of involving teachers in the study of their own practice, is a typical example of those collaborative projects that could be located in the bottom-right quadrant. Furthermore, it was noticeable that many of the collaborations that were located in this quadrant had taken place in South America. We found no clear reasons to explain this fact, but it seems worth noting that it deserves a deeper analysis.

Concerning the top-right quadrant, the NSF Project (Dawson 2008) represents an example of a large scale collaborative project, throughout Micronesia, led by forty mathematics educators who acted as mentors for ten

Table 1 Classification of collaborations where only partial information was provided (n = 34)

Collaboration	
Large number of teachers (>100) but no temporal scale	4
Small number of teachers (≤100) but no temporal scale	20
Collaborations over days, months or years (≤1 year) but numbers of participants not specified.	0
Collaborations over a long period (>1 year) but numbers of participants not specified.	7

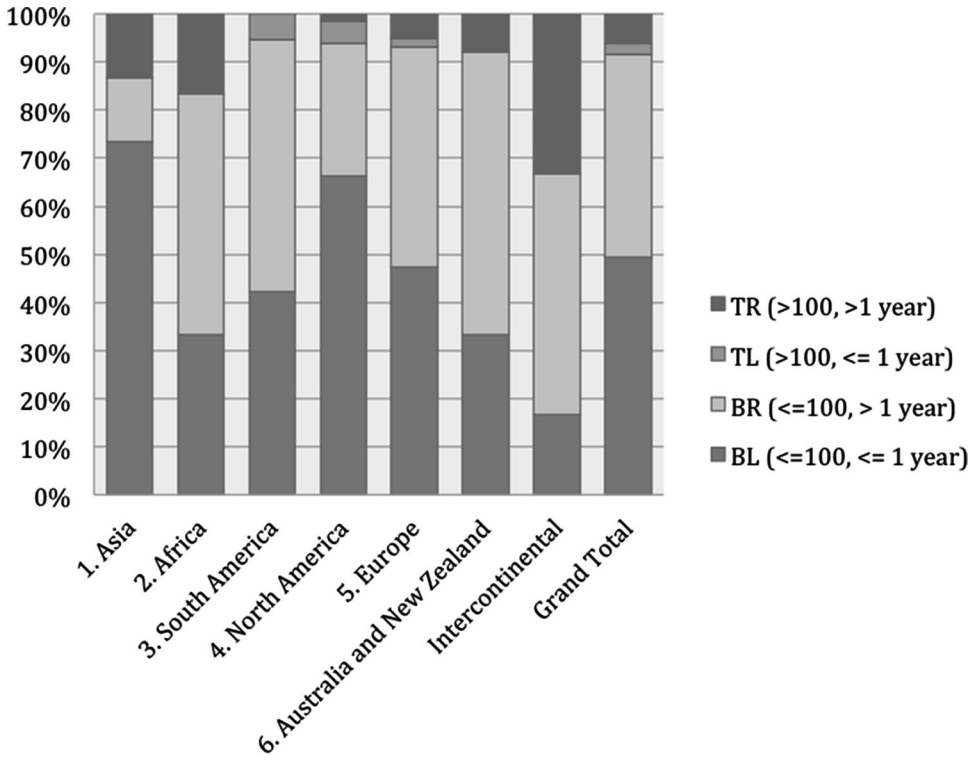

Fig. 6 Distribution of types of collaboration by continent

communities of elementary and secondary teachers of mathematics. Dawson stresses, in particular, how, while the project involved large numbers of people, places and distances involved, the mentors played a crucial role in enabling developmental activities related to mathematics learning and teaching, alongside supporting the mutual trust and respect to permeate the project community.

We analysed the distribution of the different classifications of collaborative initiatives by continent (Fig. 6).

This graphic reveals the international variability in the type of collaborative work that is being reported in the sources. In addition, although the vast majority of collaborations involved fewer than 100 participants, there were a higher proportion of reported collaborations of over one-year duration in Australia and New Zealand compared to shorter collaborations. This balance was reversed in North America whereas in Europe there was an almost equal distribution.

4.1.5 The composition of collaborative groups and the roles of the participants

4.1.5.1 Rationale Within any collaborative group, it is likely that the participants in that group take on different roles at different times. For this reason, it is important to describe the participants within the groups and to analyse how they are *involved*, *interact* and *support* each other in their activity. In other words, we need to consider how they *work* together. In Sect. 4.3, we articulate how the teachers and 'others' *learn* in and from this work.

4.1.5.2 Findings Our survey, like that of Adler and colleagues earlier, indicates that the majority of sources are written by researchers working in the field of teacher education. It follows that such collaborative groups mainly comprise (mathematics) teacher-educator-researchers and pre- or in-service teachers, who might also participate as researchers. However, the composition of these communities varies between collaborations. In some cases, other participants are included in the communities, such as: the school principals (Vale, Davies, Weaven, Hooley, Davidson, and Loton 2010); members of specific cultural communities (Howard and Perry 2007); community leaders (Owens 2008); education assistants and officers (Hurst, Armstrong, and Young 2011); district leaders (Jackson, Cobb, Wilson, Webster, Dunlap, and Appelgate 2015); curriculum leaders (Groves 2013) and doctoral and masters degree students (Sensevy, Forest, Quilio, and Morales 2013).

We now consider the *involvement of participants*, and in particular the different ways in which they have become part of the community. For example, Goos (2014), from her own experience and perspective as a researcher, analysed fourteen different projects and categorised these in three different ways that indicated how researchers and teachers had begun their collaborative work thus: "Researcher seeks teacher participants; Funding body (school system) invites

and/or selects schools to participate; and Mutual colleague initiates interaction—thereafter Teacher may seek Researcher" (p. 192). However, our survey has revealed that this information is not always declared in the sources.

Often, these teachers may already have a professional relationship with the researchers, or they may have a particular interest or motivation to participate in national/local initiatives. Sometimes, "the willingness to be part" of the project represents a criterion established by the researchers themselves (Elipane 2012). There are also examples of voluntary collaborative work within networks "meetings for groups of teachers from different schools or colleges who get together to mutually support one another" (de Geest, Back, Hirst, and Joubert 2009, p. 23).

An example of the creation of an online group as an alternative to the "official" university learning systems is presented by Goos (2012), who established a Yahoo Groups bulletin board with the aim of fostering the professional socialisation of beginning teachers. This created a space where pre-service teachers could voluntarily contribute, without being graded for assessment purposes and where the role of researchers was limited to modelling and encouraging professional dialogue about issues raised by pre-service teachers during face-to-face classes and practicum sessions.

Another example of an online virtual community is Connect-ME (Dalgarno and Colgan 2007), which simultaneously represents a website, a repository of exemplary resources, a virtual meeting place, and a 24 h helpline for novice elementary teachers. One of the peculiarities of this community is that it is shaped and self-regulated by both student teachers and graduate teachers.

Although we are aware that teachers do take part in communities born in a more spontaneous way, such as within online fora (for example synchronous and asynchronous open discussion is being convened using Twitter using hash tags such as #mathscpdchat and #mathpd), our survey revealed no actual examples of such collaboration.

The ways in which the participants *interact and support* each other is closely linked to the roles they play within the community. Although it was not always possible to deduce from our sources how the participants' roles and the relationships within the communities were established, some sources explicitly address this theme. It is clear that teachers can adopt different roles, depending on the initiative. As this survey is focused on collaborative work, the sources reveal aspects of the teachers' more active roles during various phases of collaborations.

For example, Gilbert and Gilbert (2013) present a model of professional development that involved teachers in collaboration with researchers, within professional learning communities, in designing and piloting curriculum revisions. Another example of this kind was within the Comenius project "Understanding of mathematics classroom culture in different countries" (Hospesovà, Machàckovà, and Tichà 2006) where the key feature of collaboration between teachers and researchers was the "equal status of all members of the team in all areas of work, i.e., when preparing, carrying out and analysing instructional experiments" (p. 100). The authors report that this "equal status" was underpinned by the idea that all the members of the community had the same level of responsibility, despite their different roles and interests, and the equal status was fostered through the community's involvement in a collaborative work characterised by the following phases: (a) discussion on the mathematical background of the teaching experiment; (b) independent preparation of experimental lessons by teachers; (c) teachers' identification of the teaching episodes to be analysed and individual reflections on the video-recordings of these episodes; (d) joint reflection, by the whole community, on the chosen episodes.

This idea of "equal status of all members" can also be found in projects aimed at establishing communities of inquiry, where teachers are drawn into the developmental process so that they become "*co-learners in partnership* with didacticians" (Goodchild 2013). However, these sources reveal that there are a number of tensions that need to be resolved to achieve equality of participants' status.

During the 2006 IGPME Research Forum "Teachers researching with University Academics", three teacher researchers presented their experiences to highlight those themes and results that are sometimes not so explored by academics (Lebethe, Eddy, and Bennie 2006). These teachers declared: "This proved to be an extremely complex experience as our first hurdle was to convince the traditionally-minded academics who were presenting a generic Research Methods course that our planned research was legitimate and acceptable. Despite the difficulties that each of us experienced along the way, we were pleased when each of us received recognition from the academy that our work was worth the award of distinction" (p. 100).

Another role that can be assumed by teachers, which highlights a deeper level of involvement, is that of *teacher-researchers*, in which they are fully engaged in all phases of the research process—from planning to implementation to data analysis to dissemination; for example, within a US federal-funded Master's programme that explicitly aimed to develop teacher-researchers through collaborative action research projects for which the teachers had decided their individual foci (Kyei-Blankson 2014).

Other projects involve the teachers as mentors for other teachers, or leaders of sub-groups of teachers, and go further to seek to analyse the roles of the "expert". For example, Ding, Jones, Pepin, and Sikko (2014) focused on the expert teacher's voice within a school-based teacher professional development study being conducted in a local

laboratory school located in a western suburb of Shanghai. Their analysis of the expert teacher's input in discussions with junior school teachers highlighted that expert teachers represent key stakeholders in the research community, a conclusion that has also been made by Kieran, Krainer, Shaughnessy, and Clements (2013).

An interesting distinction is between the projects that implicitly perceive a static role for the teachers and the projects that explicitly involve a dynamic evolution of the teachers' role (from participants, to leaders, to teacher educators, to teacher-researchers, and so on…). Sometimes, the dynamic evolution of the teacher's role is an integral part of professional development programmes/research projects, since it is directly aligned to the development of teacher autonomy. When this evolution is described and discussed, researchers are able to report on aspects of the professional change in the teaching practices over time. Elements of this evolution can be seen in the Norwegian projects described in Sect. 2.3.

Few papers document and explore the teachers' perceptions about their participation and role, explicitly highlighting the importance of all participants understanding the differing but parallel aims of their participation in the projects. In recognition of this finding at a fairly early stage in analysis, the survey team explicitly sought teachers' perspectives wherever they were aware of activity and projects around the world. This has resulted, as mentioned already above, in a set of narratives which are still being analysed and will be reported on elsewhere.

A focus on the researchers shows they may also play different roles. The vast majority of sources have authors who also have roles as instructors/educators within the courses or the developmental activities and are studying the processes in which they are involved. In some cases, this dual role is recognised and scrutinised by the researchers themselves, as in the paper by Goos (2012), who stresses that, when the researcher plays dual roles as a facilitator of teacher discussion and the researcher of its effectiveness, the impact on teacher professional learning is significant. Potari (2013) also highlights the crucial dual roles played by the academic researcher, "as a teacher in the context of mathematics teacher education and as a researcher who produces new research tools in the activity of her own research" (p. 517).

Researchers may also not be teacher educators, but periodically meet the teachers to compare and share ideas and reflections, as in the study reported by Kotelawala (2010), who analysed three teachers' joint planning of curriculum units and after-school meetings to share ideas and reflect on their teaching struggles and successes during a school year.

The analysis of the dynamics and relationships between teachers and researchers and of the ways in which these relationships may, or may not, shift over time is another interesting aspect that some research scrutinises and highlights. Besamusca and Drijvers (2013), for example, observe that, although at the beginning of their project, the researchers intended for authority between members to be equally divided as the project progressed, an unintended shift of the power happened, making the researchers the authority figures. They report, "This change was unintended and likely due to the members settling into their basic roles. In other words, the researcher, who initiated and guided the project, was the natural authority figure, while the teachers, who applied for the project, naturally followed his lead" (p. 10).

4.2 Theme 2: theories and methodologies framing the studies

4.2.1 Theories that frame the studies

4.2.1.1 Rationale Although some collaborations, such as many of the NCETM groups (Sect. 2, Example 2), do not appear to base their work on any explicit theory, it is likely that a collaboration of and for mathematics teachers might ground its work in a variety of theoretical perspectives some of which are implicit or tacit. In all cases however, we assume that approaches to developing teaching practice are underpinned by a set of beliefs, which may stand for 'theory'.

A first set of theories discussed in the papers relate to classroom teaching and learning, such as learning mathematics through problem solving, or how children learn about ratio. A second set relate to theories about adults learning in collaborative groups such as collaborative inquiry or Lesson Study (Otani 2009; Isoda 2015). We would expect some theories related to teacher identity, knowledge and beliefs to appear in this set. The extent to which the collaboration draws on either of these sets of theories clearly varies, as does the extent to which the theory is shared amongst the participants of the collaboration. Our emphasis, given the scope of the survey, is on the second set of theories. It must be said that many theories relevant to mathematics teachers' collaboration, are much broader in their scope. In our accounts below, we are interested mainly in how they apply in our survey.

For example, implicit in the second set of theories, is some perspective on what it means to learn and the nature of knowledge in collaborative settings. Within our scope, we take this to mean collaboratively learning to teach mathematics, with concomitant knowledge relating to mathematics, to teaching and to collaboration. Papers do not always address this explicitly, but we might assume there to be some kind of 'social' theory behind the goals for research relating to teachers learning through collaboration and acknowledging the dimension of socialisation which appears to play a central part in such learning. Such 'social'

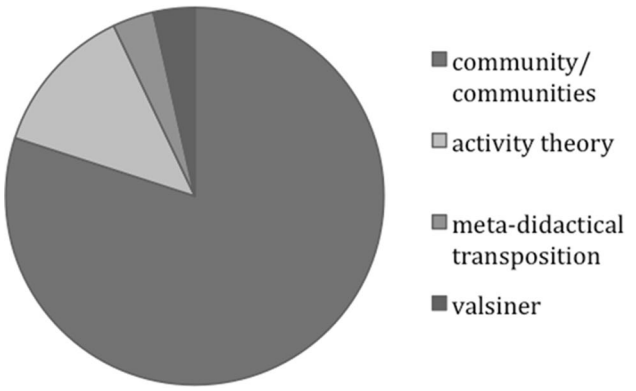

Fig. 7 The nature of theoretical perspectives concerning the collaborations where explicitly stated (n = 55)

theory may take some form of constructivism (e.g. Cobb and Yackel 1996; Ernest 1991) or some form of sociocultural theory (e.g. Lerman 2000). In the papers surveyed, a sociocultural focus is more common, often relating explicitly to Vygotskian theory, and can be seen in different forms such as Social Practice Theory, Learning Communities, Communities of inquiry, Activity Theory or Valsiner's Zone Theory. We recognise that such areas of theory have very different ecologies, with for example Activity Theory having a long history developing from Vygotskian theory over three generations. Valsiners' theories also relate specifically to Vygotskian theory but have a much more recent origin. Other theories are also much more recently conceptualised. When knowledge is addressed explicitly, it is often with reference to Shulman's "Pedagogical Content Knowledge" (e.g. Ball, Hill, and Bass 2005; Rowland, Huckstep, and Thwaites 2005).

Further, we are interested in how the theoretical framework or perspective informs what the teachers actually did, how they were included in and influenced through collaboration, how the collaborations were enacted and ultimately, how the findings informed the development of theoretical knowledge and practice about mathematics teachers' collaborative work. The sections below report on what we found.

4.2.1.2 Findings As expected, the survey has revealed a diversity of different research theories or perspectives that frame the design and enactment of the collaborations. Figure 7 emphasises the nature of theoretical references. References to "community or communities" were many and varied. References to particular theoretical perspectives on collaboration or community were far fewer.

In Sect. 4.1.2 above, we reported that many collaborations identified in the survey had a particular mathematical focus (e.g., algebraic thinking, teaching of fractions, geometry etc.). In most cases the articles included a section within the literature review that addressed ways in which the mathematical focus and developmental goals were related. For example, Hunter and Back (2011) includes a discussion of the development of algebraic reasoning in their paper aimed at exploring how professional development in the form of Lesson Study supported the teachers to 'notice' opportunities for developing students' early algebraic reasoning.

In a number of collaborations, the research literature that underpinned the focus for the work formed part of the discourse of the collaborative group (Potari 2013; Allmond and Huntly 2013). The range of ways in which this was accomplished included: making research literature available to the participants (through a reader or uploaded to an online learning community resource); reading and presenting research papers; reflecting on ways in which teachers could use research findings in their teaching; and designing mathematical examples and tasks, and justifying their choices based on research. Moreover, Potari and Jaworski (2002) and Turner (2008) report on studies where a specific theoretical tool (respectively "teaching triad" and "knowledge quartet") was provided to teachers as a framework to focus and deepen their reflections, positively affecting the ways in which they engaged with learning situations in their own contexts.

Within the references to community are a small number of explicit references to different types of community, in some cases drawing explicitly on related theory as set out below.

Communities of various kinds: As demonstrated above, a large number of papers refer to *community* in some form. Of 85 papers, which refer explicitly to theory, 68 papers (80 %) refer to "community". In this section we look only at the cases in which community is specified more particularly.

One of the first researchers who analysed the idea of community was Wenger (1998) who, focused on "communities of *practice*" in a commercial setting, and characterised them as groups of people who are involved in shared practices. He suggested three dimensions of sharing: mutual engagement, joint enterprise and shared repertoire. Members are brought together by joining in common activities with familiar resources and by sharing what they have learned through their mutual engagement in these activities. The practice of *teaching* fits these ideas with Wengers' dimensions providing ways of analyzing aspects of this practice.

For example, Goos and Bennison (2008) describe a study whose aim was to investigate how a community of practice, focused on 'becoming a teacher of secondary school mathematics', emerged during a pre-service teacher education programme and was sustained after students graduated and began their first year of full-time teaching in schools. The authors write:

> Bulletin board discussions of one pre-service cohort are analysed in terms of Wenger's (1998) three defining features of a community of practice … Emergence of the online community was associated with our own role in facilitating professional dialogue, the voluntary and unstructured nature of participation, initial face-to-face interaction that created familiarity and trust, and the convenience of using email rather than logging on to a website. The study shows that the emergent design of the community contributed to its sustainability in allowing the pre-service and beginning teachers to define their own professional goals and values. (p. 42).

Wenger (1998) distinguished between the following *modes of belonging*, to capture the different forms of participation within a community:

- *Engagement*: doing things together, talking, producing artefacts.
- *Imagination*: constructing an image of ourselves, of our communities, and of the world, in order to orient ourselves, to reflect on our situation, and to explore possibilities.
- *Alignment*: a mutual process of coordinating perspectives, interpretations, and actions so they realise higher goals.

The mode of alignment can be seen as (implicitly) preserving the norms and expectations of a community over time, such that practices become perpetuated, even if they are not the most desirable for effective working. The mode of alignment is helpful analytically as both a research and a developmental tool: research might use it to identify key elements of practice which afford or constrain collaborative activity; participants might challenge their alignment with various norms, asking questions and looking critically at their own and others' practices with a view to developing or improving practice. Such a process has been called "critical alignment" (Jaworski 2006). In the case presented in Sect. 2.3 above, we see critical alignment occurring as teachers and didacticians collaborated to resolve tensions and foster ways of working that suited both of their groups.

Community of Practice (as in Wenger 1998) was first introduced by Lave and Wenger (1991) in a book entitled "Situated learning: Legitimate peripheral participation". Situated Cognition is a theoretical perspective which suggests that, contrary to cognitivist perspectives dealing with the individual cognising subject (often rooted in the work of Piaget), cognition is always 'situated' in a setting or context in which it takes place (Lave 1988, presented case studies of cognition in supermarket shopping and in groups of alcoholics anonymous. See also Kirshner and Whitson 1997). Where learning mathematics is concerned, the institutional setting is often seen as vital to students' cognitive processing. So, for teachers learning through collaboration, the settings in which their learning takes place (for example, CPD course or inquiry-based project) are fundamental to what is learned as is the mathematics on which activity is based. Such theoretical considerations demand a sociocultural approach to the study, even if this is not acknowledged, and support a critical stance into affordances and constraints.

Community has been recognised (e.g. by Engeström 1999) as central to seeing work and practice as "activity", deriving from the work of Vygotsky and Leont'ev. The community has an important mediational role with regard to a *subject* achieving its *object* in an activity system. The critical dimension (as in critical alignment) can be seen in Engeström's concept of "expansive learning" whereby, through the actions of certain members of a community, questioning the status quo, an expansive cycle is initiated leading to a quantum of development in the system (e.g., Jaworski & Goodchild 2006). Again we can see elements of theory providing tools through which to analyse elements of collaborative practice.

However, none of the above theoretical ideas, in and of themselves, interpret community in direct relation to *mathematics teaching*, so when these theories are used in the studies we have surveyed, although it is always with an interpretation into mathematics education, this is sometimes left implicit.

A significant number of papers refer to teachers participating in some form of *Inquiry Community*. Sometimes this is treated as a community that engaged in inquiry, with little theorisation beyond a description of inquiry-based tasks or activities, as in the case described in 2.2 above. Others go further to theorise inquiry community, often developing from Wenger's community of practice. For example Jaworski (2006) draws on Wenger (1998), as well as Wells (1999—Inquiry Community) and Cochran-Smith and Lytle (1999—Inquiry as Stance) to define *Community of Inquiry*, taking Wenger's dimensions (of mutual engagement, joint enterprise and shared repertoire) and modes of belonging (engagement, imagination and alignment) as a basis. Inquiry is defined with reference to Wells (1999) as referring to questioning, problem-solving, wondering, investigating, taking nothing for granted, and looking critically. For example, Goodchild (2008), working in the project Learning Communities in Mathematics (Sect. 2.3), describes how the creation of a community of inquiry with teachers, leading to critical alignment in practice, can be seen as "good research" in that it includes teachers as full members of the community and treats issues and tensions in developing practice as central to the life and work of community members.

Activity theory: Constructs from Activity Theory are used in a number of papers, largely as an analytical tool. For example, *community* is a key element in Engeström's (e.g., 1999) third generation framework in Activity Theory, in which he presents his "Expanded Mediational Triangle" deriving from first and second generation versions of Vygotsky's mediational triangle. Here some subject achieves an object or goal through the mediation of an instrument or artefact (or tool). As well as the mediation of artefacts (in our studies, such as text books, on-line systems or mathematical symbols), Engeström suggests that *Rules*, *Community* and *Division of Labour* are also important mediators in an activity system. Thus, in taking activity theory as a basis for research into mathematics teachers' learning through collaboration, the idea of the community in which learning occurs is central to the concept of mediation. Several studies use an Activity Theory frame through which to address the situative aspects of the study. The frame is in some cases Engeström's triangle; in others it is a three-layer framework attributed to Leont'ev consisting of Activity related to Motive, Actions related to Goals, and Operations related to Conditions. According to Leont'ev, Activity is always motivated, although the motive might not be explicit. Within motive we have actions which are always explicitly goal related. Action and goals depend upon operations and conditions within activity.

An example can be seen in the work of Sakonidis and Potari (2014) for whom community involves their joint activity as mathematics teacher educators and academic researchers collaborating with both experienced and novice teachers. This community is a mediating force for its participants' learning. Adopting an Activity Theory (AT) perspective, Sakonidis and Potari analysed their activity, identifying its nature and the transformations that frame our professional learning. The activity theory perspective made them aware of aspects of their own practice which allowed further development to take place.

Valsiner's zone theory: Also deriving from Vygotsky is Valsiner's *Zone Theory*, relating to Vygotsky's Zone of Proximal Development (ZPD). Valsiner's zones, the Zone of Free Movement (ZFM) and the Zone of Promoted Action (ZPA), expand the ZPD by focusing explicitly on the free movement of a learner in their learning zone and the restrictions imposed on a learner by some 'other' who seeks particular outcomes for the learner. These seem to have obvious application to teachers and learners and have been applied in mathematics education and teacher education.

For example, Goos (2005) addresses the question of how pre-service teachers learn from experience during their educational programmes and make the transition from pre-service to beginning teaching of secondary school mathematics. Her analytical use of zone theory helps in understanding how teachers' professional identities emerge in practice, when they are working and learning in communities of professional courses. The Zone of Free Movement represents the students' (behaviour, motivation, perceived abilities), curriculum and assessment requirements, and the availability of teaching resources and suggests which teaching actions are possible. The Zone of Promoted Action represents the efforts of a university-based teacher educator, school-based supervising teacher, or more experienced teaching colleague to promote particular teaching skills or approaches within the community of teachers involved in the educational programme if development of their identity as a teacher is to occur.

Meta-didactical transposition: Meta-didactical transposition is a theoretical tool related to the Anthropological Theory of Didactics (ATD—Chevallard 1985) to study the evolution of teachers' praxeologies, when working in collaboration with the research community, within different teacher education programmes (*M@t.abel*, *ArAl*, *MM-lab*). Introduced by an Italian research team, it focuses on new didactical and methodological practices (e.g., mathematics laboratory, problem solving, class discussions, reflective analysis of classroom processes) (Aldon, Arzarello, Cusi, Garuti, Martignone, Robutti, Sabena, and Soury-Lavergne 2013; Arzarello, Robutti, Sabena, Cusi, Garuti, Malara, and Martignone 2014; Clark-Wilson, Aldon, Cusi, Goos, Haspekian, Robutti, and Thomas 2014). The associated research has been on communities of in-service teachers involved in professional development programmes with researchers. This theoretical frame provides a model to develop research on teachers' communities and a paradigm to support the design of teachers' educational programmes. The most important feature of the model is to describe the complex dynamics that characterise activities when communities of teachers and researchers collaborate, in particular the fact that during collaboration there can be evolution and changes in teaching practices by members of the communities. These changes are considered in the institutional context where the professional development of teachers takes place, according to some specific educational and pedagogical goals (e.g. promoting teachers' knowledge of new curricula, enhancing their use of digital technologies, and so on).

Links and connections between theories: Whereas researchers in Mathematics Education have been working for some years on linking theoretical perspectives (e.g., Prediger et al. 2008) the survey is mainly concerned with the identification of the most influential theories and perspectives that have contributed to our knowledge concerning mathematics teachers working and learning in collaboration. We therefore focus on the theories and perspectives mentioned above.

For example, the MDT and Valsiner frameworks analyse the development of teachers' professionalism, the first from

the perspective of the anthropological theory of didactics (Chevallard 1985), the second from the sociocultural perspective of Valsiner based in the work of Vygotsky. Both are interested in the emerging of teaching practices useful for teachers' action in the classroom. Both of them are focused on the processes activated in teachers' educational programmes, and particularly the first observes teachers working together in communities of practice and professional development. In addition these two theoretical ideas take account of the dynamics of the situations by considering how roles and identities evolve during collaborative work over time.

The various theories relating to situated cognition and to communities of practice and of inquiry are rooted in Vygotskian theory and based on sociocultural principles in which knowledge grows through mediation between the people concerned. Mediation through tools and signs is the basis of Activity Theory, which also focuses fundamentally on community as a mediating force in an activity system. Engeström emphasises the various elements of mediation (mediating artefacts, rules, community and division of labour), which are linked in promoting or impeding the object of activity. When there is tension or contradiction between different elements of mediation, there is a perturbation in the system, which can lead ultimately to expansive learning through which the system develops. The notion of critical alignment in communities of practice has been seen to create similar possibilities for change, as does Engeström's expansive learning. Jaworski (1998) offers a relevant example in the practice of the teacher Julie, a member of a small community of inquiry: Julie had reached a critical point in her classroom research and was almost at the point of abandoning it, when another member of the group, Sam, invited her to his school to observe in his classroom. This activity led to Julie's renewed energy and her subsequent success in her classroom research aims. Potari (2013) writes of using activity theory in analysing teachers' participation in a community of inquiry and of the teachers grappling with contradictions revealed in their practice. She writes:

> The contradictions that emerge may again be the driving force for professional learning and the development of teaching. However, these contradictions could also be an obstacle for the teacher to adopt practices that she considers can promote students' mathematical learning (p. 517).

4.2.2 *Methodologies of work with teachers within the studies*

4.2.2.1 Rationale This section explores the methods and approaches reported by the studies in terms of teachers learning through collaboration. These methods are of two kinds, sometimes integrated, and not always well distinguished. The first kind is a research methodology: this is the methodology on which the research is constructed by the researchers. The second is a developmental methodology, which underpins the activities in which teachers take part in order to develop their practice. This can be the methodology used by teacher-educators in programmes for teaching development, or it can be methodology developed by groups of teachers in working together for development. An example of the first kind is the use of classroom observation techniques and interviews to gain access to teachers' thinking and practice in a developmental programme. An example of the second kind is Lesson Study, which is a developmental methodology used by teachers and their academic colleagues in developing lessons in schools. An example that crosses the two (a sort of 'hybrid') is Design Research, which is a research methodology in designating the purposes of research and methods used, and a developmental methodology in developing the learning of teachers and students through cycles of design activity. In this section we shall focus more generally on the second and hybrid kinds.

4.2.2.2 Findings First a brief word about methodology of the first kind, i.e. research methodologies. Qualitative research dominates the sources. However, qualitative approaches embrace a number of different paradigms and associated methods. In some studies, researchers use observation and interview to gain access to teachers' teaching and associated thinking through a grounded approach. Several sources use case study methods. In one such example, the authors chose a case study approach to obtain rich, contextual data, which consisted of video recordings of the planning meetings, lessons and post-lesson discussions and audio recordings of interviews with the teachers (Foster, Wake, and Swan 2014). A very few studies use quantitative methods, based on questionnaires with structured response formats.

4.2.2.2.1 General terms used in survey studies

We now focus on methodologies of the second kind, i.e. developmental methodologies (or hybrid). First we address briefly some of the general terms and concepts used in studies in the survey, appearing in paper abstracts, keywords, or both.

Collaboration: The papers that in the abstract refer explicitly to collaboration among teachers, or of teachers with researchers, comprise about one-third of the total number of papers; this means that these papers describe an experience where teachers are involved in activities where they work or learn in collaboration with colleagues. It does not mean that other papers do not deal with collaboration, but that they do not refer to it in the abstract. For example, there are papers

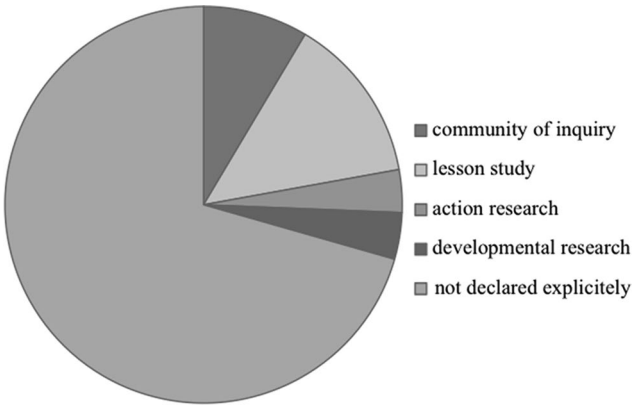

Fig. 8 Relative numbers of studies declaring methodology in the given areas

such as do Carmo de Sousa (2011), where the term collaboration is not mentioned in the abstract, but the intention of the author is to report on a collaborative and participatory group of teachers working together on the analysis of the current mathematics curriculum of São Paulo prepared by Mathematics teachers working in the Interactive Centre for Research in Mathematics Education. We know this through more detailed scrutiny of the paper itself, but have not been able to go into such detail for all of our 316 papers.

Professional development: About half of the papers refer explicitly to teachers' professional development using different keywords, such as professional development, teachers' education, PD, CPD, and so on. We referred, in Sect. 2.1.2, with reference to Simon (2008), to two categories of professional development, those involving content and process goals, and those focusing on process alone. In both cases, these refer to explicit projects or programmes involving teachers but led by 'others', often teacher-educator-researchers. Even though not all programmes involving collaboration were designated as 'professional development', the involvement, and possibly the leadership of these 'others' leads to a professional development ethos in the programme. Examples are included below.

Change: 20 % of papers are focused on experiences of involvement of teachers for whom the goal is to change their practices, didactics, pedagogical issues, methodologies of working in classroom, use of devices, and so on. These papers refer to the necessity of changing something in the school, with and through teachers' activity possibly in collaboration. It is not always clear where the goal for change originates; whether it is the teachers' goal or the goal of school management or of external 'others'.

Community of inquiry: More than 10 % of papers refer to communities of inquiry of teachers working together of with researchers, mentors, and stakeholders. We have dealt with *Community of Inquiry* theoretically above; however, often the terms community and inquiry are used to describe practices without any (overt) theoretical considerations.

4.2.2.2.2 Specific areas of methodology relating to development and research in the studies

In our sections below we address specific areas of methodology relating to development: Action Research, Design Research, Developmental Research and, Learning Study. These are distinguished by being either developmental or hybrid approaches.

It is problematic to produce a meaningful graph, representing exactly the data on these areas of methodology, due to degrees of overlapping. However, an approximate representation is shown in Fig. 8.

4.2.2.2.3 Lesson study

This should be read in conjunction with the example on Japanese Lesson Study in Sect. 2.1 above. As a collaborative way of working, Japanese Lesson Study is a highly structured approach that has become well known in many countries and has gained significant exposure outside Japan following the TIMSS Video Study and publication of The Teaching Gap (Stigler and Hiebert 1999). There are two important features of the educational setting in Japan that reduce the likelihood of lesson study leading to a proliferation of local, context-specific theories. First, Japanese teachers work with a common national curriculum and textbooks, which provide a stable reference point for elaboration of theories that inform classroom practice in reproducible ways. Secondly, university academics, pre-service teachers, and practising teachers engage together in lesson study, and it is the participation of university-based researchers that establishes conditions for integration of local theories into a shared theory of teaching. This evolving theory is recursively reinterpreted across generations of teachers as newcomers join in lesson study cycles with more experienced colleagues. In this way, lesson study creates intergenerational learning communities that have shared histories and shared ways of working, thus developing lesson study as a reproducible science of teaching (Isoda 2015). It is also argued that lesson study builds a synergy between the theory-building goals of research and the practical craft of teaching (Y. Shimizu 2013).

In Japanese lesson study, collaborative work involves preparing a plan for the research lesson that is oriented towards a long term goal or research theme, teaching and observation of the lesson (where the observers may include teachers from the same school, teachers from other schools, or knowledgeable 'others'), and post-lesson discussion and reflection that concludes with invited comments from a "knowledgeable other" (Takahashi 2014). While each of these phases of the lesson study cycle appears straightforward and perhaps not very different from other

collaborative professional development approaches, their essential features are not always well understood outside the cultural context in which lesson study has developed for over a century (Doig and Groves 2011).

More than 20 % of the sources we identified in our survey reported on lesson study, most often in relation to how this approach has been taken up in countries other than Japan. The largest group of sources reported on lesson study research carried out in the United States, but there were also contributions from Australia, the UK, and countries in Asia, Africa, Europe and South America. Only rarely did we find explicit investigations of the lesson study mechanisms, or collaborative ways of working, that support teacher change. One example was noted in Alston, Potari, and Myrtil's (2005) research into teachers' discussions about mathematical and pedagogical ideas as they planned lessons and then reflected on their implementation. Teachers' growing attention to the importance of tools for learning mathematics—concrete aids for building mathematical representations or solutions—was identified as a crucial element of their lesson study reflections.

In countries outside Japan, reasons for the uptake of lesson study are often concerned with improving specific teaching practices or enhancing knowledge for teaching—so that teacher collaboration via lesson study is seen as a means to this end. For example, in the US, Inoue (2011) worked with a small group of 4th and 5th grade teachers to incorporate neriage (the consensus building phase of Japanese mathematical inquiry lessons) into their mathematics lessons, while teacher questioning techniques were the focus of a Malaysian study conducted by Ong, Lim, and Ghazali (2010). In a UK study, teachers in nine secondary schools formed lesson study groups to improve the teaching of problem solving, and the research focus was on expanding the interpretation of mathematical knowledge for teaching to include knowledge of problem solving processes and pedagogies (Foster et al. 2014).

In addition to those studies where lesson study was a means of initiating teacher collaboration in order to bring about change in teacher knowledge or practices, we identified reports of how lesson study is implemented in different countries. In the latter contexts, interest often centres on how lesson study could bridge "the gap between policy at the national level and teaching at the classroom level", as in the South African study reported by Coe, Carl, and Frick (2010) to bring about large scale changes in teaching practice. For example, Baba and Nakai (2011) reported on progress in institutionalising lesson study in Zambia, with the aim of shifting teaching methods from transmissive to participatory approaches and enhancing teacher professionalism. However, other reports have pointed to the dangers of introducing lesson study as a government-sponsored top-down initiative. Kusanagi (2014) argues that lesson study in Indonesia has been bureaucratised because of the centralised regulation of teaching and the emphasis placed on preparing students for national examinations. Cultural differences between notions of teacher professionalism and collegiality also contributed to differences in the implementation of lesson study in Indonesian and Japanese schools. Kusanagi went as far to claim that, in the Indonesian case study school, "there was little evidence that lesson study produced collaborative learning among teachers to share practices" (2014, p. 99).

Nationwide lesson study projects initiated by ministries of education in several countries have developed cascade models in collaboration with Japanese experts. In the case of Thailand, Inprasitha (2015) reported the process of scaling a lesson study project from 13 schools to 600 schools using Thai editions of Japanese textbooks and associated theories. The ministry of education in Chile and the Japan International Cooperation Agency engaged fifteen universities in a lesson project within their teacher recurrent education program (Isoda, Arcavi, and Mena Lorca 2012; Isoda and Olfos 2011). In Mexico, an elementary teacher education program was developed for all teacher education colleges through the adaptation of Japanese textbooks and associated theories (Avalos and Isoda 2013). Common to these projects is the challenge to reform educational cultural practice and materials by using lesson study *and* Japanese resources (textbooks, research findings etc.) as tools. Such tools are the basis for new creations and are necessary for ensuring the quality of lesson studies done by thousands of teachers and a number of researchers.

Teaching and learning are cultural activities, and the expectation that lesson study can simply be transferred from one cultural context to another overlooks the implications of lesson study's cultural underpinnings. Ebaeguin and Stephens (2014) drew on Hofstede's dimensions of national culture to develop a cultural perspective on why lesson study "works" in Japan, noting the significance of "high respect for collegial relationships among teachers, attention to detail in planning, and a view that real benefits are more likely to be achieved in the longer term" (p. 206). Ebaeguin (2015) then used this analysis to develop a culturally embedded approach to promoting teacher growth through Lesson Study in a Philippine public school. These studies suggest that research could usefully focus on understanding better how Lesson Study can be recontextualised in different cultural contexts (see White and Lim 2008).

Another approach to the question of why lesson study works, involves looking at this collaborative practice through different theoretical lenses in order to examine its features and impact in diverse contexts. Lewis, Perry, and Hurd (2009) developed a theoretical model of lesson study, drawing on situated learning theories to explain its

potential effectiveness outside Japan. Doig and Groves (2011) liken the collaborative processes of Japanese lesson study to participation in a community of practice, while Wake, Foster, and Swan (2013) use cultural-historical activity theory to understand the nature of teacher professional learning in terms of learning at the boundary between the classroom and the lesson study group. These could be regarded as examples of theoretical recontextualisation as Western researchers attempt to connect lesson study with more general theories of teacher learning and development.

4.2.2.2.4 Action research/design research/learning study/developmental research

Action research, design research, learning study and developmental research have an important characteristic in common: this is an iterative or cyclic process, in which learning from activity in any one cycle feeds forward to successive cycles. Beyond this there are differences in how the cycles are conceived and implemented. Action research often involves teachers working on some aspects of their own practice and learning from successive cycles; for example, the teacher Julie in Jaworski's study (1998), mentioned above, engaged in four cycles of activity to learn about her students' use of mathematical dialogue in classroom activity. Learning from each cycle prompted her activity in the next. Action research can be collaborative when several teachers engage in planned action cycles, as in the study of Krainer and Zehetmeier (2013), who present IMST, a large-scale and long-term project, developed in the years 2000–2004. This project, which supported about 50 innovative programmes at Austrian upper secondary schools each school year, was focused on the creation of small professional communities aimed at supporting each participant in the definition and consequent investigation of specific issues, by means of action research.

Design research in education, also called design-based research or design experiment, is essentially an intervention in which some artefact, such as, for example, a lesson plan, a mathematics task for the classroom or a piece of software, is designed drawing on previously developed theory. In a sort of teaching experiment, the artefact is tried out and improved iteratively in the light of what happened (Cobb, Confrey, diSessa, Lehrer, and Schauble 2003; Edelson 2002; Kelly 2004). The teaching experiment is not seen as a test of a fully formed design, but rather as a learning situation for researchers and teachers (Gravemeijer and van Eerde 2009). Our survey found a few examples of studies of teachers involved in collaborations in which researchers framed their work within the structure of design research in the sense of Cobb et al. (2003) (e.g. Kynigos and Kalogeria 2012; Swan, Pead, Doorman, and Mooldijk 2013).

According to Wood and Berry (2003) the artefact of design research can be a model or approach for teacher professional development. Educators applying a professional development model might develop the model through a number of iterations with one or several groups of teachers. One example can be seen in Swan et al. (2013), who present a design-research process aimed at the design, implementation, analysis and refinement of a professional development intervention program, and associated resources, conceived to support the investigation of effective inquiry-based-learning pedagogies by mathematics teachers in the UK. Another example of a study focused on design research involving collaboration is the one presented by Sensevy et al. (2013), who refer to *cooperative engineering* as a subset of design research. This is characterised by an intertwining of the goals of designing learning environments and of developing theories of teaching and learning and by the need to consider all the "games" played by the different agents involved in the process (that is what every agent "sees" and "knows") irreducible to any other one. The outcomes of design research are not only the artefact, but also theoretical understanding in terms of, for example, insights that can be used by others in their own work (McKenney and Visscher-Voerman 2013).

A methodology which builds on both Lesson Study and Design Research is known as Learning Study (Lo, Marton, Pang, and Pong 2004). Learning studies borrow much of their rationale from Lesson Study and in addition bring in the iterative element of Design Research. The studies have a clear theoretical origin, with which teachers are made familiar, often involving Variation Theory (Marton and Booth 1997), and clear goals for students' learning (for example enhancing learning of fractions). Teachers plan lessons together with the researchers, and lessons are taught and observed as in Lesson Study. Students are tested on their knowledge and understanding before and after each cycle of lessons and the results fed back into the next cycle (e.g., Kullberg et al. 2016). It was interesting that our survey did not return any papers relating to Learning Study. This is possibly because "collaboration" does not appear in abstracts or as a keyword, despite being a central feature of work in Learning Study.

In contrast with the rather well defined areas of study in this section above, there is an important group of studies that have adopted a "developmental research methodology". However, not all studies characterise such an option, in a clear and precise way. From those sources in which the authors go beyond merely naming the chosen approach, the notion of "developmental research methodology" has often referred to Freudenthal (1991); Goodchild (2008) or Gravemeijer (1994). Goodchild particularly (e.g., 2008, p. 208) has drawn on Freudenthal and Gravemeijer to offer a model of developmental research involving cycles that are on the one hand developmental and on the other hand research cycles (Fig. 9).

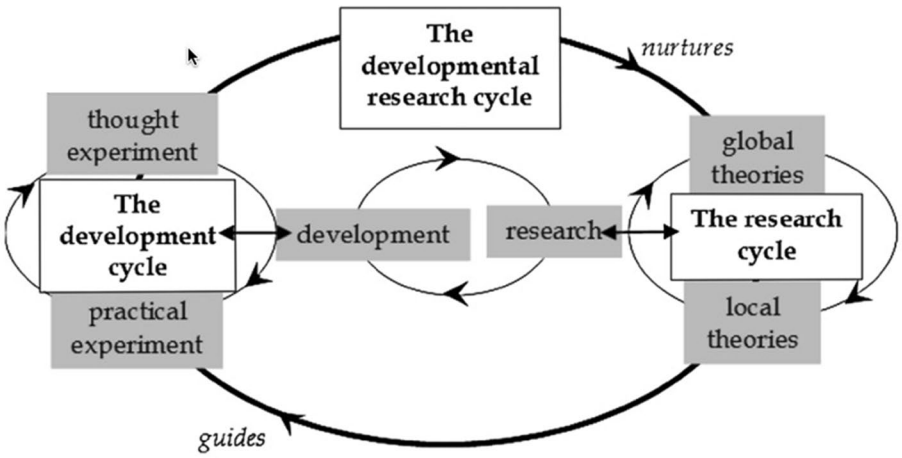

Fig. 9 The development research cycle (Goodchild 2008, p.208)

Freudenthal (1991) has argued that developmental research means "experiencing the cyclic process of development and research so consciously, and reporting on it so candidly that it justifies itself, and that this experience can be transmitted to others to become like their own experience" (p. 161). This quotation emphasises the research process alongside a study of the process of development.

Developmental research has been described as research that not only studies the developmental process, but also contributes to that development (Jaworski 2003). Such research very typically involves collaboration between teachers and educators, both of whom act as researchers. The LCM project in Norway, described in Sect. 2.3 above, is a clear example of developmental research.

Another example can be seen in the study conducted by Erfjord, Hundeland, and Carlsen (2012). It is pointed out that in their project, there was a cyclical relationship between the work and activities arranged for the mathematics teachers' professional development and the research conducted by mathematics educators and teachers. Therefore, at least in a context of collaboration, a developmental research approach implies a multifaceted work that opens the possibility of studying the complex phenomenon of mathematics teacher learning, mainly in situations on which teachers engaged with a real teaching situation alongside researchers, students and 'others'.

4.2.2.2.5 Narrative analysis of mathematics classes

A *narrative* is a written story that reveals insights to the activity of the writer and presents the writer's 'voice'. Ponte et al. (2003) indicate that narratives are stories in which the author relates, in sequence, a series of events that make sense for him or her, through an internal logic. *Narrative inquiry* is a research methodology that uses narrative as data to see into social practice. Clandinin and Connelly (2000) assert that three dimensions are intertwined in a narrative: space, time and social interactions. They point out that "narrative inquiry is a way of understanding experience" (p. 20).

Related to the use of narrative in an educational context, Chapman (2008) recognises that narrative could be considered as: a tool for collecting data; an object for analysis in studying teaching; a basis for, or tool in teacher professional development or teacher education; and as a basis for reflective thinking. Thus we see the hybrid nature of narrative as a tool both for research and for development in learning and teaching.

Ponte et al. (2003) link the uses of narrative, as a tool, to understanding mathematics teachers' learning through collaboration. In order to carry out their study, they opted to look at written narratives produced by teachers about the experiences developed by them. The analysis of the narrative reveals that the context of collaboration helped these teachers to overcome some difficulties, but also generated some challenges.

Lobo da Costa and Prado (2011) report on their work with mathematics teachers in a setting of collaborative group discussions (face to face, and in virtual fora) about mathematical content in curricula. For their study they collected data by using narrative and life stories on reflections upon practices. From analysis of this data it became clear that teachers found reflection on mathematical knowledge to be the first step for rethinking their practices. The teachers also recognised "the existence of a new way of learning based on the exchange of experiences that took place in the context of training which was developed as a collaborative work and by establishing and strengthening a confidence climate for learning" (p. 11)

Thus we see evidence of teachers' learning through collaboration revealed through their narratives, the analysis of which leads to researchers understanding key elements of teachers' learning.

4.2.2.2.6 Other methodological approaches
We found papers referring to a number of methodologies for which we will not devote space in this article, but which deserve consideration. Just briefly, we point to the three most important in terms of the numbers of papers involved. They are *Professional Learning Communities*, *Video Clubs*, and *On-line Communities*. In all three cases, the nature of the collaboration is captured in the words used to describe it. We found eight papers that referred to *Professional Learning Communities* or PLC. In all of these, the communities described and their activity overlap in differing ways with areas of community and forms of research discussed above. There seems to be no one theoretical formulation of PLC linking these papers. Although many papers referred to the collection of data through video recording, we found six papers that referred to Video Clubs or forms of Video Professional Development. These refer to the use of video recordings of classroom learning and teaching in which groups of teachers (and 'others') meet to view and discuss practices and issues, and from which teachers develop their understanding of teaching processes. On-line Communities are mentioned in seven papers in which virtual meetings of teachers and 'others' in an electronic medium, synchronous or asynchronous, are described. The on-line meetings provide a convenient medium through which communication can take place without the need to meet physically in one place.

4.3 Theme 3: research findings and knowledge generated

This theme considers the outcomes or findings reported in the set of documents comprising the dataset. The sub themes, or dimensions, are taken from the data, much of which provides concrete and detailed outcomes, for example such as teachers reporting that they had come to recognise that students have interesting ideas about mathematics, and that it was valuable for them to carefully consider such ideas (van Es and Sherin 2009).

Interestingly, whereas a priori we may have expected to find or notice significant differences in the ways in which the collaborations played out owing to cultural and contextual differences, little of these differences were reported or visible apart from a small number of papers which analyse how and why Japanese lesson study is difficult to implement faithfully in other countries, as discussed above. In fact, the majority of the outcomes fall into two main areas which relate to a) the design and set up of the processes of collaboration and, frequently also professional development, in which the teachers were involved and b) teacher learning.

4.3.1 Reflections on collaborating

All sources in the dataset reported on initiatives that were deliberately and explicitly designed as collaborations, or which were seen as being so by the survey team. Our preliminary analysis suggests that about 20 % of the papers explicitly considered whether, how and why the collaborations were effective. This section draws out the findings related to the design of the collaborative activity in terms of the way it was initiated, membership, activities carried out and opportunities for learning.

4.3.1.1 Factors that support collaborative ways of working Some authors identified particular aspects of the way in which the collaboration was set up and developed as effective. Several studies referred to the participants and their roles in the collaboration, with many stating that diversity of role amongst the group members is valuable. For example Redmond, Brown, and Sheehy (2011) suggested that university academics' perspectives help teachers and others to see and interpret local practices in new ways. It seems too, that teachers value the presence of other teachers, such as in the study by Olsen and Kirtley (2005) in which the authors stated that the "interaction between high school teachers and elementary teachers with their different expertise was critical" (p. 31). Of course, in many collaborations, the roles of participants shift over time, as described, for example, by Jaworski (2005).

In a number of studies, shared goals and interests seemed to be important for the development and sustainability of the collaboration. For example, Geiger and Goos (2006) identify a "shared interest in the improvement of educational practice" (p. 260) as one of the conditions that led to a successful collaboration.

A further common factor identified in the sources in relation to successful collaborations was the development of an environment in which the teachers felt safe to talk and valued, such as in the CSCL-mediated environment described by Nason, Chalmers, and Yeh (2012) and in the study by Pires and Martins (2009).

We now turn to the design of the activities in which teachers were engaged; these relate to opportunities for teachers rather than their reported learning, which is the focus of Sect. 4.3.2.

A number of papers identified particular activities as effective. Unsurprisingly, given that this survey is about collaboration, it seems that opportunities for teachers to interact with other teachers were highly valued. For example, King and Murata (2005), concluded that the many opportunities teachers had to engage with colleagues and explore knowledge of mathematics and beliefs about mathematics were important.

Other sources identified the analysis of video recordings of teaching as providing the sorts of opportunities that might be valued: for example Alsawaie and Alghazo (2009), state that "[o]ur results support the use of video lesson analysis in teacher preparation" (p. 239).

4.3.1.2 Factors that inhibit collaborative ways of working Although the majority of the sources suggested that the collaborative working and learning had achieved some success, there were some that described aspects that did not work well and some that questioned the value of collaboration per se. For example, Campbell (2009), in writing about mathematics teachers working within a professional learning community, concluded that he had doubts about "the real benefits that mathematics teachers can take away from these collaborative experiences" (p. 962).

Overall the main barriers to effective collaborations appear to be (a) teachers' lack of ownership (b) time and (c) institutional constraints other than time, such as curriculum and assessment structures.

With respect to teachers' sense of ownership, it seems that in some collaborations, teachers felt that the leaders of the collaboration did not take into account the teachers' needs and interests, and that the responsibility for the collaboration did not lie with them. The paper by Besamusca and Drijvers, (2013) provides such an example. Related to this, some papers suggested that teachers did not feel confident in expressing their views (e.g. Hospesovà et al. 2006) and others discussed the tension for some teachers between their familiar isolated ways of working and the unfamiliar collaborative ways of working (e.g. Puchnera and Taylor 2006).

It is well recognised that teachers are busy and that finding the time to engage in professional activities outside the classroom is problematic, so it is perhaps not surprising that lack of time was identified in a number of sources as a barrier to successful collaboration. There are two aspects to the issue of time; the first is finding the time in teachers' day to day activities, as identified by, for example, Berg (2011) and the second is the overall time allocated to the project. In terms of the time for the project, Cavanagh and McMaster (2015), for example, state that "it is clear that it takes considerable time for a learning community to begin to coalesce so that pre-service teachers can develop their ability to notice and reflect on lessons" (p. 488).

Various papers referred to institutional factors other than time that had a negative influence on the collaborative work of the teachers. Jaworski (2008), for example, cites the influences of school and educational system as such factors, explaining that in her project, established ways of working and project aims were sometimes in tension, leading to scope for critical alignment.

4.3.2 Impacts on teachers' knowledge, thinking, and practice

Many of the sources report on what and how the teachers participating in the collaborations learned. The first section below outlines the findings in terms of their reported changes in knowledge, understanding and beliefs and the second is concerned with reported changes in their practice.

4.3.2.1 Teachers' knowledge and thinking Many of the sources provided some general statement about teacher learning, such as "[b]y the end of this study, it was found that the prospective teachers had made considerable advances to their repertoires of PCK" (Nason et al. 2012, p. 238). Specific and concrete examples were frequently also provided, as summarised below.

A first set of claims about specific learnings relates to learning to participate in a collaborative group. Some teachers, for example, learned to listen to, and value, the views of others over time (e.g. Muñoz-Catalán et al. 2010a). Some claims refer to joint work, developing in accordance with the views of the community as they work together, such as reported by Menezes (2011), who states that "[t]he study shows that teachers develop professionally, manifest developments in their forms of collaboration in the group from providing aid and assistance to joint work, and in parallel deepen their teaching knowledge and professional practices" (p. 225).

A number of sources suggest that teachers have learnt more about how to reflect on their own, and others', teaching, sometimes supported by the use of video recordings. For example, Pires and Martins (2009), stated that "[t]he program has … allowed the development of their ability to reflect (oral and written) on practices" (p. 47).

Many sources reported teachers learning about teaching. Once again, some of this learning appears to have been non-specific, such as Peng's (2007) report that teachers had gained an expanded knowledge of mathematical topics and their teaching of these topics.

A number of sources stated that teachers had gained the confidence to try out new approaches in the classroom, such as Warren (2008), who reported: "as teachers came to an understanding of the mathematical knowledge, not only did their confidence in mathematics increase but so also did their willingness to experiment in the classroom." (p. 43). Experimentations included, for example, trying out inquiry processes (e.g. Allmond and Huntly 2013).

Other learning about specific teaching included, for example, improved questioning (Norton and McCloskey 2008) and sequencing mathematical tasks (Huang, Su, and Xu 2014).

Many authors reported that teachers had learned the importance of attending to students' mathematical thinking and understanding, and of developing an awareness of their students' needs (e.g. Posthuma 2012). There appeared to be evidence of a shift in the ways in which teachers noticed students' understandings, sometimes supported by the use of video recordings (e.g. van Es and Sherin 2009).

Finally, a number of studies claimed that teachers had learned some mathematics, which included specific mathematics such as ratios and fractions (McDougall and Nason 2005), fractions and the use of linear measurement representations (Lewis and Perry 2014); the quadratic formula (Kotelawala 2010); skills of using dynamic geometry in teaching (Meng and Sam 2011). On a sort of meta-level, reported teacher learning includes thinking about mathematical representations (King and Murata 2005); the mathematics behind relational thinking and ability to identify how to integrate it into their teaching (Bao and Stephens 2013) and mathematical argumentation (Boavida 2008).

4.3.2.2 Teachers' practices Whereas the section above reports on perceived learning of teachers, this section reports on what teachers began to *do* differently, thus going beyond what we reported in the previous section. Some reports state that teaching or lecturing improved (e.g. Barton, Oates, Paterson, and Thomas 2014) and others provide more specific examples of changes in practice.

There is some reported change in terms of lesson planning, such as in the report by Slavit and Nelson (2009). They state that: "the teachers in this case study exhibited explicit connections between their collaborative inquiry and their instructional practice. The rich tasks and collaboratively-developed lesson plans illustrate this important connection at the level of classroom practice." (p. 218)

Other authors suggest changes with respect to interaction with the students. The study by Silver, Charalambous, Strawhun, and Stylianides (2006), for example, stated that "teachers appeared to have become less inclined to do all the thinking for their students. There appeared to be a shift in the direction of stronger support for having students share and discuss multiple solutions when solving a mathematics problem." (p. 296)

There is some evidence regarding the character of changes related to classroom practices such as, for example, questioning. Ong et al. (2010), for example, "found that the experienced mathematics teachers moved away from routine factual questions which focused on procedures and final answers which were used in the beginning and by the end of the study, they were able to generate questions to probe the pupils' thinking." (p. 86).

4.3.2.3 How teachers learned In Sects. 4.3.1 and 4.3.2 above, we discussed outcomes reported in the sources related to the design of the collaboration and/or learning environment, and particularly pointed out how these designs provide opportunities for teacher learning. Here we summarise the outcomes in terms, perhaps, of the opportunities the teachers took up and hence *how* they learned.

Consistent with other literature addressing teaching development (e.g. Zaslavsky 2008; Jaworski and Huang 2014) a large number of papers claimed that teachers learned through reflection: reflecting on their own teaching, on student learning and on others' teaching such as, for example, in the case of the study by Olson (2005), who stated that "each case-study teacher who participated in the lesson study changed unique aspects of their pedagogy when they reflected on their own practices from a new perspective" (p. 597). Related to this are outcomes claiming that learning occurred through raised awareness (e.g. Sakonidis and Potari 2014). We are aware that the nature of reflection and associated raising of awareness are key aspects of teacher learning which deserve further consideration (see Footnote 2).

Others suggested that teachers learned through discussion and conversation. King and Murata (2005), for example, write that in lesson study meetings "teachers' differing experiences and ideas about scaffolding and representations surfaced, where teachers wrestled with diverse opinions, analysed student strategies, and developed their new ideas and perspectives" (p. 748).

There are further outcomes related to how teachers learned which are about *what* they reflected on or discussed. Several papers suggest that teachers learned by looking into others' classrooms in some way, either by peer observation or through the use of video. Posthuma (2012), for example, states that teachers in her study "learned from watching their fellow participants on video to change their teaching to become more learner-centred" (p. 6). Another set of papers claimed that teachers learned by focusing on students: student conceptions, student errors and student strategies as in King and Murata (2005).

5 Discussion

The ICME 13 survey team was tasked to conduct a survey on mathematics teachers working and learning through collaboration. This article is a preliminary analysis of what we found in the research literature. In this final section we return to the research questions and discuss the main findings expressed in the sections above. We focus together on Research Questions 1 and 2, and separately then on Questions 3 and 4.

5.1 Research questions 1 and 2

What is the nature of collaborative working (to include the different roles that teachers can play) and how does this relate to situation, culture and context?

Who are the people who engage collaboratively to promote the effective learning and teaching of mathematics, what are their roles, and how do they relate to each other within the different communities?

Our survey reveals that collaborative working as reported in the research literature takes many forms and involves different groups of people with differing roles. In terms of teachers learning through collaboration the most significant 'others' are teacher-educator-researchers who, typically, are those authoring the research reports. In most cases it is these 'others' who initiate and lead the collaborative activity, whether it is explicitly CPD (Continuing Professional Development) or some other type of programme. In many of these collaborations, the 'others' have a specific agenda for teachers' learning; in other programmes a more collegial approach is intended, with aims for equity between the partners of collaboration, and learning for both groups related to the focuses of collaboration. Many of the sources do not declare how the collaboration between teachers and others was initiated; this is perhaps because sources written by academics for publication in journals and conference proceedings have dominated, and these authors may not have deemed the origins of the collaboration to be important and/or relevant. However, as collaborations are initiated and further developed, an understanding of the rationale that informs the negotiation of the focus and goals for the collaborative work, and the roles that participants play in it, appears to be a crucial aspect of collaborative activities that have been reported as successful. This understanding will be able to inform the ways in which we, as a community, set up new collaborations.

In some cases there are parallel sets of aims—research aims and teaching development aims. For example, teachers' aims might commonly be concerned with improving learners' mathematical outcomes, whereas researchers' aims might be more concerned with eliciting or developing teachers' professional learning in the same context. When the research is an outsider study of a teaching development programme, it makes sense for research and development to be separate. However, many studies reveal that there is strong overlap between the activity of researchers and other participants in a teaching development programme, with some individuals taking both roles (e.g. teacher-researcher; teacher-educator-researcher). In many such cases research is a strong contributor to the developmental process.

However, the degree to which the aims of the different participants were articulated within the sources and, more importantly, shared between participants was only visible in the survey data if the authors had chosen to comment on these aspects. These findings suggest to researchers and authors the importance of deciding carefully where to focus attention in providing evidence for learning and development within a project.

The studies in which collaboration is initiated by teachers and sustained by them, even when certain 'others' are involved, seem to build on teachers' initial confidence in their aims for development, with associated learning emerging through evolution of unexpected or contradictory findings. Often, sustaining of activity is fostered through the external support offered by the 'others'. Success might be seen in terms of participants becoming more aware of their respective roles and responsibilities.

Very few studies have revealed unsuccessful collaborations. This could be because it is mainly the successes that are reported. However, in cases where unsuccessful collaborations were acknowledged, barriers to success were seen to be related to issues with culture or context. Particularly unsuccessful were projects in which activity developed elsewhere was imposed in a culture in which it was found unacceptable. Where collaborations crossed teacher-'other' boundaries, some studies revealed the emergence of tensions between communities, which although seen as challenging were also reported to be critical to the emergent learning of both groups.

Given that most of the reported collaborations were initiated and/or led, not by the teachers themselves but by 'others', for whom (perhaps) knowledge and leadership were more confidently expressed, the question of teachers' 'voice' is important. We see teachers' voice emerging in different ways in different projects. Britzman (2003) suggests that "Voice is meaning that resides in the individual and enables that individual to participate in a community. … Voice suggests relationships: … the individual's relationship to the other, since understanding is social" (p. 44). The key words here for our survey seem to be "since understanding is social". Despite Britzman's focus on the individual, we found evidence in the studies of something we might call "community voice". This is most often expressed in studies which analyse collaboration between groups of teachers and 'others'. Where there is a genuine aim for mutual learning, it can be seen that as teachers gain confidence, and are supported by each other, their voice emerges and this results in learning for their colleagues, the 'others', as well as for themselves.

Many of the more reportedly successful projects make clear the learning of teachers through teachers' own words, providing further evidence of the importance of the community voice. This can involve learning with other teachers from their own or others´ practices. It is through such examples that we gain deeper insights into what learning can look like for teachers who are involved; for example, seeing teachers involved in working with video, resulting in their learning. In the studies, we have seen and

characterised a wide range of different kinds of 'work' such as working with video.

Some of the issues reported above relate to theoretical or methodological perspectives in the studies surveyed. This is the focus of Research Question 3.

5.2 Research question 3

What methodological and theoretical perspectives are used to guide and inform collaborative working and learning?

Few authors reported explicitly on how their theoretical frame concerning teachers' collaborative work shaped the design of research methodologies/approaches that were at the heart of the direct activities with teachers. In fact, collaboration itself was rarely theorised, except by reference to some theory relating to the community in which collaboration took place. In Sect. 4.2 we reported that Lesson Study accounts for more than 20 % of the declared methodological perspectives. Lesson study has its own consistencies, whether in the form developed over 150 years in Japan in which theories relate to curriculum and published mathematical texts, or more recent developments of Lesson Study in other parts of the world. However, theory relating to collaboration in lesson study is largely implicit, based in the practices and ways of working that lesson studies employ. In contrast Learning Study, of which teachers collaboration is a central theme, employs an overt theoretical approach (in Variation Theory) into which teachers are inducted by researchers before the classroom lessons are designed.

In direct connection with collaboration, we have seen that ideas of 'community' permeate the studies. In most cases, ideas of collaboration and community are taken for granted and not theorised. Where they are related to theory, Wenger's (1998) theory of Community of Practice dominates and is extended in some cases to theory of Community of Inquiry in which Wenger's concept of 'alignment' is extended to 'critical alignment' through co-learning inquiry (Jaworski 2006). Several researchers are using Activity Theory to analyse data relating to community learning, especially where issues, tensions or disagreements between participants with differing roles are observed. With just a few papers, Activity Theory is used explicitly to make sense of alternative perspectives, with differing groups of participants acting against each other, and potentially acting against the developmental aims of the project. In such cases, it is likely that the views of the 'others' may dominate at the expense of teacher voice and of any real sustainable development for the teachers. It is therefore incumbent on the 'others', usually teacher-educator-researchers, who tend to have the most confidence in their knowledge and actions, to work to support teachers' voice and promote teachers' full participation in project outcomes.

This brings us to methodologies reported in the studies. As we explained above, there were methodologies relating to both research and development. While the former articulated the research methods used to collect and analyse data, the latter focused on the ways in which 'work' in the activities of the studies was organised and how development was promoted. Largely we have focused on the second of these since we were especially interested in how collaborative work led to teacher learning and teaching development. Thus, reflections on 'communities' above, from theoretical perspectives relate strongly to methodology within the studies. We have seen differing forms of community and of roles within communities, usually related to the nature of the project and its goals. There have often been times when we would have liked more detail on nature, roles and goals in order to have a clearer insight to the learning which took place and how teachers were positioned within the project.

5.3 Research question 4

What learning can be reported and how does it relate to collaboration?

This has been the most difficult question to address. Given the focus of the survey is on collaborative work in *mathematics* education, central to all of the sources is a focus on the development of mathematics knowledge and pedagogy. Although it is hard to quantify, we sense that not all studies recognise (or possibly report) the special and particular "need" for teachers to work collaboratively on mathematics, either at their own level or in working through tasks that are intended for pupils. Whilst this might be because the focus and aims for the reported research are on another dimension of the teachers' collaborative work, this might also relate to the fact that, for many teachers, "the mathematical nature of their work was a given, which was implicit and unquestioned" (Jaworski 1998, p. 25).

The three examples we chose to include in Sect. 2 all report aspects of teachers' learning. In Japanese Lesson Study, the historical development of LS practices and their use throughout the Japanese system over lengthy time periods is indicative of teachers' learning within this developing system. In the NCETM teacher studies, we see the teachers' voice reporting vividly what teachers have learned from their work, which often has a research focus. In the LCM Project, writings quote the voices of teachers declaring what they have learned and in some cases how this learning has taken place.

However, learning is not easy to observe. Although many of the studies report developing or changing practices which claim that learning took place, sometimes with quotations from the teachers involved, we have no consistent clarity on the ways in which learning has occurred or on the issues that have been involved for teachers. In particular we cannot overwhelmingly claim that the reported learning is due to collaboration.

5.4 Implications for future research and study

As well as the issues associated with teachers' learning, its recognition and articulation, there are many other questions and issues that we have not been able to respond to from our study to date.

One of these concerns the big issues of scalability and sustainability. A challenge in very large projects is to gather and analyse data in such ways that the finer details of teaching and learning are not lost. A further challenge is in the scaling up of modes of practice that seem to have success in terms of teachers' learning—the very nature of what fosters this learning might be lost at scale, even if scale were possible. The sustaining of successful practices is also a challenge when a project ends. We see few projects which declare aims for sustainability in the initial project design.

Another area where further research is needed concerns the spectrum of classroom resources from modes of display to interactive digital devices, their contribution to teachers' practice, teachers' learning and to teaching development. There is much research, with associated theory, that addresses learning with and through digital technology, for example, but we found few studies that addressed this alongside teachers working through collaboration.

While there were a number of studies which addressed teachers' focus on teaching mathematical topics, with attention to research and theory relating to the topic, there were few studies which looked at teachers' working on mathematics themselves and on their learning of mathematics, in order to address what is needed in learning the topic.

We found many studies concerning teachers' working in both primary and secondary schools, and just a few which discussed the two groups learning together. There was some evidence that teachers from the two levels working together had advantages for both groups in terms of understanding students' progression across the two levels.

Although we have referred to teachers' voice and the importance of nurturing and revealing teachers' voice, the number of studies in which this is made explicit is too few to be conclusive. In future studies, the particular aims of the teachers could be made more explicit, and possibly theorised.

We end with some key questions that raise issues for future consideration.

- In what ways does or can collaboration in mathematics teaching lead to learning and how is such learning recognised?
- What is the impact of collaborative practices on the understanding of *mathematics* by teachers and their students?
- How can research findings from small scale studies involving collaboration be used to promote sustainable practices at a larger scale?
- What approaches to research and development can lead to the teachers' voice being heard more prominently?
- How do or can teachers working and learning through collaboration make use of powerful (digital) resources to promote mathematics learning?

We look forward to possibilities for exploring these questions and issues in the future.

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Appendix 1

Journals searched

Educational Studies in Mathematics
Journal of Mathematics Teacher Education
Journal of Research in Mathematics Education

International Journal of Science and Mathematics Education
For the Learning of Mathematics
Research in Mathematics Education
Mathematical Thinking and Learning
PNA
NOMAD
Mathematics Education Research Journal
Mathematics Teacher Education and Development
African Journal of Research in Mathematics, Science and Education
Pythagoras
Eurasia
ZDM - Mathematics Education
BOLEMA: Mathematics Education Bulletin
Journal of Science and Mathematics Education in Southeast Asia (RECSAM)
Mathematics Teacher
Teaching Children Mathematics
Teaching Mathematics in the Middle School
RDM (Recherches en Didadactique de Mathématiques)
RELIME
Japan Society of Mathematical Education: The Reports of Mathematical Education
Journal of Science and Mathematics Education in Southeast Asia
The Mathematics Educator
Asia Pacific Journal of Education
Pedagogies: An International Journal
Zetetiké
Quadrante
Professional Development in Education
Journal of In-service Education
The Teacher Educator
The Journal of Educational Research
Teaching and Teacher Education

Conference proceedings searched

International Group for the Psychology of Mathematics Education (IGPME)
Mathematics Education Research Group of Australasia (MERGA)
Canadian Mathematics Education Study Group (CMESG)
International Group for the Psychology of Mathematics Education-North American Chapter (PME-NA)
Inter-American Committee of Mathematics Education (CIAEM)
CIBEM
SGC
RELME
Southern African Association for Research in Mathematics, Science and Technology Education (SAARMSTE)
Association for Mathematics Education of South Africa (AMESA)

Other sources

Clark-Wilson, A., Robutti O. & Sinclair, N. (Eds.). (2014). *The Mathematics Teacher in the Digital Era*. Dordrecht, The Netherlands: Springer.
Bednarz, Nadine, Fiorentini, Dario, & Huang, Rongjin (Eds.). (2011). *International Approaches to Professional Development for Mathematics Teachers*. Ottawa: University of Ottawa Press.
Bishop, A.J., Clements, M. A., Keitel, C., Kilpatrick J. & Leung, F. K. S. (Eds.), (2013) *Third International Handbook of Mathematics Education*. New York: Springer.
de Geest, Els, Back, Jenni, Hirst, Christine, & Joubert, Marie. (2009). *Final Report: Researching Effective CPD in Mathematics Education*. Sheffield: National Centre for Excellence in the Teaching of Mathematics.
Even, Ruhama, & Loewenberg Ball, Deborah (Eds.). (2009). *The professional education and development of teachers of mathematics: The 15th ICMI study*. Berlin: Springer.
Hmelo-Silver, C. E., Chinn, C. A., Chan, C., & O'Donnell, A. M. (Eds.). (2013). *The International Handbook of Collaborative Learning*. New York: Routledge.
Jaworski, Barbara, & Wood, Terry (Eds.). (2008). *The International Handbook of Mathematics Teacher Education (Vol 4) The Mathematics Teacher Educator as a Developing Professional*. Netherlands: Sense Publishers.
Krainer, Konrad, & Wood, Terry (Eds.). (2008). *The International Handbook of Mathematics Teacher Education (Vol 3) Participants in Mathematics Teacher Education: Individuals, Teams, Communities and Networks*. Netherlands: Sense Publishers.

Appendix 2

See Table 2.

Table 2 Database fields employed for the collation of the source data

Database field	Aims for the analysis
Full bibliographic information of the source.	To provide a statistical summary of papers by journal titles.
Title of paper	
Year of publication	To indicate the frequency of papers by year and to explore trends
Language	To highlight trends in the published languages
Authors—Country Geographic-cultural area	To highlight the predominance of specific geographical-cultural context and transnational research projects**.
Participants: (1) Who? (2) How/why is collaboration initiated? (3) How are roles and relationships established?	To identify the typical contexts for teacher working through collaboration and the characteristics of this collaboration.
Research aims (Who decides?) University researcher/multiple aims Teacher development aims (Who decides?)	To highlight trends in how the research aims are declared.
Scale of collaboration (also time scale)	To highlight trends in both the size of collaboration and temporal factors.
Institutional context	To comment on trends in institutional factors.
Phase of schooling in which the participants are located	To comment on the distribution of collaborations by educational phase.
Theoretical perspective (about topic, the process in which teachers are involved, the professional development)	To comment on the more dominant theoretical frames and their evolutions
Research methods and development activities (authors/participating teachers)	To comment on trends in methods and developing activities.
Outcomes	To identify prominent results, that could give insight of the effective ways of fostering collaboration and that could influence the future research.
Abstract/summary	To have a synthetic description of the paper: its main ideas and outcomes.
Keywords	Stats on how research is located with respect to existing and emerging keywords?

** It was, of course, impossible to search every possible country or cover geographical regions. So indications of predominance relate very strongly to where we have searched and where not. We try to make this clear in our reporting

References

Adler, J., Ball, D., Krainer, K., Lin, F. L., & Novotna, J. (2005). Reflections on an emerging field: researching mathematics teacher education. *Educational Studies in Mathematics, 60*, 359–381.

Aldon, G., Arzarello, F., Cusi, A., Garuti, R., Martignone, F., Robutti, O., et al. (2013). The meta-didactical transposition: a model for analysing teachers education programmes. In L. A. M., & A. Heinze (Eds.), *Proceedings of PME 37* (Vol. 1, pp. 97–124). Kiel, Germany: PME.

Allmond, S., & Huntly, K. (2013). Achievements and challenges encountered by classroom teachers involved in a research project: a reflection (Symposium Paper 2). In V. Steinle, L. Ball, & C. Bardini (Eds.), *Mathematics education: yesterday, today and tomorrow (Proceedings of the 36th annual conference of the Mathematics Education Research Group of Australasia)* (pp. 759–762), Melbourne: MERGA.

Alsawaie, O. N., & Alghazo, I. M. (2009). The effect of video-based approach on prospective teachers' ability to analyze mathematics teaching. *Journal of Mathematics Teacher Education, 13*(3), 223–241. doi:10.1007/s10857-009-9138-8.

Alston, A., Potari, D., & Myrtil, T. (2005). An analysis of teachers' mathematical and pedagogical activity as participants in lesson study. In G. M. Lloyd, M. Wilson, J. L. M. Wilkins, & S. L. Behm (Eds.), *Proceedings of the 27th annual meeting of the north American chapter of the international group for the psychology of mathematics education* (pp. 418–425). Roanoke, Virginia.

Arzarello, F., Robutti, O., Sabena, C., Cusi, A., Garuti, R., Malara, N., et al. (2014). Meta-didactical transposition: a theoretical model for teacher education programmes. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: an international perspective on technology focused professional development* (pp. 347–372). Dordrecht: Springer.

Avalos, C., & Isoda, M. (2013). *Matemáticas para la Educación Normal: Guía para el aprendizaje y enseñanza de la aritmética*. Estado de México: Pearson Educación de México.

Baba, T., & Nakai, K. (2011). Teachers' institution and participation in a lesson study project in Zambia: implication and possibilities. In *Africa-Asia University Dialogue for Educational Development*. Report of the international experience sharing seminar (Vol. 2). University of Hiroshima.

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: who knows mathematics well enough to teach third grade, and how can we decide? *American Educator* (Fall 2005), 14–46.

Bao, L., & Stephens, M. (2013). Using a modified form of lesson study to develop students' relational thinking in years 4, 5 & 6 In V. Steinle, L. Ball, & C. Bardini (Eds.), *Mathematics education: yesterday, today and tomorrow (Proceedings of the 36th annual conference of the Mathematics Education Research Group of Australasia)* (pp. 74–81), Melbourne: MERGA.

Barton, B., Oates, G., Paterson, J., & Thomas, M. (2014). A marriage of continuance: professional development for mathematics lecturers. *Mathematics Education Research Journal, 27*(2), 147–164. doi:10.1007/s13394-014-0134-7.

Berg, C. V. (2011). In-service teachers' professional development: which systemic aspects are involved? *Research in Mathematics Education, 13*(2), 223–224.

Besamusca, A., & Drijvers, P. (2013). The impact of participation in a community of practice on teachers' professional development concerning the use of ICT in the classroom. In A. Lindmeier, & A. Heinze (Eds.), *37th conference of the international group for the psychology of mathematics education* (Vol. 2, pp. 81–88). Kiel, Germany: PME.

Boavida, A., Maria, Roque, (2008). Colaborando a propósito da argumentação na aula de Matemática. *Quadrante, 15*(1–2).

Britzman, D. P. (2003). *Practice makes practice*. Albany: State University of New York Press.

Brodie, K. (2012). Opportunities for mathematics teacher learning. In T.-Y. Tso (Ed.), *Proceedings of the 36th conference of the international group for the psychology of mathematics education* (Vol. 1, pp. 101–106). Taipei, Taiwan: PME.

Brodie, K., & Shalem, Y. (2011). Accountability conversations: mathematics teachers' learning through challenge and solidarity. *Journal of Mathematics Teacher Education, 14*(6), 419–439. doi:10.1007/s10857-011-9178-8.

Campbell, M. P. (2009). Mathematics teachers and professional learning communities: understanding professional development in collaborative settings. In S. L. Swars, D. W. Stinson, & S. Lemons-Smith (Eds.), *Proceedings of the 31st annual meeting of the north American chapter of the international group for the psychology of mathematics education* (pp. 956–964). Atlanta, Georgia

Carlsen, M. (2010). Orchestrating mathematical activities in the kindergarten: the role of inquiry. *NoMAD, 15*(3), 51–72.

Cavanagh, M. (2012). From arithmetic to algebra: sequences and patterns as an introductory lesson in seventh grade mathematics. In J. Dindyal, L. P. Cheng, & S. F. Ng (Eds.), *Proceedings of the 35th annual conference of the Mathematics Education Research Group of Australasia, Singapore* (pp. 63–71).

Cavanagh, M., & McMaster, H. (2015). A professional experience learning community for secondary mathematics: developing pre-service teachers' reflective practice. *Mathematics Education Research Journal, 27*(4), 471–490. doi:10.1007/s13394-015-0145-z.

Cestari, M. L., Daland, E., Eriksen, S., & Jaworski, B. (2006). Working in a developmental research paradigm: the role of didactician/researcher working with teachers to promote inquiry practices in developing mathematics learning and teaching. In M. Bosche (Ed.), *Proceedings of the Fourth Congress of the European Society for Research in Mathematics Education* (pp. 1348–1358). San Feliu de Guíxols, Spain: CERME.

Chapman, O. (2008). Narratives in mathematics teacher education. In D. Tirosh & T. Wood (Eds.), *The international handbook of mathematics teacher education (Vol 2): Tools and processes in mathematics teacher education* (pp. 15–38). Dordrecht: Sense Publishers.

Chen, C.-H., & Chang, C.-Y. (2012). An exploration of mathematics teachers' discourse in a teacher professional learning community. In T.-Y. Tso (Ed.), *Proceedings of the 36th conference of the international group for the psychology of mathematics education* (Vol. 2, pp. 123–130).

Chevallard, Y. (1985). *La transposition didactique*. Grenoble: La Pensée Sauvage.

Clandinin, D. J., & Connelly, F. M. (2000). *Narrative inquiry: experience and story in qualitative research*. San Francisco: Jossey-Bass.

Clark-Wilson, A., Aldon, G., Cusi, A., Goos, M., Haspekian, M., Robutti, O., et al. (2014). The challenges of teaching mathematics with digital technologies—the evolving role of the teacher. In P. Liljedahl, C. Nichol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the joint meeting of PME 38 and PME-NA 36* (Vol. 1, pp. 87–116). Vancouver: University of British Columbia.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9–13.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*(3–4), 175–190.

Cochrane-Smith, M., & Lytle, S. (2009). *Inquiry as stance: practitioner research in the next generation*. Columbia: Teachers College Press.

Cochran-Smith, M., & Lytle, S. L. (1999). Relationships of knowledge and practice: teacher learning in communities. *Review of Research in Education, 24*, 249–305.

Coe, K., Carl, A., & Frick, L. (2010). Lesson study in continuing professional teacher development: a South African case study. *Acta Academica, 42*(4), 206–230.

Coles, A. (2012). Using video for professional development: the role of the discussion facilitator. *Journal of Mathematics Teacher Education, 16*(3), 165–184. doi:10.1007/s10857-012-9225-0.

Cooper, T. J., Baturo, A., & Grant, E. (2006). Collaboration with teachers to improve mathematics learning: pedagogy at three levels. In J. Novotna, H. Moraova, M. Kratka, & N. Stehlikova (Eds.), *Proceedings of the 30th conference of the international group for the psychology of mathematics education* (Vol. 2, pp. 361–367) Prague, Czech Republic: Charles University.

Cordingly, P., Bell, M., Evans, D., & Firth, A. (2005a). *The impact of collaborative CPD on classroom teaching and learning review: what do teacher impact data tell us about collaborative CPD?* London: EPPI-Centre, Social Science Research Unit, Institute of Education, University of London.

Cordingly, P., Bell, M., Rundell, B., Evans, D., & Curtis, A. (2003). *The impact of collaborative CPD on classroom teaching and learning: how does collaborative continuing professional development (CPD) for teachers of the 5-16 age range affect teaching and learning?* London: EPPI-Centre, Social Science Research Unit, Institute of Education, University of London.

Cordingly, P., Bell, M., Thomason, S., & Firth, A. (2005b). *The Impact of collaborative CPD on classroom teaching and learning: review: what do teacher impact data tell us about collaborative CPD?* London: EPPI-Centre, Social Science Research Unit, Institute of Education, University of London.

Da Ponte, J. P. (2008). Investigar a nossa prática: uma estratégia de formação e de construção do conhecimento professional. *PNA, 2*(4), 153–180.

Dalgarno, N., & Colgan, L. (2007). Supporting novice elementary mathematics teachers' induction in professional communities and providing innovative forms of pedagogical content knowledge development through information and communication technology. *Teaching and Teacher Education, 23*(7), 1051–1065.

Dawson, S. (2008). Trust and respect: a path laid while walking. In B. Jaworski, & T. Wood (Eds.), *International handbook of mathematics teacher education* (Vol. 4, pp. 315–334). Rotterdam, The Netherlands: Sense Publishers.

de Carvalho Borba, M. C., & Llinares, S. (2012). Online mathematics teacher education: overview of an emergent field of research. *ZDM, 44*(6), 697–704. doi:10.1007/s11858-012-0457-3.

de Geest, E., Back, J., Hirst, C., & Joubert, M. (2009). *Final report: researching effective CPD in mathematics education*. Sheffield: National Centre for Excellence in the Teaching of Mathematics.

Ding, L., Jones, K., Pepin, B., & Sikko, S. A. (2014). An expert teacher's local instruction theory underlying a lesson design study through school-based professional development. In C. Nichol, P. Liljedahl, S. Oesterle, & D. Allan (Eds.), *Joint meeting of*

PME 38 and PME-NA 36 (Vol. 2, pp. 401–408). Vancouver: University of British Columbia.

do Carmo de Sousa, M. (2011). *Quando professores de Matemática analisam o currículo que ministram.* Paper presented at the XIII CIAEM-IACME, Recife, Brasil.

Doig, B., & Groves, S. (2011). Japanese lesson study: professional development through communities of inquiry. *Mathematics Teacher Education and Development, 13*(1), 77–93.

Dowling, D. (2013). *Hungary for calculation: developing approaches to calculation in the new curriculum using Hungarian methodology as our inspiration (NCETM CTP4213).* Sheffield: NCETM. Retrieved from https://www.ncetm.org.uk/files/20365123/CTP4213+Final+Report.pdf.

Ebaeguin, M. (2015). Promoting teacher growth through lesson study: a culturally embedded approach. In M. Marshman, V. Geiger, & A. Bennison (Eds.), *Mathematics education in the margins: Proceedings of the 38th annual conference of the Mathematics Education Research Group of Australasia* (pp. 213–220). Sunshine Coast: MERGA.

Ebaeguin, M., & Stephens, M. (2014). Cultural challenges in adapting lesson study to a Philippines setting. *Mathematics Teacher Education and Development, 16*(1), 43–64.

Edelson, D. (2002). Design research: what we learn when we engage in design. *Journal of the Learning Sciences, 11*(1), 1–24. doi:10.1207/S15327809JLS1101_4.

Elipane, L. E. (2012). Infrastructures within the student teaching practicum that nurture elements of lesson study. In T.-Y. Tso (Ed.), *Proceedings of the 36th conference of the international group for the psychology of mathematics education* (Vol. 2, pp 219–226). Taipei, Taiwan: PME.

Ell, F., & Meissel, K. (2011). Working collaboratively to improve the learning and teaching of mathematics in a rural New Zealand community. *Mathematics Education Research Journal, 23*(2), 169–187.

Ellis, K. (2013). *Developing fluency and understanding of fractions through rich mathematical tasks (NCETM CTP1813).* Sheffield: NCETM. Retrieved from https://www.ncetm.org.uk/public/files/17489864/Final+Report+CTP1813.pdf.

Engeström, Y. (1999). Activity theory and individual and social transformation. In Y. Engeström, R. Miettinen, & R. L. Punamäki (Eds.), *Perspectives on activity theory* (pp. 19–38). Cambridge: Cambridge University Press.

Erfjord, I., Hundeland, P., & Carlsen, M. (2012). Kindergarten teachers' accounts of their developing mathematical practice. *ZDM, 44*(5), 653–664. doi:10.1007/s11858-012-0422-1.

Ernest, P. (1991). *The philosophy of mathematics education*. London: Falmer Press.

Fede, B., Civil, M., & Toscano, R. (2014). Exploring mathematics together: figuring the worlds of teachers and prospective teachers. In S. Oesterle, P. Liljedahl, C. Nicol, & D. Allan (Eds.), *Proceedings of the joint meeting of PME 38 and PME-NA 36* (Vol. 3, pp. 41–48). Vancouver: University of British Columbia.

Foster, C., Wake, G., & Swan, M. (2014). Mathematical knowledge for teaching problem solving: lessons from lesson study. In S. Oesterle, P. Liljedahl, C. Nichol, & D. Allan (Eds.), *Proceedings of the joint meeting 3–97 of PME 38 and PME-NA 36* (Vol. 3, pp. 97–104). Vancouver: University of British Columbia.

Freudenthal, H. (1991). *Revisiting mathematics education*. Dordrecht: Kluwer Academic Publishers.

Fried, M. N., & Amit, M. (2005). A spiral task as a model for in-service teacher education. *Journal of Mathematics Teacher Education, 8*(5), 419–436. doi:10.1007/s10857-005-3850-9.

Geiger, V., & Goos, M. (2006). Living in the gap: a tale of two different types of researchers. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures, and learning spaces: proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia* (pp. 254–261). Adeleide: MERGA.

Gellert, U. (2008). Routines and collective orientations in mathematics teachers' professional development. *Educational Studies in Mathematics, 67*(2), 93–110.

Gestoso de Souza, A. P., & Anunciato de Oliveira, R. M. (2013). *Creating stories for children with Mathematical contents in a collaborative group: teaching learnings.* Paper presented at the Proceeding of the VII Ibero-American Congress on Mathematics Education.

Gilbert, M., & Gilbert, B. (2013). Connecting teacher learning to curriculum. In *Proceedings of the 37th conference of the international group for the psychology of mathematics education* (Vol. 2, pp. 337–344). Kiel, Germany: PME.

Goodchild, S. (2008). A quest for 'good' research. In B. Jaworski & T. Wood (Eds.), *The international handbook of mathematics teacher education (Vol 4) The mathematics teacher educator as a developing professional* (pp. 201–220). Dordrecht: Sense Publishers.

Goodchild, S. (2013). Imagination and teaching development. In *Proceedings of the 37th conference of the international group for the psychology of mathematics education* (Vol. 2, pp. 369–376). Kiel, Germany: PME.

Goodchild, S. (2014). Mathematics teaching development: learning from developmental research in Norway. *ZDM, 46*(2), 305–316. doi:10.1007/s11858-013-0567-6.

Goodchild, S., Fuglestad, A. B., & Jaworski, B. (2013). Critical alignment in inquiry-based practice in developing mathematics teaching. *Educational Studies in Mathematics, 84*(3), 393–412.

Goos, M. (2005). A sociocultural analysis of the development of preservice and beginning teachers' pedagogical identities as users of technology. *Journal of Mathematics Teacher Education, 8*(1), 35–59. doi:10.1007/s10857-005-0457-0.

Goos, M. (2012). Creating opportunities to learn in mathematics education: a sociocultural journey. In T.-Y. Tso (Ed.), *Proceedings of the 36th conference of the international group for the psychology of mathematics education* (Vol. 1, pp. 67–82). Taipei, Taiwan: PME.

Goos, M. (2014). Researcher–teacher relationships and models for teaching development in mathematics education. *ZDM, 46*(2), 189–200. doi:10.1007/s11858-013-0556-9.

Goos, M., & Bennison, A. (2008). Developing a communal identity as beginning teachers of mathematics: emergence of an online community of practice. *Journal of Mathematics Teacher Education, 11*(1), 41–60. doi:10.1007/s10857-007-9061-9.

Gravemeijer, K. (1994). Educational development and developmental research in mathematics education. *Journal for Research in Mathematics Education, 25*(5), 443–471. doi:10.2307/749485.

Gravemeijer, K., & van Eerde, D. (2009). Design research as a means for building a knowledge base for teachers and teaching in mathematics education. *The Elementary School Journal, 109*(5), 510–524.

Groves, S. (2013). Implementing the Japanese problem solving lesson structure. In V. Steinle, L. Ball, & C. Bardini (Eds.), *Mathematics education: yesterday, today and tomorrow: proceedings of the 36th annual conference of the Mathematics Education Research Group of Australasia* (pp. 712–714). Melbourne: MERGA.

Heffernan, J. (2013). *Improving continuity and progression in mathematics across the KS2/3 boundary (NCETM CTP3113).* Sheffield: NCETM. Retrieved from https://www.ncetm.org.uk/public/files/20637602/CTP3113+final+report.pdf.

Hoek, D., & Gravemeijer, K. (2011). Changes of interaction during the development of a mathematical learning environment. *Journal of Mathematics Teacher Education, 14*(5), 393–411.

Hospesovà, A., Machàckovà, J., & Tichà, M. (2006). Joint reflection as a way to cooperation between researchers and teachers. In *Proceedings of the 30th conference of the international group for the psychology of mathematics education* (Vol. 1, pp. 99–103). Prague, Czech Republic: PME.

Howard, P., & Perry, B. (2007). A school-community model for enhancing Aboriginal students' mathematical learning. In J. W. K. Beswick (Ed.), *Mathematics: essential research, essential practice: Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia* (pp. 402–411). Hobart: MERGA.

Huang, R., Su, H., & Xu, S. (2014). Developing teachers' and teaching researchers' professional competence in mathematics through Chinese Lesson Study. *ZDM, 46*(2), 239–251. doi:10.1007/s11858-013-0557-8.

Hunter, J., & Back, J. (2011). Facilitating sustainable professional development through lesson study. *Mathematics Teacher Education and Development, 13*(1), 94–114.

Hurst, C., Armstrong, T., & Young, M. (2011). Making a difference for indigenous children. In J. Clark, B. Kissane, J. Mousley, T. Spencer, & S. Thornton (Eds.), *Mathematics: traditions and [New] practices: proceedings of the 34th annual conference of the Mathematics Education Research Group of Australasia and the Australian Association of Mathematics Teachers* (pp. 373–381). Adelaide: AAMT and MERGA.

Inoue, N. (2011). Zen and the art of neriage: facilitating consensus building in mathematics inquiry lessons through lesson study. *Journal of Mathematics Teacher Education, 14*(1), 5–23.

Inprasitha, M. (2015). Transforming education through lesson study: Thailand's decade-long journey. In S. Cho (Ed.), *Selected regular lectures from the 12th International Congress on Mathematical Education* (pp. 343–354). Switzerland: Springer.

Inprasitha, M., Isoda, M., Wang-Iverson, P., & Yeap, B. (Eds.). (2015). *Lesson study: challenges in mathematics education (Vol. 3, Series on Mathematics Education)*. Singapore: World Scientific.

Isoda, M. (2015). The science of lesson study in the problem solving approach. In M. Inprasitha, M. Isoda, P. Wang-Iverson, & B. Yeap (Eds.), *Lesson study: challenges in mathematics education* (pp. 81–108). Singapore: World Scientific.

Isoda, M., Arcavi, A., & Mena Lorca, A. (2012). *El Estudio de Clases Japonés en Matemáticas: su importancia para el mejoramiento de los aprendizajes en el escenario global* Valparaíso: Ediciones Universitarias de Valparaíso.

Isoda, M., & Olfos, R. (2011). *Enseñanza de la Multiplicación : desde el estudio de clases japonés a las propuestas iberoamericanas*. Valparaiso: Universitarias de Valparaíso.

Isoda, M., Stephens, M., Ohara, Y., & Miyakawa, T. (Eds.). (2007). *Japanese lesson study in mathematics: its impact, diversity and potential for educational improvement*. Singapore: World Scientific.

Jackson, K., Cobb, P., Wilson, J., Webster, M., Dunlap, C., & Appelgate, M. (2015). Investigating the development of mathematics leaders' capacity to support teachers' learning on a large scale. *ZDM, 47*(1), 93–104. doi:10.1007/s11858-014-0652-5.

Japan Society of Mathematics Education. (2000). Special issue: exploring secondary school mathematics education of Japan: practical studies in the 1990s. *Journal of the Japan Society of Mathematics Education, 82*(9), 15–151.

Japan Society of Mathematics Education. (2010). Mathematics education theories for lesson study: problem solving approach and the curriculum through extension and integration. *Journal of the Japan Society of Mathematics Education, 92*(11), 83–157.

Jaworski, B. (1998). Mathematics teacher research: process, practice and the development of teaching. *Journal of Mathematics Teacher Education, 1*(1), 3–31. doi:10.1023/a:1009903013682.

Jaworski, B. (2003). Research practice into/influencing mathematics teaching and learning development: towards a theoretical framework based on co-learning partnerships. *Educational Studies in Mathematics, 54*, 249–282.

Jaworski, B. (2005). Learning communities in mathematics: creating an inquiry community between teachers and didacticians. *Research in Mathematics Education, 7*(1), 101–119. doi:10.1080/14794800008520148.

Jaworski, B. (2006). Theory and practice in mathematics teaching development: critical inquiry as a mode of learning in teaching. *Journal of Mathematics Teacher Education, 9*(2), 187–211.

Jaworski, B. (2008). Building and sustaining inquiry communities in mathematics teaching development: teachers and didacticians in collaboration. In K. Krainer & T. Wood (Eds.), *The international handbook of mathematics teacher education* (Vol. 3, pp. 309–330). Rotterdam: Sense Publishers.

Jaworski, B., & Goodchild, S. (2006). Inquiry community in an activity theory frame. In J. Novotna, H. Moraova, M. Kratka, & N. Stehlikova (Eds.), *Proceedings of the 30th conference of the international group for the psychology of mathematics education* (Vol. 3, pp. 353–360). Prague, Czech Republic: PME.

Jaworski, B., Goodchild, S., Eriksen, S., & Daland, E. (2011). Mediating mathematics teaching development and pupils' mathematics learning: the life cycle of a task. In O. Zaslavsky & P. Sullivan (Eds.), *Constructing knowledge for teaching secondary mathematics: tasks to enhance prospective and practicing teacher learning* (pp. 143–160). Boston: Springer.

Jaworski, B., & Huang, R. (2014). Teachers and didacticians: key stakeholders in the processes of developing mathematics teaching. *ZDM, 46*(2), 173–188. doi:10.1007/s11858-014-0574-2.

Joubert, M. (2013). Using digital technologies in mathematics teaching: developing an understanding of the landscape using three "grand challenge" themes. *Educational Studies in Mathematics, 82*, 341–359. doi:10.1007/s10649-012-9430-x.

Joubert, M., & Sutherland, R. (2010). *Understanding teacher enquiry.* Paper presented at the British Society for Research into Learning Mathematics, University of Nottingham, June 2010.

Kelly, A. (2004). Design research in education: yes, but is it methodological? *Journal of the Learning Sciences, 13*(1), 115–128.

Kieran, C., Krainer, K., Shaughnessy, J., & Clements, M. (2013). Linking research to practice: teachers as key stakeholders in mathematics education research. In A. J. Bishop, C. Keitel, J. Kilpatrick, & F. K. Leung (Eds.), *Third international handbook of mathematics education* (Vol. 27, pp. 361–392). New York: Springer.

King, E., & Murata, A. (2005). Mathematics lesson study teachers learn about representations to promote communication: a case study. In G. M. Lloyd, M. Wilson, J. L. M. Wilkins, & S. L. Behm (Eds.), *Proceedings of the 27th annual meeting of the north American chapter of the international group for the psychology of mathematics education* (pp. 741–748). Roanoke, Virginia.

Kirshner, D., & Whitson, J. A. (1997). *Situated cognition: social, semiotic, and psychological perspectives*. Mahwah: Erlbaum.

Kotelawala, U. (2010). Collaborative planning for a unit on the quadratic formula. *The Mathematics Teacher*, 669–674.

Krainer, K., & Zehetmeier, S. (2013). Inquiry-based learning for students, teachers, researchers, and representatives of educational administration and policy: reflections on a nation-wide initiative fostering educational innovations. *ZDM, 45*(6), 875–886. doi:10.1007/s11858-013-0537-z.

Krammer, K., Ratzka, N., Klieme, E., Lipowsky, F., Pauli, C., & Reusser, K. (2006). Learning with classroom videos: conception and first results of an online teacher-training program. *ZDM, 38*(5), 422–432.

Kullberg, A., Mårtensson, P., & Runesson, U. (2016). What is to be learned? Teachers' collective inquiry into the object of learning. *Scandinavian Journal of Educational Research*, *60*(3), 309-322.

Kusanagi, K. N. (2014). The Bureaucratising of lesson study: a Javanese case. *Mathematics Teacher Education and Development, 16*, 84–103.

Kyei-Blankson, L. (2014). Training math and science teacher–researchers in a collaborative research environment: implications for math and science education. *International Journal of Science and Mathematics Education, 12*(5), 1047–1065. doi:10.1007/s10763-013-9444-6.

Kynigos, C., & Kalogeria, E. (2012). Boundary crossing through in-service online mathematics teacher education: the case of scenarios and half-baked microworlds. *ZDM, 44*(6), 733–745. doi:10.1007/s11858-012-0455-5.

Lave, J. (1988). *Cognition in practice: mind, mathematics and culture in everyday life*. New York: Cambridge University Press.

Lave, J., & Wenger, E. (1991). *Situated learning: legitimate peripheral participation*. New York: Cambridge University Press.

Lebethe, A., Eddy, N., & Bennie, K. (2006). Opening the space of possibilities: tales from three teachers. In J. Novotna, H. Moraova, M. Kratka, & N. Stehlikova (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 103–106). Prague, Czech Republic: PME.

Lerman, S. (2000). The social turn in mathematics education research. In J. Boaler (Ed.), *Multiple perspectives in mathematics teaching and learning* (pp. 19–44). Westport: Greenwood Publishing Group.

Lewis, C., & Perry, R. (2014). Lesson study with mathematical resources: a sustainable model for locally-led teacher professional learning. *Mathematics Teacher Education and Development, 16*(1), 22–42.

Lewis, C., Perry, R., & Hurd, J. (2009). Improving mathematics instruction through lesson study: a theoretical model and North American case. *Journal of Mathematics Teacher Education, 12*(4), 285–304.

Lin, P.-J. (2007). The effect of a mentoring development program on mentors' conceptualizing mathematics teaching and mentoring. In J.-H. Woo, H.-C. Lew, K.-S. Park, & D.-Y. Seo (Eds.), *Proceedings of the 31st conference of the international gruop for the psychology of mathematics education* (Vol. 3, pp. 201–208). Seoul, Korea: PME.

Lo, M. L., Marton, F., Pang, M. F., & Pong, W. Y. (2004). Toward a pedagogy of learning. In F. Marton & A. Tsui (Eds.), *Classroom discourse and the space of learning* (pp. 189–232). Mahwah: Lawrence Erlbaum Associates.

Lobo da Costa, N. M., & Prado, M. E. (2011). *Formação Continuada do Professor de Matemática – o trabalho colaborativo e o desenvolvimento profissional docento*. Paper presented at the XIII CIAEM-IACME, Recife, Brasil.

Marquesin, D. F. B., & Nacarato, A. M. (2011). A prática do saber e o saber da prática em geometria: análise do movimento vivido por um grupo de professoras dos anos iniciais do Ensino Fundamental. *Zetetiké, 19*(35), 102–137.

Martins, C., & Santos, L. (2012). Development of reflection ability in continuous training in mathematics (PFCM). In T.-Y. Tso (Ed.), *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 193–200). Taipei, Taiwan: PME.

Marton, F., & Booth, S. (1997). *Learning and awareness*. Mahwah: Lawrence Erlbaum Associates.

McDougall, M., & Nason, R. (2005). Growth of teacher knowledge within an on-line collaborative learning environment. In P. C. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, et al. (Eds.), *Building connections: research, theory and practice: Proceedings of the 28th annual conference of the mathematics education research group of Australasia, Sydney* (pp. 529–536): MERGA.

McKenney, S., & Visscher-Voerman, I. (2013). Formal education of curriculum and instructional designers. *Journal of the International Society for Design and Development in Education, 2*, 1–20.

Menezes, L. (2011). Collaborative research as a strategy of professional development of teachers. In B. Ubuz (Ed.), *Proceedings of the 35th conference of international group for the psychology of mathematics education* (Vol. 3, pp. 252–232). Ankara, Turkey: PME.

Meng, C. C., & Sam, L. C. (2011). Enhancing pre-service secondary mathematics teachers' skills of using the geometer's sketchpad through lesson study. *Journal of Science and Mathematics Education in Southeast Asia, 34*(1), 90–110.

Miyazaki, T. (2015). Is changing teaching practice the mission impossible. A case study of continuing professional development for primary school teachers in Senegal. *Compare: A Journal of Comparative and International Education,*. doi:10.1080/03057925.2015.1043238.

Morris, J. C., & Miller-Stevens, K. (2016). *Advancing collaboration theory: models, typologies, and evidence*. London: Routledge Publishing.

Muñoz-Catalán, M. C., Carrillo, J., & Climent, N. (2010a). Mathematics teacher change in a collaborative environment: to what extent and how. *Journal of Mathematics Teacher Education, 13*(5), 425–439. doi:10.1007/s10857-010-9157-5.

Muñoz-Catalán, M. C., Climent, N., Carrillo, J., & Contreras, L. C. (2010b). Cognitive processes associated with the professional development of the mathematics teacher. *PNA, 43*(3), 87–97.

Nacarato, A. M., & Grando, R. C. (2013). The learning of teachers in a research community: math class as an object of study. In *Proceedings of the VII Ibero-American Congress on Mathematics Education* (pp. 4883–4890).

Nason, R., Chalmers, C., & Yeh, A. (2012). Facilitating growth in prospective teachers' knowledge: teaching geometry in primary schools. *Journal of Mathematics Teacher Education, 15*(3), 227–249. doi:10.1007/s10857-012-9209-0.

Nickerson, S., & Moriarty, G. (2005). Professional communities in the context of teachers' professional lives: a case of mathematics specialists. *Journal of Mathematics Teacher Education, 8*, 113–140.

Norton, A. H., & McCloskey, A. (2008). Teaching experiments and professional development. *Journal of Mathematics Teacher Education, 11*(4), 285–305. doi:10.1007/s10857-008-9076-x.

Nyaumwe, L. J. (2009). Peer influence on mathematics student teacher development of teaching skills during school experience. In M. Schäfer & C. McNamara (Eds.), *Proceedings of the 17th Annual Meeting of the Southern African Association for Research in Mathematics, Science and Technology Education, Rhodes* (Vol. 2, pp. 658–667).

Olsen, J. C. (2005). Do Teachers change their practices while participating in a lesson study?. In P. C. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, et al. (Eds.), *Building Connections: research, theory and practice: Proceedings of the 28th Annual Conference of the Mathematics Education Research Group of Australasia, Sydney* (pp. 592–600): MERGA.

Olsen, J. C., & Kirtley, K. (2005). The transition of a secondary mathematics teacher: from a reform listener to a believer. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Annual Conference for the Psychology of Mathematics Education* (Vol. 4, pp. 25–32). Melbourne, Australia: PME.

Ong, E. G., Lim, C. S., & Ghazali, M. (2010). Examining the changes in novice and experienced mathematics teachers'

questioning techniques through the lesson study process. *Journal of Science and Mathematics Education in Southeast Asia, 33*(1), 86–109.

Otani, M. (2009). In search of theoretical perspective on the "lesson study" in mathematics. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 105–107). Thessaloniki, Greece: PME.

Owens, K. (2008). Culturality in mathematics education: a comparative study. *NoMAD, 13*(4), 7–28.

Peng, A. (2007). Knowledge growth of mathematics teachers during professional activity based on the task of lesson explaining. *Journal of Mathematics Teacher Education, 10*(4), 289–299. doi:10.1007/s10857-007-9041-0.

Peter-Koop, A., Santos-Wagner, V., Breen, C., & Begg, A. (2003). *Collaboration in teacher education*. Dordrecht: Springer.

Pires, M. V., & Martins, C. (2009). *Olhares sobre um plano de formação contínua em Matemática*. Paper presented at the VI CIBEM or Ibero-American Congress on Mathematics Education, Puerto Monte, Chile.

Ponte, J. P., Segurado, I., & Oliveira, H. (2003). A collaborative project using narratives: what happens when pupils work on mathematical investigations? In A. Peter-Koop, V. Santos-Wagner, C. Breen, & A. Begg (Eds.), *Collaboration in teacher education: examples from the context of mathematics education* (pp. 85–97). Dordrecht: Kluwer.

Posthuma, B. (2012). Mathematics teachers' reflective practice within the context of adapted lesson study. *Pythagoras 33,* 3. doi:10.4102/pythagoras.v33i3.140.

Potari, D. (2013). The relationship of theory and practice in mathematics teacher professional development: an activity theory perspective. *ZDM, 45*(4), 507–519. doi:10.1007/s11858-013-0498-2.

Potari, D. & Jaworski, B. (2002) Tackling complexity in mathematics teacher development: using the teaching triad as a tool for reflection and enquiry. *Journal of Mathematics Teacher Education*, *5*(4), 351–380.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: first steps towards a conceptual framework. *ZDM, 40*(2), 165–178. doi:10.1007/s11858-008-0086-z.

Puchnera, L., & Taylor, A. (2006). Lesson study, collaboration and teacher efficacy: stories from two school-based math lesson study groups. *Teaching and Teacher Education, 22*(7), 922–924.

Redmond, T., Brown, R., & Sheehy, J. (2011). Reflecting on participation in research communities of practice: situating change in the development of mathematics teaching. In J. Clark, B. Kissane, J. Mousley, T. Spencer, & S. Thornton (Eds.), *Mathematics: traditions and [new] practices: proceedings of the 34th annual conference of the Mathematics Education Research Group of Australasia and the Australian Association of Mathematics Teachers* (pp. 657–659). Adelaide: AAMT and MERGA.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: the knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education, 8*, 255–281.

Sakonidis, C., & Potari, D. (2014). Mathematics teacher educators'/researchers' collaboration with teachers as a context for professional learning. *ZDM, 46*(2), 293–304. doi:10.1007/s11858-014-0569-z.

Sembiring, R. K., Hadi, S., & Dolk, M. (2008). Reforming mathematics learning in Indonesian classrooms through RME. *ZDM, 40*(6), 927–939.

Sensevy, G., Forest, D., Quilio, S., & Morales, G. (2013). Cooperative engineering as a specific design-based research. *ZDM, 45*(7), 1031–1043. doi:10.1007/s11858-013-0532-4.

Shimizu, Y. (2013). Working at the intersection of research and practice: a perspective on the study and improvement of mathematics lessons. In V. Steinle, L. Ball, & C. Bardini (Eds.), *Mathematics education: yesterday, today and tomorrow: Proceedings of the 36th annual conference of the Mathematics Education Research Group of Australasia* (pp. 2–12). Melbourne: MERGA.

Shimizu, S., & Chino, K. (2015). History of lesson study to develop good practices in Japan. In M. Inprasitha, M. Isoda, P. Wang-Iverson, & B. H. Yeap (Eds.), *Lesson study: challenges in mathematics education* (pp. 123–140). Singapore: World Scientific.

Silver, E. A., Charalambous, C. Y., Strawhun, B., Font, B. F., & Stylianides, G. J., (2006). Focusing on teacher learning: revisiting the issue of having students consider multiple solutions for mathematics problems. In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez (Eds.), Proceedings of the *28th annual meeting of the north American chapter of the international group for the psychology of mathematics education* (pp. 294–301). Mérida, Yucatán, México.

Simon, M. (2008). The challenge of mathematics teacher education in an era of mathematics education reform. In B. Jaworski & T. Wood (Eds.), *The international handbook of mathematics teacher education: The mathematics teacher educator as a developing professional* (Vol. 4, pp. 17–30). Dordrecht: Sense Publishers.

Sinclair, N., & Robutti, O. (2014). Teaching practices in digital environments. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 598–601). Dordrecht: Springer.

Slavit, D., & Nelson, T. H. (2009). Collaborative teacher inquiry as a tool for building theory on the development and use of rich mathematical tasks. *Journal of Mathematics Teacher Education, 13*(3), 201–221. doi:10.1007/s10857-009-9136-x.

Stahl, G., Koschmann, T., & Suthers, D. (2006). Computer-supported collaborative learning: an historical perspective. In R. K. Sawyer (Ed.), *Cambridge handbook of the learning sciences* (pp. 409–426). Cambridge: Cambridge University Press.

Stephens, M., & Isoda, M. (2007). Introduction to the English translation. In M. Isoda, M. Stephens, Y. Ohara, & T. Miyakawa (Eds.), *Japanese lesson study in mathematics: its impact, diversity and potential for educational improvement* (pp. xv–xxiv). Singapore: World Scientific.

Stigler, J. W., Gonzales, P., Kawanaka, T., Knoll, S., & Serrano, A. (1999). The TIMSS videotape classroom study: methods and findings from an exploratory research project on eighth-grade mathematics instruction in Germany, Japan, and the United States (NCES 1999-074). Washington DC: US Department of Education, Office of Educational Research and Improvement, National Center for Education Statistics.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: best ideas from the world's teachers for improving education in the classroom*. New York: Free Press.

Swan, M., Pead, D., Doorman, M., & Mooldijk, A. (2013). Designing and using professional development resources for inquiry-based learning. *ZDM, 45*(7), 945–957. doi:10.1007/s11858-013-0520-8.

Takahashi, A. (2014). The role of the knowledgeable other in lesson study: examining the final comments of experienced lesson study practitioners. *Mathematics Teacher Education and Development, 16*(1), 4–21.

Turner, F. (2008). Growth in teacher knowledge: individual reflection and community participation. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda (Eds.), *Proceedings of the Joint Meeting of PME 32 and PME-NA XXX* (pp. 353–360). México: Cinvestav-UMSNH.

Vale, C., Davies, A., Weaven, M., Hooley, N., Davidson, K., & Loton, D. (2010). Leadership to improve mathematics outcomes in low

SES schools and school networks. *Mathematics Teacher Education and Development, 12*(2), 47–71.

van Es, E. A., & Sherin, M. G. (2009). The influence of video clubs on teachers' thinking and practice. *Journal of Mathematics Teacher Education, 13*(2), 155–176. doi:10.1007/s10857-009-9130-3.

Verhoef, N., Tall, D., Coenders, F., & van Smaalen, D. (2014). The complexities of a lesson study in a Dutch situation: mathematics teacher learning. *International Journal of Science and Mathematics Education, 12*(4), 859–881. doi:10.1007/s10763-013-9436-6.

Wake, G., Foster, C., & Swan, M. (2013). A theoretical lens on lesson study: professional learning across boundaries. In A. Lindmeier, & A. Heinze (Eds.), *Proceedings of the 37th conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 369–376).

Warren, E. (2008). Early childhood teachers' professional learning in early algebraic thinking: a model that supports new knowledge and pedagogy. *Mathematics Teacher Education and Development, 10*, 30–45.

Watson, A., & De Geest, E. (2014). Department-initiated change. *Educational Studies in Mathematics, 87*(3), 351–368. doi:10.1007/s10649-014-9549-z.

Wells, G. (1999). *Dialogic inquiry: towards a sociocultural practice and theory of education*. Cambridge: Cambridge University Press.

Wenger, E. (1998). *Communities of practice. Learning, meaning and identity*. Cambridge: Cambridge University Press.

White, A. L. (2007). Two related approaches to teacher professional learning in the Asia Pacific Region. *Journal of Science and Mathematics Education in Southeast Asia, 30*(1), 67–83.

White, A. L., & Lim, C. S. (2008). Lesson study in Asia Pacific classrooms: local responses to a global movement. *ZDM, 40*(6), 915–925.

Wood, T., & Berry, B. (2003). What does "design research" offer mathematics teacher education? *Journal of Mathematics Teacher Education, 6*(3), 195–199. doi:10.1023/a:1025180118053.

Zaslavsky, O. (2008). Meeting the challenges of mathematics teacher education through design and use of tasks that facilitate teacher learning. In B. Jaworski & T. Wood (Eds.), *International handbook of mathematics teacher education: the mathematics teacher educator as a developing professional* (Vol. 4, pp. 93–114). Rotterdam: Sense.