Full length article

Explicit matrix inverses for lower triangular matrices with entries involving Jacobi polynomials

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Abstract

For a two-parameter family of lower triangular matrices with entries involving Jacobi polynomials an explicit inverse is given, with entries involving a sum of two Jacobi polynomials. The formula simplifies in the Gegenbauer case and then one choice of the parameter solves an open problem in a recent paper by Koelink, van Pruijssen & Román. The two-parameter family is closely related to two two-parameter groups of lower triangular matrices, of which we also give the explicit generators. Another family of pairs of mutually inverse lower triangular matrices with entries involving Jacobi polynomials, unrelated to the family just mentioned, was given by J. Koekoek & R. Koekoek (1999). We show that this last family is a limit case of a pair of connection relations between Askey–Wilson polynomials having one of their four parameters in common.

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1. Introduction

This note started as a kind of supplement to the paper [13] by Koelink, van Pruijssen and Román, but gradually it got a wider scope. As for [13] it solves an open problem there (see Theorem 2.1 and paragraph after Theorem 6.2 in [13]) to invert a lower triangular matrix with entries involving Gegenbauer polynomials. For a two-parameter family of such matrices involving Jacobi polynomials we give the explicit inverse matrix in Theorem 4.1. Specialization to Gegenbauer polynomials then gives a one-parameter family. One specialization of the parameter in the latter family gives the inversion desired in [13]. Another specialization gives a matrix inversion already handled by Brega and Cagliero [3].

Our two-parameter family of Jacobi polynomials is closely related to two commutative two-parameter groups of lower triangular matrices involving Jacobi polynomials. We also give the explicit infinitesimal generators of these two-parameter groups. Furthermore we obtain a biorthogonality relation for two explicit systems of functions on $\mathbb{Z}$ involving Jacobi polynomials with respect to an explicit bilinear form on $\mathbb{Z}$.

Another two-parameter family of pairs of mutually inverse lower triangular matrices with entries involving Gegenbauer polynomials, unrelated to the family mentioned above, is implied by Brown and Roman [4, (4.14)]. J. Koekoek and R. Koekoek [11, (17)], unaware of [4], generalized a one-parameter subfamily of this two-parameter family to entries involving Jacobi polynomials. We will show that this last family can be realized as a limit case of a pair of connection relations between Askey–Wilson polynomials having one of their four parameters in common. These Askey–Wilson connection coefficients were first given by Askey and Wilson [2, (6.5)]. The limit case connects Jacobi polynomials $P_n^{(\alpha,\beta)}$ with shifted monomials $x \mapsto (x - y)^k$.

The contents of the paper are as follows. In Section 2 some preliminaries about Jacobi polynomials are given. Degenerate cases of Jacobi polynomials are classified in Section 3. The main results about the mutually inverse lower triangular matrices are stated in Section 4. This section ends with some open problems. The computations leading to the explicit inverse matrix of the first family of lower triangular matrices are given in Section 5. The two-parameter groups and their generators are treated in Section 6. The biorthogonal systems with respect to an explicit bilinear form are the topic of Section 7. Finally, the computations giving the limit of the Askey–Wilson connection relations are done in Section 8.

The reader may start in Section 4 and then continue with Section 5 or with Sections 6 and 7 or with Section 8. The preliminary Sections 2 and 3 can be consulted when needed.

2. Preliminaries about Jacobi polynomials

Jacobi polynomials (see for instance [15, Chapter IV], [1, Chapter 6], [8, Chapter 4], [12, Section 9.8], [14, Chapter 18]) can be expressed in terms of the Gauss hypergeometric function by

$$
\begin{aligned}
P_n^{(\alpha,\beta)}(x) &:= \frac{(\alpha + 1)_n}{n!} \, {}_2F_1\left(\begin{matrix} -n, n + \alpha + \beta + 1 \\ \alpha + 1 \end{matrix}; \tfrac{1}{2}(1 - x)\right) \\
&= \sum_{k=0}^{n} \frac{(n + \alpha + \beta + 1)_k \, (\alpha + k + 1)_{n-k}}{k! \, (n - k)!} \left(\frac{x - 1}{2}\right)^k.
\end{aligned}
\tag{2.1}
$$

Note that they are well-defined for all values of α, β. Their normalization avoids artificial singularities. Jacobi polynomials satisfy a Rodrigues formula

$$
P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} (1 - x)^{-\alpha} (1 + x)^{-\beta} \left(\frac{d}{dx}\right)^n \left((1 - x)^{n+\alpha}(1 + x)^{n+\beta}\right).
\tag{2.2}
$$

For $\alpha = \beta$ Jacobi polynomials are often written as Gegenbauer polynomials:

$$C_n^{(\lambda)}(x) := \frac{(2\lambda)_n}{\left(\lambda + \frac{1}{2}\right)_n} P_n^{\left(\lambda - \frac{1}{2}, \lambda - \frac{1}{2}\right)}(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (\lambda)_{n-k}}{k! \, (n-2k)!} (2x)^{n-2k}$$

$$= \frac{2^n (\lambda)_n}{n!} x^n \, {}_2F_1\left(\begin{matrix} -\frac{1}{2}n, -\frac{1}{2}n + \frac{1}{2}; \\ 1 - \lambda - n \end{matrix} \frac{1}{x^2}\right), \tag{2.3}$$

where we also used [6, 10.9(18)]. Thus $C_n^{(0)}(x) = \delta_{n,0}$, which will be kept as a convention in this paper, although in the literature the case $\lambda = 0$ is usually rescaled in order to obtain the Chebyshev polynomials of the first kind. In the proportionality factor in the second part of (2.3) artificial singularities can occur. This factor should be understood by continuity in λ. We can rewrite the first equality in (2.3) as

$$C_{2m}^{(\lambda)}(x) = \frac{2^{2m} (\lambda)_m}{\left(\lambda + m + \frac{1}{2}\right)_m} P_{2m}^{\left(\lambda - \frac{1}{2}, \lambda - \frac{1}{2}\right)}(x),$$

$$C_{2m-1}^{(\lambda)}(x) = \frac{2^{2m-1} (\lambda)_m}{\left(\lambda + m - \frac{1}{2}\right)_m} P_{2m-1}^{\left(\lambda - \frac{1}{2}, \lambda - \frac{1}{2}\right)}(x)$$

or as

$$C_n^{(\lambda)}(x) = \frac{2^{2n} (\lambda)_n}{(n + 2\lambda)_n} P_n^{\left(\lambda - \frac{1}{2}, \lambda - \frac{1}{2}\right)}(x). \tag{2.4}$$

In the Legendre case $\alpha = \beta = 0$ we write $P_n(x) := P_n^{(0,0)}(x)$. There are symmetries

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x), \qquad C_n^{(\lambda)}(x) = (-1)^n C_n^{(\lambda)}(-x). \tag{2.5}$$

For Jacobi polynomials we will need the following generating function (see [15, (4.4.5)]):

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) \, w^n = 2^{\alpha+\beta} R^{-1} (1 - w + R)^{-\alpha} (1 + w + R)^{-\beta},$$

$$R := (1 - 2xw + w^2)^{\frac{1}{2}}, \tag{2.6}$$

convergent for $x \in [-1, 1]$, $|w| < 1$. A more simple generating function for Gegenbauer polynomials (but not the case $\alpha = \beta$ of (2.6)) is the following (see [15, (4.7.23)]):

$$\sum_{n=0}^{\infty} C_n^{(\lambda)}(x) \, w^n = (1 - 2xw + w^2)^{-\lambda} \quad (x \in [-1, 1], \ |w| < 1). \tag{2.7}$$

3. Degenerate cases of Jacobi polynomials

This section is not needed very much in the sequel. It may be skipped on first reading.

For $\alpha, \beta > -1$ Jacobi polynomials are orthogonal on the interval $(-1, 1)$ with respect to the weight function $(1 - x)^\alpha (1 + x)^\beta$, but we will not deal with this property in the paper. However, since in our formulas α, β will be allowed to be arbitrarily complex, and definitely not only larger than -1, it is relevant to see which degeneracies can occur in (2.1), i.e., when coefficients in the

sum on the right of (2.1) become zero (here we assume $n > 0$). There are two shifted factorials in the numerator of the terms which can cause this:

1. $(n+\alpha+\beta+1)_k = 0$ for some $k \in \{1, \ldots, n\}$, i.e., $(n+\alpha+\beta+1)_n = 0$, i.e., $\alpha+\beta \in \mathbb{Z}_{\leq -2}$, $n+\alpha+\beta+1 \leq 0$ and $2n+\alpha+\beta \geq 0$. Then $(n+\alpha+\beta+1)_k = 0$ for $k = -n-\alpha-\beta, \ldots, n$.
2. $(\alpha+k+1)_{n-k} = 0$ for some $k \in \{0, 1, \ldots, n-1\}$, i.e., $(\alpha+1)_n = 0$, i.e., $\alpha \in \mathbb{Z}_{\leq -1}$ and $n+\alpha \geq 0$. Then $(\alpha+k+1)_{n-k} = 0$ for $k = 0, \ldots, -\alpha-1$.

By combining these two cases we see when $(n+\alpha+\beta+1)_k(\alpha+k+1)_{n-k} = 0$ for all $k \in \{0, \ldots, n\}$:

Proposition 3.1. $P_n^{(\alpha,\beta)}(x) = 0$ *identically in x iff* $\alpha, \beta \in \mathbb{Z}_{\leq -1}$ *and* $\max(-\alpha, -\beta) \leq n \leq -\alpha - \beta - 1$.

Case 1 above causes that $P_n^{(\alpha,\beta)}(x)$ has degree lower than n in x, while case 2 causes that $P_n^{(\alpha,\beta)}(x)$ vanishes for $x = 1$ with a certain multiplicity. A similar case with vanishing at -1 then follows by (2.5). In all these cases we can look at the right-hand side of (2.1) in a different way and thus obtain a transformation formula such that the true degree or the multiplicity of vanishing at 1 or -1 can be read off from the transformed expression. The results are:

Proposition 3.2. *Let* $n > 0$. *Assume that* $P_n^{(\alpha,\beta)}(x)$ *does not vanish identically in x.*

(a) $P_n^{(\alpha,\beta)}(x)$ *has degree* $< n$ *in x iff* $\alpha + \beta \in \mathbb{Z}_{\leq -2}$, $n + \alpha + \beta + 1 \leq 0$ *and* $2n + \alpha + \beta \geq 0$. *Then the degree is* $-n - \alpha - \beta - 1$ *and*

$$P_n^{(\alpha,\beta)}(x) = \frac{(-n-\beta)_{2n+\alpha+\beta+1}}{(-n-\alpha-\beta)_{2n+\alpha+\beta+1}} P_{-n-\alpha-\beta-1}^{(\alpha,\beta)}(x).$$

(b) $P_n^{(\alpha,\beta)}(1) = 0$ *iff* $\alpha \in \mathbb{Z}_{\leq -1}$ *and* $n + \alpha \geq 0$. *Then the zero at 1 has multiplicity* $-\alpha$ *and*

$$P_n^{(\alpha,\beta)}(x) = \frac{(n+\alpha+\beta+1)_{-\alpha}\,(n+\alpha)!}{n!} \left(\frac{x-1}{2}\right)^{-\alpha} P_{n+\alpha}^{(-\alpha,\beta)}(x).$$

(c) $P_n^{(\alpha,\beta)}(-1) = 0$ *iff* $\beta \in \mathbb{Z}_{\leq -1}$ *and* $n + \beta \geq 0$. *Then the zero at* -1 *has multiplicity* $-\beta$ *and*

$$P_n^{(\alpha,\beta)}(x) = \frac{(n+\alpha+\beta+1)_{-\beta}\,(n+\beta)!}{n!} \left(\frac{x+1}{2}\right)^{-\beta} P_{n+\beta}^{(\alpha,-\beta)}(x).$$

Combinations of the cases in this last proposition can occur. Then the corresponding transformation formulas can be combined. For instance, combination of (a) and (b) yields:

(d) $\alpha, \beta \in \mathbb{Z}$ *and* $\beta + 2 \leq \alpha \leq -1$. *Then for* $\max(-\alpha, -\frac{1}{2}(\alpha+\beta)) \leq n \leq -\beta - 1$ *we have*

$$P_n^{(\alpha,\beta)}(x) = \left(\frac{1-x}{2}\right)^{-\alpha} P_{-n-\beta-1}^{(-\alpha,\beta)}(x).$$

A further combination of (d) with (c) is empty. The combination of (a) and (c) can be obtained from (d) by using (2.5):

(e) $\alpha, \beta \in \mathbb{Z}$ *and* $\alpha + 2 \leq \beta \leq -1$. *Then for* $\max(-\beta, -\frac{1}{2}(\alpha+\beta)) \leq n \leq -\alpha - 1$ *we have*

$$P_n^{(\alpha,\beta)}(x) = (-1)^{\alpha+1} \left(\frac{1+x}{2}\right)^{-\beta} P_{-n-\alpha-1}^{(\alpha,-\beta)}(x).$$

Combination of (b) and (c) yields:

(f) $\alpha, \beta \in \mathbb{Z}_{\leq -1}$. Then for $n \geq -\alpha - \beta$ we have

$$P_n^{(\alpha,\beta)}(x) = \left(\frac{x-1}{2}\right)^{-\alpha} \left(\frac{x+1}{2}\right)^{-\beta} P_{n+\alpha+\beta}^{(-\alpha,-\beta)}(x). \tag{3.1}$$

We can also consider vanishing of coefficients in the sum in (2.3). Let us rewrite this as a summation formula for Jacobi polynomials $P_n^{(\alpha,\alpha)}(x)$ and let us distinguish between cases $n = 2m$ and $n = 2m - 1$:

$$P_{2m}^{(\alpha,\alpha)}(x) = 2^{-2m}(\alpha+m+1)_m \sum_{k=0}^{m} \frac{(-1)^k \left(\alpha+m+\frac{1}{2}\right)_{m-k}}{k!\,(2m-2k)!} (2x)^{2m-2k}, \tag{3.2}$$

$$P_{2m-1}^{(\alpha,\alpha)}(x) = 2^{1-2m}(\alpha+m)_m \sum_{k=0}^{m-1} \frac{(-1)^k \left(\alpha+m+\frac{1}{2}\right)_{m-1-k}}{k!\,(2m-1-2k)!} (2x)^{2m-1-2k}. \tag{3.3}$$

From (3.2) and (3.3) Propositions 3.1 and 3.2(a) can again be derived in the case $\alpha = \beta$. Furthermore, we conclude that, if (3.2) and (3.3) are not identically zero in x, then they have no zero at $x = 0$ (in case of (3.2)) respectively no zero of multiplicity higher than one at $x = 0$ (in case of (3.3)).

4. Main results

In Section 5 it will be shown that

$$\sum_{k=0}^{n} P_{n-k}^{(\alpha_1+k,\beta_1+k)}(x)\, P_k^{(\alpha_2-k,\beta_2-k)}(x) = P_n^{(\alpha_1+\alpha_2,\beta_1+\beta_2)}(x) \tag{4.1}$$

and

$$\sum_{k=0}^{n} k\, P_{n-k}^{(\alpha_1+k,\beta_1+k)}(x)\, P_k^{(\alpha_2-k,\beta_2-k)}(x) = \frac{n(\alpha_2+\beta_2)}{\alpha_1+\alpha_2+\beta_1+\beta_2+2n} P_n^{(\alpha_1+\alpha_2,\beta_1+\beta_2)}(x)$$

$$+ \frac{\alpha_2\beta_1 - \alpha_1\beta_2 + n(\alpha_2-\beta_2)}{\alpha_1+\alpha_2+\beta_1+\beta_2+2n} P_{n-1}^{(\alpha_1+\alpha_2,\beta_1+\beta_2)}(x) \quad (n > 0). \tag{4.2}$$

For $\alpha_1 = -\alpha_2 = \alpha$, $\beta_1 = -\beta_2 = \beta$ these formulas reduce to

$$\sum_{k=0}^{n} P_{n-k}^{(\alpha+k,\beta+k)}(x)\, P_k^{(-\alpha-k,-\beta-k)}(x) = P_n(x), \tag{4.3}$$

$$\sum_{k=0}^{n} k\, P_{n-k}^{(\alpha+k,\beta+k)}(x)\, P_k^{(-\alpha-k,-\beta-k)}(x)$$

$$= -\tfrac{1}{2}(\alpha+\beta)P_n(x) + \tfrac{1}{2}(\beta-\alpha)P_{n-1}(x) \quad (n > 0). \tag{4.4}$$

From (4.3), (4.4) and (4.1) we obtain for $n > 0$ that

$$\sum_{k=0}^{n} P_{n-k}^{(\alpha+k,\beta+k)}(x)\Big((k+\beta)P_k^{(-\alpha-k,-\beta-k)}(x) + (\alpha-\beta)P_k^{(-\alpha-k,-\beta-k-1)}(x)\Big)$$

$$= -\tfrac{1}{2}(\alpha+\beta)P_n(x) + \tfrac{1}{2}(\beta-\alpha)P_{n-1}(x) + \beta P_n(x) + (\alpha-\beta)P_n^{(0,-1)}(x) = 0 \tag{4.5}$$

by [6, 10.8(36)]. Thus we have derived for $n = 0, 1, 2, \dots$ that

$$\sum_{k=0}^{n} P_{n-k}^{(\alpha+k,\beta+k)}(x) \left(\frac{k+\beta}{\alpha} P_k^{(-\alpha-k,-\beta-k)}(x) + \frac{\alpha-\beta}{\alpha} P_k^{(-\alpha-k,-\beta-k-1)}(x) \right) = \delta_{n,0} \quad (4.6)$$

and, in particular,

$$\sum_{k=0}^{n} \frac{k+\alpha}{\alpha} P_{n-k}^{(\alpha+k,\alpha+k)}(x) P_k^{(-\alpha-k,-\alpha-k)}(x) = \delta_{n,0}. \quad (4.7)$$

Now make in (4.6) the substitutions $n \to m - n$, $k \to k - n$, $\alpha \to \alpha + n$, $\beta \to \beta + n$, where the new variables m, n can be arbitrarily integer such that $m \geq n$. The resulting identity is:

$$\sum_{k=n}^{m} P_{m-k}^{(\alpha+k,\beta+k)}(x) \left(\frac{k+\beta}{n+\alpha} P_{k-n}^{(-\alpha-k,-\beta-k)}(x) + \frac{\alpha-\beta}{n+\alpha} P_{k-n}^{(-\alpha-k,-\beta-k-1)}(x) \right)$$

$$= \delta_{m,n} \quad (m \geq n). \quad (4.8)$$

In this and related formulas it turns out that the expression remains continuous in α as α tends to the apparent singularity, see Remark 4.2.

Let $\mathcal{L}_\infty$ be the group of all lower triangular $\infty \times \infty$ matrices (doubly infinite, i.e., with indices running over all integers) for which the entries depend on a complex variable x (usually polynomially), but which have the entries on the main diagonal identically 1. The identity (4.8) can be rephrased by giving two explicit elements of $\mathcal{L}_\infty$ which are inverse to each other:

Theorem 4.1. *$LM = I = ML$ where $L = L^{(\alpha,\beta)}$ and $M = M^{(\alpha,\beta)}$ are lower triangular matrices for which the lower triangular entries ($m \geq n$) are given by*

$$L_{m,n}^{(\alpha,\beta)} = P_{m-n}^{(\alpha+n,\beta+n)}(x),$$
$$M_{m,n}^{(\alpha,\beta)} = \frac{m+\beta}{n+\alpha} P_{m-n}^{(-\alpha-m,-\beta-m)}(x) + \frac{\alpha-\beta}{n+\alpha} P_{m-n}^{(-\alpha-m,-\beta-m-1)}(x). \quad (4.9)$$

In the Gegenbauer case $\alpha = \beta$ formulas (4.8) and (4.9) simplify because the term with factor $\alpha - \beta$ vanishes.

The lower triangular matrices in Theorem 4.1 can also be considered with entries m, n running over all integers $\geq n_0$ for some integer n_0, in particular with entries running over all nonnegative integers. There is no loss of generality in doing this because

$$L_{m,n}^{(\alpha,\beta)} = L_{m-k,n-k}^{(\alpha+k,\beta+k)}, \qquad M_{m,n}^{(\alpha,\beta)} = M_{m-k,n-k}^{(\alpha+k,\beta+k)}.$$

There are two different places in the literature where Theorem 4.1 can be used, for $\alpha = \beta = 1$ and $\alpha = \beta = \frac{1}{2}$, respectively:

1. The case $\alpha = \beta = 1$ (matrix entries running over all nonnegative integers) occurs in Brega and Cagliero [3, p. 471] with a proof similar as given here.
2. The case $\alpha = \beta = \frac{1}{2}$ of the matrix $L^{(\alpha,\beta)}$ in (4.9) occurs in Koelink, van Pruijssen and Román [13, Theorem 2.1] in the form of the lower triangular matrix $L(x)$ given by

$$(L(x))_{m,n} = \frac{m!\,(2n+1)!}{(m+n+1)!\,n!} C_{m-n}^{(n+1)}(x) = \frac{m!\,\left(\frac{3}{2}\right)_n}{\left(\frac{3}{2}\right)_m n!} P_{m-n}^{\left(n+\frac{1}{2},n+\frac{1}{2}\right)}(x) \quad (m \geq n \geq 0).$$

There the matrix has finite size (which does not matter for the purpose of inversion). As the authors wrote in [13, paragraph after Theorem 6.2], they tried to find an explicit inverse matrix but did not succeed. We can give the inverse by (4.9) for $\alpha = \frac{1}{2}$ as follows:

$$
\begin{aligned}
(L^{-1}(x))_{m,n} &= \frac{m!\,(m+n)!}{(2m)!\,n!}\, C_{m-n}^{(-m)}(x) \\
&= \frac{m!\,\left(-\frac{1}{2}\right)_{n+1}}{\left(-\frac{1}{2}\right)_{m+1}\, n!}\, P_{m-n}^{\left(-m-\frac{1}{2},\,-m-\frac{1}{2}\right)}(x) \quad (m \geq n \geq 0).
\end{aligned}
$$

Our result is mentioned in an Addendum at the end of [13].

Remark 4.2. In the formula for $M_{m,n}$ in (4.9) the denominator only gives an apparent singularity. We have $M_{n,n}^{(\alpha,\beta)} = 1$ and for $m > n$ we obtain by (2.1) that

$$
\begin{aligned}
M_{m,n}^{(\alpha,\beta)} = -&\sum_{k=0}^{m-n-1} \Big((m+\beta)(-\alpha-\beta-m-n+1)_k + (\alpha-\beta)(-\alpha-\beta-m-n)_k \Big) \\
&\times \frac{(-\alpha-m+k+1)_{m-n-k-1}}{k!\,(m-n-k)!} \left(\frac{x-1}{2} \right)^k \\
&- \frac{(\alpha+\beta+2m)(-\alpha-\beta-m-n+1)_{m-n-1}}{(m-n)!} \left(\frac{x-1}{2} \right)^{m-n}.
\end{aligned}
$$

Remark 4.3. A referee suggested to see if (4.5), which leads to Theorem 4.1, can be generalized by starting in the left part with

$$
\sum_{k=0}^{n} P_{n-k}^{(\alpha_1+k,\beta_1+k)}(x)\Big((x_n k + y_n) P_k^{(\alpha_2-k,\beta_2-k)}(x) + z_n P_k^{(\alpha_2-k,\beta_2-k-1)}(x) \Big) \tag{4.10}
$$

for yet unknown x_n, y_n, z_n and then find for which choices of the unknowns the generalization of the middle part of (4.5) will match with [6, 10.8(36)] and thus yield zero. The solution (up to multiplication of the three unknowns by the same possibly n-dependent factor) turns out to be

$$
\begin{aligned}
x_n &= -n - \alpha_1 - \alpha_2, \qquad y_n = \beta_2\, n + \alpha_1\beta_2 - \alpha_2\beta_1, \\
z_n &= (\alpha_2 - \beta_2)n - \alpha_1\beta_2 + \alpha_2\beta_1.
\end{aligned} \tag{4.11}
$$

Solutions independent of n can be obtained iff $\alpha_1 + \alpha_2 = 0 = \beta_1 + \beta_2$, precisely the case which we already had in (4.5). It is not clear how we can obtain a pair of mutually inverse lower triangular matrices in the general n-dependent case of (4.11) making (4.10) zero.

Remark 4.4. Just as we can go from (4.6) to $LM = I$ and backwards, we can go back and forth from $ML = I$ to the identity

$$
\sum_{k=0}^{n} P_k^{(\alpha,\beta)}(x) \left(\frac{n+\beta}{k+\alpha}\, P_{n-k}^{(-\alpha-n,-\beta-n)}(x) + \frac{\alpha-\beta}{k+\alpha}\, P_{n-k}^{(-\alpha-n,-\beta-n-1)}(x) \right) = \delta_{n,0}.
$$

Remark 4.5. By Theorem 4.1 we have a biorthogonal system of functions on $\mathbb{Z}$ given for $n \geq k$ by

$$
\phi_n^{(\alpha,\beta,x)}(k) := L_{n,k}^{(\alpha,\beta)} = P_{n-k}^{(\alpha+k,\beta+k)}(x), \tag{4.12}
$$

$$\psi_n^{(\alpha,\beta,x)}(k) := M_{-k,-n}^{(\alpha,\beta)} = \frac{\beta - k}{\alpha - n} P_{n-k}^{(-\alpha+k,-\beta+k)}(x) + \frac{\alpha - \beta}{\alpha - n} P_{n-k}^{(-\alpha+k,-\beta+k-1)}(x),$$

and otherwise zero. Here $\psi_n^{(\alpha,\beta,x)}(k)$ for $\alpha \to n$ is to be understood as in Remark 4.2. Then

$$\sum_{k \in \mathbb{Z}} \phi_m^{(\alpha,\beta,x)}(k) \, \psi_{-n}^{(\alpha,\beta,x)}(-k) = \delta_{m,n}, \tag{4.13}$$

where the sum actually only runs over $\{k \in \mathbb{Z} \mid n \leq k \leq m\}$. If α and β are shifted by the same integer j then the biorthogonal system does not essentially change, since

$$\phi_n^{(\alpha+j,\beta+j,x)}(k) = \phi_{n+j}^{(\alpha,\beta,x)}(k+j), \qquad \psi_n^{(\alpha+j,\beta+j,x)}(k) = \psi_{n-j}^{(\alpha,\beta,x)}(k-j).$$

For $\alpha = \beta = 0$ the biorthogonality relation (4.13) simplifies to

$$\sum_{k \in \mathbb{Z}} \phi_m^{(0,0,x)}(k) \, \phi_{-n}^{(0,0,x)}(-k) \, \frac{k}{n} = \delta_{m,n}.$$

Again the sum only runs over $\{k \in \mathbb{Z} \mid n \leq k \leq m\}$. The singularity for $n = 0$ in $\phi_{-n}^{(0,0,x)}(-k)\frac{k}{n}$ is only apparent because of Remark 4.2. For $n \leq 0 \leq -n < m$ the summation range is further restricted to $\{k \in \mathbb{Z} \mid n \leq k \leq -n\}$ because of Proposition 3.1. Similarly, for $n < -m \leq 0 \leq m$ the summation range is restricted to $\{k \in \mathbb{Z} \mid -m \leq k \leq m\}$.

From $ML = I$ we get a biorthogonality relation for the dual systems:

$$\sum_{k \in \mathbb{Z}} \psi_{-k}^{(\alpha,\beta,x)}(-m) \, \phi_k^{(\alpha,\beta,x)}(n) = \delta_{m,n}.$$

Remark 4.6. Brown and Roman [4, (4.12)–(4.15)] obtain inverse relations involving Gegenbauer polynomials of which a special case is close to (4.7) but not equal to it. It reads

$$\sum_{k=0}^{n} \frac{(n + 2\alpha + 1)_k}{(2\alpha + 2)_k} P_{n-k}^{(\alpha+k,\alpha+k)}(x) \, P_k^{(-\alpha-k-1,-\alpha-k-1)}(x) = \delta_{n,0}. \tag{4.14}$$

In fact, they give a more general identity

$$\sum_{k=0}^{n} \frac{\nu}{\mu k + \nu} C_k^{(\mu k + \nu)}(x) \, C_{n-k}^{(-\mu k - \nu)}(x) = \delta_{n,0}. \tag{4.15}$$

Then (4.14) is the case $\mu = -1$, $\nu = -\alpha - \frac{1}{2}$ of (4.15), while the case $\mu = 0$ of (4.15) is the very elegant formula

$$\sum_{k=0}^{n} C_k^{(\nu)}(x) \, C_{n-k}^{(-\nu)}(x) = \delta_{n,0}, \tag{4.16}$$

which is also the case $\lambda = -\nu$ of the formula

$$\sum_{k=0}^{n} C_k^{(\nu)}(x) \, C_{n-k}^{(\lambda)}(x) = C_n^{(\nu+\lambda)}(x), \tag{4.17}$$

mentioned in [14, (18.18.20)]. The identity (4.17) is a direct consequence of the generating function (2.7).

Formula (4.14) is also the case $\alpha = \beta$ of the identity

$$\sum_{k=0}^{n} \frac{(\alpha + \beta + n + 1)_k}{(\alpha + \beta + 2)_k} \, P_{n-k}^{(\alpha+k,\beta+k)}(x) \, P_k^{(-\alpha-k-1,-\beta-k-1)}(x) = \delta_{n,0}. \tag{4.18}$$

This last identity is a consequence of the pair of mutually inverse lower triangular matrices (8.11) implied by J. Koekoek and R. Koekoek [11, (17)].

In Section 8 we will show that (4.18), and hence (4.14), is related to a limit case of a connection formula for Askey–Wilson polynomials.

Remark 4.7. There remain several interesting questions. First of all, is there a larger family of explicit mutually inverse lower triangular matrices which includes both the family of Theorem 4.1 and the family (8.10) implying (4.18)? (See the attempt made in Remark 4.3.) Furthermore, are there two simple systems of special functions connected by the matrices in Theorem 4.1? If yes, can this also be seen as a limit case for $q \to 1$ of some connection formula in the q-case? Concerning the pair of mutually inverse lower triangular matrices (8.11) involving Jacobi polynomials there are analogues for some other families of orthogonal polynomials in the Askey scheme, for instance for Charlier and Meixner polynomials, as surveyed by Koekoek [10]. It would be interesting to see if these also come from limit cases of the Askey–Wilson connection relations. Finally there is the puzzling Brown–Roman formula (4.15). Does this have an extension to Jacobi polynomials for general μ? It would also be interesting to generalize (4.17) such that it is related to (4.15).

5. Computations leading to Theorem 4.1

Lemma 5.1. *If the functions f and g have derivatives up to order n then*

$$\sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) \, g^{(k)}(x) = (fg)^{(n)}(x), \tag{5.1}$$

$$\sum_{k=0}^{n} k \binom{n}{k} f^{(n-k)}(x) \, g^{(k)}(x) = n \, (fg')^{(n-1)}(x). \tag{5.2}$$

Proof. Formula (5.1) is well-known. For the proof of (5.2) rewrite its left-hand side as

$$n \sum_{j=0}^{n-1} \binom{n-1}{j} f^{(n-j-1)}(x) \, (g')^{(j)}(x)$$

and use (5.1). □

By the Rodrigues formula (2.2) we have

$$\sum_{k=0}^{n} P_{n-k}^{(\alpha_1+k,\beta_1+k)}(x) \, P_k^{(\alpha_2-k,\beta_2-k)}(x) = \frac{(-1)^n}{2^n n!} \, (1-x)^{-\alpha_1-\alpha_2} \, (1+x)^{-\beta_1-\beta_2}$$

$$\times \sum_{k=0}^{n} \binom{n}{k} \left(\frac{d}{dx} \right)^{n-k} \left((1-x)^{n+\alpha_1}(1+x)^{n+\beta_1} \right) \left(\frac{d}{dx} \right)^{k} \left((1-x)^{\alpha_2}(1+x)^{\beta_2} \right).$$

By (5.1) and again (2.2) we obtain (4.1).

Similarly, by (2.2) and (5.2) we can write for $n > 0$:

$$\sum_{k=0}^{n} k\, P_{n-k}^{(\alpha_1+k,\beta_1+k)}(x)\, P_k^{(\alpha_2-k,\beta_2-k)}(x)$$

$$= \frac{(-1)^n}{2^n n!}\, (1-x)^{-\alpha_1-\alpha_2}\, (1+x)^{-\beta_1-\beta_2} \sum_{k=0}^{n} k\binom{n}{k}$$

$$\times \left(\frac{d}{dx}\right)^{n-k}\left((1-x)^{n+\alpha_1}(1+x)^{n+\beta_1}\right)\left(\frac{d}{dx}\right)^{k}\left((1-x)^{\alpha_2}(1+x)^{\beta_2}\right)$$

$$= \frac{(-1)^n}{2^n (n-1)!}\, (1-x)^{-\alpha_1-\alpha_2}\, (1+x)^{-\beta_1-\beta_2}$$

$$\times \left(\frac{d}{dx}\right)^{n-1}\left((1-x)^{n+\alpha_1}(1+x)^{n+\beta_1}\,\frac{d}{dx}\big((1-x)^{\alpha_2}(1+x)^{\beta_2}\big)\right).$$

By straightforward computation we get

$$(1-x)^{n+\alpha_1}(1+x)^{n+\beta_1}\frac{d}{dx}\left((1-x)^{\alpha_2}(1+x)^{\beta_2}\right)$$

$$= \frac{\alpha_2+\beta_2}{\alpha_1+\alpha_2+\beta_1+\beta_2+2n}\frac{d}{dx}\left((1-x)^{\alpha_1+\alpha_2+n}(1+x)^{\beta_1+\beta_2+n}\right)$$

$$- 2\,\frac{\alpha_2\beta_1-\alpha_1\beta_2+n(\alpha_2-\beta_2)}{\alpha_1+\alpha_2+\beta_1+\beta_2+2n}\,(1-x)^{\alpha_1+\alpha_2+n-1}(1+x)^{\beta_1+\beta_2+n-1}.$$

By (2.2) we finally obtain (4.2).

6. Further matrix identities involving Jacobi polynomials

As a consequence of the generating function (2.6) we have

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} P_{n-k}^{(\alpha_1,\beta_1)}(x)\, P_k^{(\alpha_2,\beta_2)}(x)\right) w^n$$
$$= 2^{\alpha_1+\alpha_2+\beta_1+\beta_2} R^{-2}(1-w+R)^{-\alpha_1-\alpha_2}(1+w+R)^{-\beta_1-\beta_2},$$

by which the inner sum on the left-hand side as a function of $\alpha_1, \alpha_2, \beta_1, \beta_2$ only depends on $\alpha_1+\alpha_2, \beta_1+\beta_2$. In particular,

$$\sum_{k=0}^{n} P_{n-k}^{(\alpha_1,\beta_1)}(x)\, P_k^{(\alpha_2,\beta_2)}(x) = \sum_{k=0}^{n} P_{n-k}(x)\, P_k^{(\alpha_1+\alpha_2,\beta_1+\beta_2)}(x). \tag{6.1}$$

Formula (6.1) is quite similar to (4.1). We can rewrite both identities as identities in $\mathcal{L}_\infty$ (the group of doubly infinite lower triangular matrices depending on a complex variable x and with 1 on the main diagonal). Let $P^{(\alpha,\beta)}, Q^{(\alpha,\beta)} \in \mathcal{L}_\infty$ with

$$P_{m,n}^{(\alpha,\beta)} := P_{m-n}^{(\alpha,\beta)}(x), \qquad Q_{m,n}^{(\alpha,\beta)} := P_{m-n}^{(\alpha+n-m,\beta+n-m)}(x) \quad (m \geq n). \tag{6.2}$$

Both are matrices of the form $A_{m,n} = f(m-n)$ (constant on each diagonal, i.e., a Toeplitz matrix). All such matrices in $\mathcal{L}_\infty$ commute. Formulas (6.1) and (4.1) can be rephrased as:

$$P^{(\alpha_1,\beta_1)}\, P^{(\alpha_2,\beta_2)} = P^{(0,0)}\, P^{(\alpha_1+\alpha_2,\beta_1+\beta_2)}, \qquad Q^{(\alpha_1,\beta_1)}\, Q^{(\alpha_2,\beta_2)} = Q^{(\alpha_1+\alpha_2,\beta_1+\beta_2)}. \tag{6.3}$$

Also, by (2.1), $Q^{(0,0)} = I$. Put

$$P_H^{(\alpha,\beta)} := P^{(\alpha,\beta)}(P^{(0,0)})^{-1} = (P^{(0,0)})^{-1} P^{(\alpha,\beta)}, \qquad Q_H^{(\alpha,\beta)} := Q^{(\alpha,\beta)}. \tag{6.4}$$

Then, by (6.3),

$$P_H^{(\alpha_1,\beta_1)} P_H^{(\alpha_2,\beta_2)} = P_H^{(\alpha_1+\alpha_2,\beta_1+\beta_2)}, \qquad Q_H^{(\alpha_1,\beta_1)} Q_H^{(\alpha_2,\beta_2)} = Q_H^{(\alpha_1+\alpha_2,\beta_1+\beta_2)}. \tag{6.5}$$

Since $P_n(x) = C_n^{(\frac{1}{2})}(x)$ we see by (4.16) that

$$\left((P^{(0,0)})^{-1}\right)_{m,n} = C_{m-n}^{\left(-\frac{1}{2}\right)}(x) \quad (m \geq n). \tag{6.6}$$

Also note that $C_0^{(-\frac{1}{2})}(x) = 1$, $C_1^{(-\frac{1}{2})}(x) = -x$ and, for $n \geq 2$,

$$
\begin{aligned}
C_n^{\left(-\frac{1}{2}\right)}(x) &= \frac{2^n \left(-\frac{1}{2}\right)_n}{n!} x^n \, {}_2F_1\left(\begin{matrix} -\frac{1}{2}n, \frac{1}{2} - \frac{1}{2}n \\ \frac{3}{2} - n \end{matrix}; \frac{1}{x^2}\right) \\
&= \frac{2^n \left(-\frac{1}{2}\right)_n}{n!} (x^2 - 1)x^{n-2} \, {}_2F_1\left(\begin{matrix} 1 - \frac{1}{2}n, \frac{3}{2} - \frac{1}{2}n \\ \frac{3}{2} - n \end{matrix}; \frac{1}{x^2}\right) \\
&= \frac{1-x^2}{n(n-1)} C_{n-2}^{\left(\frac{3}{2}\right)}(x) = \frac{1-x^2}{2(n-1)} P_{n-2}^{(1,1)}(x).
\end{aligned}
$$

Here we used (2.3) and Euler's transformation formula [1, (2.2.7)] for ${}_2F_1$ series. Alternatively, use (2.4) and (3.1).

By (6.5) the maps sending $(\alpha, \beta) \in \mathbb{C}^2$ to $P_H^{(\alpha,\beta)}$ and to $Q_H^{(\alpha,\beta)}$ are both group homomorphisms from $\mathbb{C}^2$ into $\mathcal{L}_\infty$. The maps are entrywise analytic and entries on the right-hand sides of (6.5) are obtained from finite sums on the left-hand sides. Thus we must have

$$P_H^{(\alpha,\beta)} = \exp(\alpha A_P + \beta B_P), \qquad Q_H^{(\alpha,\beta)} = \exp(\alpha A_Q + \beta B_Q) \tag{6.7}$$

for some strictly lower triangular matrices A_P, B_P, A_Q, B_Q, and these matrices can be computed by evaluating the derivatives $\frac{\partial}{\partial \alpha} P_H^{(\alpha,\beta)}$, $\frac{\partial}{\partial \beta} P_H^{(\alpha,\beta)}$, $\frac{\partial}{\partial \alpha} Q_H^{(\alpha,\beta)}$, $\frac{\partial}{\partial \beta} Q_H^{(\alpha,\beta)}$, respectively, at $(\alpha, \beta) = (0, 0)$.

Proposition 6.1. *For $m > n$ the matrix entries of A_Q, B_Q, A_P, B_P as occurring in (6.7) are explicitly given by*

$$(A_Q)_{m,n} = -\frac{1}{m-n} \frac{(-1-x)^{m-n}}{2^{m-n}}, \qquad (B_Q)_{m,n} = -\frac{1}{m-n} \frac{(1-x)^{m-n}}{2^{m-n}}, \tag{6.8}$$

$$(A_P)_{m,n} = \frac{1}{m-n} P_{m-n}^{(0,-1)}(x), \qquad (B_P)_{m,n} = \frac{1}{m-n} P_{m-n}^{(-1,0)}(x). \tag{6.9}$$

Proof. First note that by (6.4), (6.2), (6.6) and (2.5) we have

$$(A_Q)_{m,n}(x) = (-1)^{m-n}(B_Q)_{m,n}(-x), \qquad (A_P)_{m,n}(x) = (-1)^{m-n}(B_P)_{m,n}(-x).$$

Thus for (6.8) we only have to compute B_Q. We get from (2.1) that, for $m > n$,

$$P_{m-n}^{(n-m,\beta+n-m)}(x) = \frac{(\beta+n-m+1)_{m-n}}{(m-n)!}\frac{(x-1)^{m-n}}{2^{m-n}}.$$

Differentiation with respect to β and putting $\beta = 0$ yields $(B_Q)_{m,n}$ and, by (2.5) also $(A_Q)_{m,n}$, as given in (6.8).

For (6.9) we only have to compute A_P. Denote the two equal sides of the generating function (2.6) by $f^{(\alpha,\beta)}(w)$. Then

$$\log(f^{(\alpha,0)}(w)) = \alpha \log\left(\frac{2}{1-w+R}\right) - \log R,$$

$$\frac{\partial}{\partial\alpha}\log(f^{(\alpha,0)}(w)) = \log\left(\frac{2}{1-w+R}\right),$$

$$\frac{\partial^2}{\partial w\,\partial\alpha}\log(f^{(\alpha,0)}(w)) = \frac{R-w+x}{R(R-w+1)} = \frac{R+w+1}{2wR} - \frac{1}{w}$$

$$= \frac{1}{w}(f^{(0,-1)}(w)-1) = \sum_{n=1}^{\infty}P_n^{(0,-1)}(x)\,w^{n-1}.$$

Since $\frac{\partial}{\partial\alpha}\log(f^{(\alpha,0)}(0)) = 0$, we conclude that

$$\frac{\partial}{\partial\alpha}\log(f^{(\alpha,0)}(w)) = \sum_{n=1}^{\infty}n^{-1}P_n^{(0,-1)}(x)\,w^n,$$

$$\frac{\partial}{\partial\alpha}f^{(\alpha,0)}(w)\bigg|_{\alpha=0} = \sum_{n=0}^{\infty}\left(\sum_{k=1}^{n}k^{-1}P_k^{(0,-1)}(x)\,P_{n-k}(x)\right)w^n,$$

$$\frac{\partial}{\partial\alpha}P_n^{(\alpha,0)}(x)\bigg|_{\alpha=0} = \sum_{k=1}^{n}k^{-1}P_k^{(0,-1)}(x)\,P_{n-k}(x).$$

Since

$$P_{m-n}^{(\alpha,0)}(x) = P_{m,n}^{(\alpha,0)} = \sum_{k=n}^{m}(P_H^{(\alpha,0)})_{m,k}\,P_{k-n}(x),$$

we have

$$\frac{\partial}{\partial\alpha}P_{m-n}^{(\alpha,0)}(x)\bigg|_{\alpha=0} = \sum_{k=n}^{m-1}(A_P)_{m,k}\,P_{k-n}(x).$$

We conclude that $(A_P)_{m,n}$ is as given by (6.9). $\quad\square$

Compare the definition (4.9) of $L_{m,n}^{(\alpha,\beta)}$ with the definitions (6.2) of $P_{m,n}^{(\alpha,\beta)}$ and $Q_{m,n}^{(\alpha,\beta)}$. It follows that

$$Q_{m,n}^{(\alpha+m,\beta+m)} = L_{m,n}^{(\alpha,\beta)} = P_{m,n}^{(\alpha+n,\beta+n)}. \tag{6.10}$$

Proposition 6.2. *We have*

$$P_H^{(\alpha,\beta)}L^{(0,0)} = L^{(\alpha,\beta)} = L^{(0,0)}Q_H^{(\alpha,\beta)}. \tag{6.11}$$

Proof. The second equality follows from

$$L_{m,n}^{(\alpha,\beta)} = (Q_H^{(\alpha+m,\beta+m)})_{m,n} = \sum_{k=n}^{m} (Q_H^{(m,m)})_{m,k} (Q_H^{(\alpha,\beta)})_{k,n} = \sum_{k=n}^{m} L_{m,k}^{(0,0)} (Q_H^{(\alpha,\beta)})_{k,n}.$$

The first equality follows from

$$(P^{(0,0)} P_H^{(\alpha,\beta)} L^{(0,0)})_{m,n} = (P^{(\alpha,\beta)} L^{(0,0)})_{m,n} = \sum_{k=n}^{m} P_{m,k}^{(\alpha,\beta)} L_{k,n}^{(0,0)} = \sum_{k=n}^{m} P_{m,k}^{(\alpha,\beta)} P_{k,n}^{(n,n)}$$

$$= \sum_{j=n}^{m} P_{m,j}^{(0,0)} P_{j,n}^{(\alpha+n,\beta+n)} = \sum_{j=n}^{m} P_{m,j}^{(0,0)} L_{j,n}^{(\alpha,\beta)} = (P^{(0,0)} L^{(\alpha,\beta)})_{m,n}.$$

Here we used (6.4), (6.3) and (6.10). □

Remark 6.3. It follows from (6.11) that

$$P_H^{(\alpha,\beta)} = L^{(0,0)} Q_H^{(\alpha,\beta)} (L^{(0,0)})^{-1}.$$

So the two-parameter groups $(\alpha, \beta) \mapsto P_H^{(\alpha,\beta)}$ and $(\alpha, \beta) \mapsto Q_H^{(\alpha,\beta)}$ are conjugate by $L^{(0,0)}$ in the group $\mathcal{L}_\infty$. Note that the inverse of $L^{(0,0)}$ is given by (4.9) as

$$M_{m,n}^{(0,0)} = \frac{m}{n} P_{m-n}^{(-m,-m)}(x).$$

7. Biorthogonal systems with respect to bilinear forms

In this section we build on the results of Theorem 4.1, Remark 4.5 and Section 6 in order to obtain systems of functions on $\mathbb{Z}$, involving the functions (4.12), which are biorthogonal with respect to some explicit bilinear form on $\mathbb{Z}$.

If $\alpha_1 + \alpha_2 = -n = \beta_1 + \beta_2$ then the left-hand side of (6.1) can be evaluated by an elementary expression, where we will use Proposition 3.1 and formula (3.1). Indeed, if $n = 2m > 0$ then

$$\sum_{k=0}^{2m} P_{2m-k}^{(\alpha_1,\beta_1)}(x) P_k^{(\alpha_2,\beta_2)}(x) = \sum_{k=0}^{2m} P_{2m-k}^{(-m,-m)}(x) P_k^{(-m,-m)}(x)$$

$$= 2 P_{2m}^{(-m,-m)}(x) = 2 \left(\frac{x^2 - 1}{4} \right)^m \qquad (\alpha_1 + \alpha_2 = -2m = \beta_1 + \beta_2), \tag{7.1}$$

and if $n = 2m - 1$ then

$$\sum_{k=0}^{2m-1} P_{2m-1-k}^{(\alpha_1,\beta_1)}(x) P_k^{(\alpha_2,\beta_2)}(x) = \sum_{k=0}^{2m-1} P_{2m-1-k}^{(-m+1,-m+1)}(x) P_k^{(-m,-m)}(x)$$

$$= P_{2m}^{(-m+1,-m+1)}(x)$$

$$+ P_{2m-2}^{(-m+1,-m+1)} P_1^{(-m,-m)}(x) = x \left(\frac{x^2 - 1}{4} \right)^{m-1}$$

$$(\alpha_1 + \alpha_2 = -2m + 1 = \beta_1 + \beta_2). \tag{7.2}$$

These results can be rephrased as identities in $\mathcal{L}_\infty$. Define $R \in \mathcal{L}_\infty$ by

$$
R_{m,n} := \begin{cases}
1 & \text{if } m = n, \\
2 \left(\dfrac{x^2 - 1}{4} \right)^{\frac{1}{2}(m-n)} & \text{if } m > n \text{ and } m - n \text{ even,} \\
x \left(\dfrac{x^2 - 1}{4} \right)^{\frac{1}{2}(m-n-1)} & \text{if } m > n \text{ and } m - n \text{ odd.}
\end{cases}
$$

Also define $\mathcal{J} \colon \mathcal{L}_\infty \to \mathcal{L}_\infty$ by $(\mathcal{J}A)_{m,n} := A_{-n,-m}$. Then $\mathcal{J}(AB) = (\mathcal{J}B)(\mathcal{J}A)$. By (4.9) we have for $m > n$ that

$$
\left((\mathcal{J} L^{(-\alpha,-\beta)}) L^{(\alpha,\beta)} \right)_{m,n} = \sum_{k=0}^{m-n} P_{m-n-k}^{(-\alpha-m,-\beta-m)}(x)\, P_k^{(\alpha+n,\beta+n)}(x),
$$

which can be evaluated by (7.1) and (7.2). Thus we have obtained that

$$
(\mathcal{J} L^{(-\alpha,-\beta)})\, L^{(\alpha,\beta)} = R. \tag{7.3}
$$

Proposition 7.1. *The inverse S of R in $\mathcal{L}_\infty$ (for which we will also use a notation μ_x) is explicitly given by*

$$
S_{m,n} = \mu_x(m, n) = \begin{cases}
1 & \text{if } m = n, \\
\left(\dfrac{-1 - x}{2} \right)^{m-n} + \left(\dfrac{1 - x}{2} \right)^{m-n} & \text{if } m > n.
\end{cases} \tag{7.4}
$$

Proof. It is sufficient to show that

$$
\sum_{k=0}^{n} R_{n-k,0}\, S_{k,0} = \delta_{n,0}.
$$

This follows because the generating functions

$$
\sum_{n=0}^{\infty} R_{n,0}\, w^n = \frac{\left(1 + \frac{1}{2}(x+1)w\right)\left(1 + \frac{1}{2}(x-1)w\right)}{1 - \frac{1}{4}(x^2 - 1)w^2},
$$

$$
\sum_{n=0}^{\infty} S_{n,0}\, w^n = \frac{1 - \frac{1}{4}(x^2 - 1)w^2}{\left(1 + \frac{1}{2}(x+1)w\right)\left(1 + \frac{1}{2}(x-1)w\right)},
$$

are inverse to each other. These generating functions, convergent for $x \in [-1, 1]$, $|w| < 1$, are immediately computed by geometric series. $\square$

From (7.3) and $S = R^{-1}$ we obtain that

$$
L^{(\alpha,\beta)} S(\mathcal{J} L^{(-\alpha,-\beta)}) = I. \tag{7.5}
$$

Here we used that in $\mathcal{L}_\infty$ the implication $AB = I \Rightarrow BA = I$ holds. Formula (7.5) can be rewritten as

$$
\sum_{k,\ell=-\infty}^{\infty} L_{m,k}^{(\alpha,\beta)}\, S_{k,\ell}\, L_{-n,-\ell}^{(-\alpha,-\beta)} = \delta_{m,n},
$$

where the sum only runs over k, ℓ such that $n \le \ell \le k \le m$. With the notation (4.12) and with μ_x given by (7.4) we have obtained:

Proposition 7.2.

$$\sum_{k,\ell=-\infty}^{\infty} \phi_m^{(\alpha,\beta,x)}(k)\, \phi_{-n}^{(-\alpha,-\beta,x)}(-\ell)\, \mu_x(k,\ell) = \delta_{m,n}, \tag{7.6}$$

where the sum only runs over k, ℓ such that $n \le \ell \le k \le m$.

It is of interest to compare (7.6) with the biorthogonality relation (4.13). Formula (7.6) can also be considered as a biorthogonality relation, but this time with respect to the bilinear form μ_x on $\mathbb{Z}$.

Remark 7.3. From Theorem 4.1 and formula (7.5) we obtain

$$M_{m,n}^{(\alpha,\beta)} = S(\mathcal{J}L^{(-\alpha,-\beta)}).$$

Equivalently, we obtain from (4.13) and (7.6) that

$$\psi_{-n}^{(\alpha,\beta,x)}(-k) = \sum_{\ell \in \mathbb{Z}} \mu_x(k,\ell)\, \phi_{-n}^{(-\alpha,-\beta,x)}(-\ell)$$

with sum running over $n \le \ell \le k$.

If we consider the left-hand side of (7.3) with the two factors interchanged then we can evaluate it by an earlier result. Indeed,

$$\left(L^{(\alpha,\beta)}\, (\mathcal{J}L^{(-\alpha,-\beta)})\right)_{m,n} = \sum_{k=n}^{m} P_{m-k}^{(\alpha+k,\beta+k)}(x)\, P_{k-n}^{(-\alpha-k,-\beta-k)}(x) = P_{m-n}(x)$$

by (4.1). Hence

$$L^{(\alpha,\beta)}\, (\mathcal{J}L^{(-\alpha,-\beta)}) = P^{(0,0)}. \tag{7.7}$$

$P^{(0,0)}$ has an inverse $T = (P^{(0,0)})^{-1}$ in $\mathcal{L}_\infty$, which was already computed after (6.6) and which we also write as ν_x:

$$T_{m,n} = \nu_x(m,n) = \begin{cases} 1 & \text{if } m = n, \\ -x & \text{if } m = n+1, \\ \dfrac{1-x^2}{2(n-1)}\, P_{n-2}^{(1,1)}(x) & \text{if } m \ge n+2. \end{cases} \tag{7.8}$$

From (7.7) we obtain

$$(\mathcal{J}L^{(-\alpha,-\beta)})\, T\, L^{(\alpha,\beta)} = I. \tag{7.9}$$

With the notation (4.12) and with ν_x given by (7.8), the identity (7.9) takes the form

$$\sum_{k,\ell=-\infty}^{\infty} \phi_{-k}^{(-\alpha,-\beta,x)}(-m)\, \phi_\ell^{(\alpha,\beta,x)}(n)\, \nu_x(k,\ell) = \delta_{m,n}. \tag{7.10}$$

Just as (7.6), we can consider (7.10) as a biorthogonality relation for two systems of functions on $\mathbb{Z}$ (the duals of the ones in (7.2)) with respect to a bilinear form on $\mathbb{Z}$, here ν_x.

8. Limits of a connection formula for Askey–Wilson polynomials

Askey–Wilson polynomials [2] are defined by

$$p_n(\cos\theta; a_1, a_2, a_3, a_4 \mid q)$$

$$:= \frac{(a_1a_2, a_1a_3, a_1a_4; q)_n}{a_1^n} \; {}_4\phi_3\left(\begin{matrix} q^{-n}, a_1a_2a_3a_4q^{n-1}, a_1e^{i\theta}, a_1e^{-i\theta} \\ a_1a_2, a_1a_3, a_1a_4 \end{matrix}; q, q\right). \tag{8.1}$$

They are symmetric in a_1, a_2, a_3, a_4. The connection coefficients $c_{n,k}$ in

$$p_n(\cos\theta; b_1, b_2, b_3, a_4 \mid q)$$

$$= \sum_{k=0}^{n} c_{n,k}(b_1, b_2, b_3, a_4; a_1, a_2, a_3, a_4 \mid q) \, p_k(\cos\theta; a_1, a_2, a_3, a_4 \mid q) \tag{8.2}$$

are explicitly given in Askey and Wilson [2, (6.5)]:

$$c_{n,k}(b_1, b_2, b_3, a_4; a_1, a_2, a_3, a_4 \mid q)$$

$$= \frac{q^{k(k-n)}(q; q)_n}{a_4^{n-k}(q; q)_{n-k}(q; q)_k} \frac{(b_1b_2b_3a_4q^{n-1}; q)_k}{(a_1a_2a_3a_4q^{k-1}; q)_k}(b_1a_4q^k, b_2a_4q^k, b_3a_4q^k; q)_{n-k}$$

$$\times \; {}_5\phi_4\left(\begin{matrix} q^{k-n}, b_1b_2b_3a_4q^{n+k-1}, a_1a_4q^k, a_2a_4q^k, a_3a_4q^k \\ b_1a_4q^k, b_2a_4q^k, b_3a_4q^k, a_1a_2a_3a_4q^{2k} \end{matrix}; q, q\right). \tag{8.3}$$

See also Ismail and Zhang [9, Section 3] and Ismail [8, Section 16.4], where the connection coefficients are given more generally for $a_4 \neq b_4$. However, note that in [9, (3.13)] and [8, (16.4.3)] one should read $c_{n,k}(\mathbf{b}, \mathbf{a})$ instead of $c_{n,k}(\mathbf{a}, \mathbf{b})$.

Now put

$$a_4 := q^{\alpha+1}/b_1, \qquad b_3 := q^{\beta+1}/b_2 \tag{8.4}$$

in (8.2) and (8.3), and multiply both sides of (8.2) by $1/(q; q)_n$. By (8.1) and (2.1) we see that

$$\lim_{q\to 1} \frac{1}{(q; q)_n} p_n(\cos\theta; b_1, b_2, q^{\beta+1}/b_2, q^{\alpha+1}/b_1 \mid q)$$

$$= \left(\frac{(1-b_1b_2)(b_2-b_1)}{b_1b_2}\right)^n \frac{(\alpha+1)_n}{n!}$$

$$\times \; {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{b_2(1-2b_1\cos\theta+b_1^2)}{(1-b_1b_2)(b_2-b_1)}\right)$$

$$= \left(\frac{(1-b_1b_2)(b_2-b_1)}{b_1b_2}\right)^n P_n^{(\alpha,\beta)}\left(1 - 2\frac{b_2(1-2b_1\cos\theta+b_1^2)}{(1-b_1b_2)(b_2-b_1)}\right).$$

By (8.2) we also see that

$$\lim_{q\to 1} p_k(\cos\theta; a_1, a_2, a_3, q^{\alpha+1}/b_1 \mid q)$$

$$= \left(\frac{(1-a_1a_2)(1-a_1a_3)(b_1-a_1)}{a_1b_1}\right)^k$$

$$\times \,_1F_0\left(\begin{matrix}-k\\-\end{matrix}; \frac{(b_1-a_1a_2a_3)(1-2a_1\cos\theta+a_1^2)}{(1-a_1a_2)(1-a_1a_3)(b_1-a_1)}\right)$$

$$= \left(\frac{(1-a_1a_2)(1-a_1a_3)(b_1-a_1)-(b_1-a_1a_2a_3)(1-2a_1\cos\theta+a_1^2)}{a_1b_1}\right)^k.$$

For the $_5\phi_4$ in (8.3) we get

$$\lim_{q\to 1}\frac{(q^{\alpha+k+1};q)_{n-k}}{(q;q)_{n-k}}$$

$$\times \,_5\phi_4\left(\begin{matrix}q^{k-n},q^{n+k+\alpha+\beta+1},q^{\alpha+k+1}a_1/b_1,q^{\alpha+k+1}a_2/b_1,q^{\alpha+k+1}a_3/b_1\\q^{\alpha+k+1},q^{\alpha+k+1}b_2/b_1,q^{\alpha+\beta+k+2}/(b_1b_2),q^{\alpha+2k+1}a_1a_2a_3/b_1\end{matrix};q,q\right)$$

$$= \frac{(\alpha+k+1)_{n-k}}{(n-k)!}$$

$$\times \,_2F_1\left(\begin{matrix}-n+k,n+k+\alpha+\beta+1\\\alpha+k+1\end{matrix};\frac{b_2(b_1-a_1)(b_1-a_2)(b_1-a_3)}{(b_1-b_2)(b_1b_2-1)(b_1-a_1a_2a_3)}\right)$$

$$= P_{n-k}^{(\alpha+k,\beta+k)}\left(1-2\frac{b_2(b_1-a_1)(b_1-a_2)(b_1-a_3)}{(b_1-b_2)(b_1b_2-1)(b_1-a_1a_2a_3)}\right).$$

For the other factors in (8.3) we get

$$\lim_{q\to 1}\frac{q^{k(k-n)}}{a_4^{n-k}(q;q)_k}\frac{(q^{n+\alpha+\beta+1};q)_k}{(q^{\alpha+k}a_1a_2a_3/b_1;q)_k}(q^{\alpha+k+1}b_2/b_1,q^{\alpha+\beta+k+2}/(b_1b_2);q)_{n-k}$$

$$= \frac{(n+\alpha+\beta+1)_k}{k!}\left(\frac{b_1}{b_1-a_1a_2a_3}\right)^k\left(\frac{(b_1-b_2)(b_1b_2-1)}{b_1b_2}\right)^{n-k}.$$

Also put

$$x = 1-2\frac{b_2(1-2b_1\cos\theta+b_1^2)}{(1-b_1b_2)(b_2-b_1)},$$

$$y = 1-2\frac{b_2(b_1-a_1)(b_1-a_2)(b_1-a_3)}{(b_1-b_2)(b_1b_2-1)(b_1-a_1a_2a_3)}.$$
(8.5)

Then we obtain the following limit case of (8.2) as $q\to 1$:

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^n\frac{(n+\alpha+\beta+1)_k}{k!}P_{n-k}^{(\alpha+k,\beta+k)}(y)\left(\frac{x-y}{2}\right)^k.$$
(8.6)

Now interchange the a and b parameters in (8.2):

$$p_n(\cos\theta;a_1,a_2,a_3,a_4\mid q) = \sum_{k=0}^n c_{n,k}(a_1,a_2,a_3,a_4;b_1,b_2,b_3,a_4\mid q)$$

$$\times p_k(\cos\theta;b_1,b_2,b_3,a_4\mid q),$$
(8.7)

and use (8.3) with the a and b parameters interchanged and with the order of summation reversion formula [7, Exercise 1.4(ii)] applied to the $_5\phi_4$:

$$
c_{n,k}(a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4 \mid q) = \frac{(-1)^{n-k} q^{-\frac{1}{2}(n-k)(n+k-1)} (q; q)_n}{a_4^{n-k} (q; q)_{n-k} (q; q)_k}
$$

$$
\times \frac{(a_1 a_2 a_3 a_4 q^{n-1}; q)_n}{(b_1 b_2 b_3 a_4 q^{k-1}; q)_k (b_1 b_2 b_3 a_4 q^{2k}; q)_{n-k}} (b_1 a_4 q^k, b_2 a_4 q^k, b_3 a_4 q^k; q)_{n-k}
$$

$$
\times {}_5\phi_4 \left(\begin{matrix} q^{k-n}, q^{1-k-n}/(b_1 b_2 b_3 a_4), q^{1-n}/(a_1 a_4), q^{1-n}/(a_2 a_4), q^{1-n}/(a_3 a_4) \\ q^{1-n}/(b_1 a_4), q^{1-n}/(b_2 a_4), q^{1-n}/(b_3 a_4), q^{1-2n}/(a_1 a_2 a_3 a_4) \end{matrix} ; q, q \right). \quad (8.8)
$$

Now substitute (8.4) in (8.7) and (8.8) and let x and y be given by (8.5). By similar computations as for obtaining (8.6) we get as the limit of (8.7) for $q \to 1$ the following identity:

$$
\left(\frac{x-y}{2} \right)^n = \sum_{k=0}^n \frac{\alpha + \beta + 2k + 1}{\alpha + \beta + k + 1} \frac{n!}{(\alpha + \beta + k + 2)_n}
$$

$$
\times P_{n-k}^{(-\alpha - n - 1, -\beta - n - 1)}(y) P_k^{(\alpha, \beta)}(x). \quad (8.9)
$$

Formula (8.9) was earlier given by J. Koekoek and R. Koekoek [11, (21)]. As an alternative to their direct derivation (independently of the Askey–Wilson connection coefficients) one can compute that

$$
\frac{\int_{-1}^1 (x-y)^n P_k^{(\alpha, \beta)}(x) (1-x)^\alpha (1+x)^\beta \, dx}{\int_{-1}^1 (P_k^{(\alpha, \beta)}(x))^2 (1-x)^\alpha (1+x)^\beta \, dx}
$$

$$
= \frac{\alpha + \beta + 2k + 1}{\alpha + \beta + k + 1} \frac{2^n n!}{(\alpha + \beta + k + 2)_n} P_{n-k}^{(-\alpha - n - 1, -\beta - n - 1)}(y)
$$

by substituting the Rodrigues formula for Jacobi polynomials in the numerator on the left-hand side, then performing repeated integration by parts, then using Euler's integral representation for hypergeometric functions and finally reversing the order of summation in the resulting terminating hypergeometric series.

Remark 8.1. Formula (8.9) can also be obtained as the special case $\nu = -n$ of formula (3.1) in Cohl [5]. In that formula he gives an explicit expansion of $(z - x)^{-\nu}$ in terms of Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$, where the expansion coefficients turn out to be constant multiples of the expressions $(z-1)^{\alpha + 1 - \nu} (z+1)^{\beta + 1 - \nu} Q_{n+\nu-1}^{(\alpha + 1 - \nu, \beta + 1 - \nu)}(z)$ (the Q-functions being Jacobi functions of the second kind). His case $\nu = 1$ occurs in Szegő [15, (9.2.1)]. Cohl's formula could also have been proved in the way just sketched for (8.9).

From (8.6) and (8.9) we see (as also observed in [11]) that $AB = I = BA$, where A and B are the lower triangular matrices given for $m \geq n \geq 0$ by

$$
\begin{aligned}
A_{m,n} &= \frac{(\alpha + \beta + m + 1)_n}{n!} P_{m-n}^{(\alpha + n, \beta + n)}(y), \\
B_{m,n} &= \frac{\alpha + \beta + 2n + 1}{\alpha + \beta + n + 1} \frac{m!}{(\alpha + \beta + n + 2)_m} P_{m-n}^{(-\alpha - m - 1, -\beta - m - 1)}(y).
\end{aligned} \quad (8.10)
$$

In particular, we obtain from $AB = I$ the identities (4.18) and (4.14), while conversely from (4.18) with (α, β) running through all $(\alpha + j, \beta + j)$ $(j \in \mathbb{Z}_{\geq 0})$ the full set of scalar identities in

$AB = I$ for (α, β) can be derived. Similarly we obtain from $BA = I$ that

$$\sum_{k=0}^{n} \frac{\alpha + \beta + 2k + 1}{\alpha + \beta + 1} \frac{(\alpha + \beta + 1)_k}{(\alpha + \beta + n + 2)_k} P_k^{(\alpha,\beta)}(y) P_{n-k}^{(-\alpha-n-1,-\beta-n-1)}(y) = \delta_{n,0}. \quad (8.11)$$

Formula (8.11) also follows from (8.9) by putting $x = y$, as already observed in [11, (22)]. Conversely (see [11, p. 13]), from (8.11) with (α, β) running through all $(\alpha + j, \beta + j)$ $(j \in \mathbb{Z}_{\geq 0})$ the full set of scalar identities in $BA = I$ for (α, β) can be derived.

Note added in proof

The application in [13] of the case $\alpha = \beta = \frac{1}{2}$ of Theorem 4.1 has now been generalized to general parameter values $\alpha = \beta > -\frac{1}{2}$ in the preprint E. Koelink, A.M. de los Rios, P. Román, Matrix-valued Gegenbauer polynomials, arXiv:1403.2938.

Acknowledgments

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